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Vibration of Composite Shells of Revolution Using Equivalent Single Layer Approach

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Abstract

In this work higher order models of elastic shells of revolution are developed using the generalized elastodynamic variational principle. Following the Unified Carrera Formula (CUF), the stress and strain tensors, as well as the displacement vector, were expanded into series in terms of the coordinates of the shell thickness. As a result, all the equations of the elastodynamics were transformed into the corresponding equations for the expansion coefficients in a series in terms of the coordinates of the shell thickness. The resulting equations have been used for theoretical analysis and calculation of the eigenvalues and eigenmodes of the higher order shells of revolution.

1. Introduction

Shell models are important structural elements that are widely used in aviation and airspace engineering and technology. There are many books and thousands of articles on the theoretical analysis and modeling of the shells of revolution. Below, we will mention only a few of them that we used in preparing this article. The milestone book by Timoshenko and Woinowsky-Krieger [19] is a standard reference to the classical plate shells theory. Composited and laminated shells were considered in Ambartsumyan [2], Jin et al. [13], Qatu [17], Reddy [18].

In most of the publications mentioned above classical shell theories based on the Kirchhoff-Love and Timoshenko-Midlin hypotheses are used. There is another approach to the theory of shells, which consists in expanding the components of the stress-strain field into series of polynomials in thickness. This approach was first proposed by Cauchy and Poisson in the nineteenth century. Significant extensions and developments of this approach for shells of arbitrary geometry were made by Kilchevskiy [15]. He created the so-called generalized tensor series for the expansion of three-dimensional equations of elasticity in terms of the thickness of the shell. Then the Legendre polynomials were proposed for the development of new theories of higher orders. This approach has significant advantages, since the Legendre polynomials are orthogonal and, as a result, simpler equations are obtained. There are many books and research papers devoted to the application of the polynomial series to the development of higher order theories of bars, plates, and shells. Among others, the books of Khoma [14], Pelekh and Lazko [18] and the papers of Czekanski and Zozulya [11], Zozulya

[21, 22]. Carrera's Unified Formulations (CUF) approach can be viewed as a generalization of the polynomial decomposition method for beams, plates and shells, including sandwich structures and multi-field loads. Hundreds of articles are available at (CUF) on the various extensions and applications of Carrera and many more. Among them, the following are mentioned here: review papers Carrera [1-3] deals with multilayer anisotropic plates and shells, plates, and shells, shells of revolution are considered in Carrera and Zozulya [6-10]. For more information and references related to the polynomial series approach for developing models of multilayer anisotropic composite plates and shells and their thermal and finite element analysis, see Carrera et al. [4, 5], as well as the works mentioned above.

In this work, 2-D models of higher order shells of revolution are developed based on the 3-D equations of the dynamic theory of elasticity. Proposed models are based on the generalized variational principle and expansion of the 3-D equations of elasticity into generalized series in terms of cross-sectional coordinates in thickness. Numerical calculations of the eigenvalues and eigenmodes were performed using the computer algebra software **Mathematica**, and the results of calculation are presented in the form of tables and plots. These numerical results can be used as a benchmark example for finite element dynamic analysis of the elastic higher order shells.

2. Statement of the problem.

Let an elastic shell of revolution occupy a region $V=\Omega\times[-h,h]$ in a 3-D Euclidian space, where Ω is the middle surface and $2h$ is the thickness of the shell. The classical theory of elasticity assumes that the body consists of interconnected points and continuously fills the occupied volume. The position of a point during deformation is determined by the displacements vector $\mathbf{u}(\mathbf{x},t)=u_i(\mathbf{x},t)\mathbf{e}_i$ as functions of their coordinates and time. The stress-strain state of the shell as elastic continua is defined in terms of the symmetrical stress $\boldsymbol{\sigma}(\mathbf{x},t)=\sigma_{ij}(\mathbf{x},t)\mathbf{e}_i\otimes\mathbf{e}_j$ and strain $\boldsymbol{\varepsilon}(\mathbf{x},t)=\varepsilon_{ij}(\mathbf{x},t)\mathbf{e}_i\otimes\mathbf{e}_j$ tensors. Boundary of the shell is piece-wise smooth and consists of sections ∂V_p and ∂V_u to which the vectors of traction $\mathbf{p}(\mathbf{x},t)=p_i(\mathbf{x},t)\mathbf{e}_i$ and displacements $\mathbf{u}(\mathbf{x},t)$ respectively, are assigned. The elastic shell may be subjected to volume forces $\mathbf{b}(\mathbf{x},t)$.

Following our previous publications, here we introduce vector notations and represent the above functions that determine the stress-strain state of elastic media in the vector form.

$$\begin{aligned}&\mathbf{u}=|u_1,u_2,u_3|^T,\quad \mathbf{p}=|p_1,p_2,p_3|^T,\quad \mathbf{b}=|b_1,b_2,b_3|^T,\\&\boldsymbol{\sigma}=|\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{12},\sigma_{23},\sigma_{13}|^T,\quad \boldsymbol{\varepsilon}=|\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},\varepsilon_{12},\varepsilon_{23},\varepsilon_{13}|^T\end{aligned}\tag{1}$$

These quantities are not independent, they are related by the equations of linear elasticity. Here we will show that all the equations of linear elastodynamics including boundary conditions can be obtained from the generalized variational principle (see Gurtin [12]). For this purpose, let us introduce the generalized functional, that depends on the functions $\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}$ defined above in the form

$$\Phi=\int_{t_0}^{t_1}\left(\int_V\left(\boldsymbol{\sigma}\cdot(\mathbf{D}\cdot\mathbf{u}-\boldsymbol{\varepsilon})+W(\boldsymbol{\varepsilon})-\mathbf{b}\cdot\mathbf{u}+\frac{\rho}{2}\partial_t\mathbf{u}\cdot\partial_t\mathbf{u}\right)dV-\int_{\partial V_p}\mathbf{p}\cdot\mathbf{u}dS-\int_{\partial V_u}\boldsymbol{\sigma}\cdot\mathbf{n}\cdot(\mathbf{u}-\boldsymbol{\varphi})dS\right)dt\tag{2}$$

Here $\mathbf{D}$ is a matrix differential operator, whose form depends on chosen coordinate system (see Carrera and Zozulya [6] for references), ρ is a material density, $W(\boldsymbol{\varepsilon})$ is a potential energy function.

Let us consider a variation of the functional (2), taking into account that all the above functions are independent. The variational principles of elastodynamics and the details of the corresponding functionals variation are considered in Gurtin [12]

After some transformations and simplifications, we get

$$\delta\Phi = \int_{t_0}^{t_1}\left(\int_V \left(\delta\boldsymbol{\sigma}\cdot\left(\mathbf{D}\cdot\mathbf{u}-\boldsymbol{\varepsilon}\right)-\delta\mathbf{u}\cdot(\mathbf{D}\cdot\boldsymbol{\sigma}+\mathbf{b}-\rho\,\partial_t^2\mathbf{u})+\delta\boldsymbol{\varepsilon}\cdot\left(\frac{\partial W}{\partial\boldsymbol{\varepsilon}}-\boldsymbol{\sigma}\right)\right)dV + \right.$$
$$\left. + \int_{\partial V_p}(\delta\mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}-\mathbf{p})dS - \int_{\partial V_u}\delta\boldsymbol{\sigma}\cdot\mathbf{n}\cdot(\mathbf{u}-\boldsymbol{\varphi})dS\right)dt = 0 \tag{3}$$

Here we take into account that

$$\int_V \boldsymbol{\sigma}\cdot\mathbf{D}\cdot\delta\mathbf{u}dV = \int_{\partial V_p}\boldsymbol{\sigma}\cdot\delta\mathbf{u}\cdot\mathbf{n}dS + \int_{\partial V_u}\boldsymbol{\sigma}\cdot\delta\mathbf{u}\cdot\mathbf{n}dS - \int_V \mathbf{D}\cdot\boldsymbol{\sigma}\cdot\delta\mathbf{u}dV \tag{4}$$

Variation of the kinetic energy of a linear elastic media

$$\mathrm{K} = \frac{\rho}{2}\int_V \partial_t\mathbf{u}\cdot\partial_t\mathbf{u}dV \tag{5}$$

presented by the equation, see Gurtin [12]

$$\int_{t_0}^{t_1}\delta\mathrm{K}dt = -\rho\int_{t_0}^{t_1}dt\int_V \partial_t^2\mathbf{u}\cdot\delta\mathbf{u}dV \tag{6}$$

In view of variations $\delta\mathbf{u}$, $\delta\boldsymbol{\sigma}$ and $\delta\boldsymbol{\varepsilon}$ are independent, all equations of elastodynamics and the corresponding boundary conditions follow from the equation (3), they have the form:

$$\begin{aligned} &\mathbf{D}\cdot\boldsymbol{\sigma}-\mathbf{b}=\rho\ddot{\mathbf{u}}, \quad \boldsymbol{\varepsilon}=\mathbf{D}\cdot\mathbf{u}, \\ &\boldsymbol{\sigma}=\frac{\partial W}{\partial\boldsymbol{\varepsilon}}, \quad \boldsymbol{\sigma}\cdot\mathbf{n}-\boldsymbol{\psi}=0, \qquad \mathbf{u}-\boldsymbol{\varphi}=0. \end{aligned} \tag{7}$$

In the case of linear orthotropic elastic media, potential energy function can be presented in the following general form

$$W(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}^T\cdot\mathbf{C}\cdot\boldsymbol{\varepsilon} \tag{8}$$

where $\mathbf{C}$ is the 6×6 matrix of elasticity moduli of the form

$$\mathbf{C} = \begin{vmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{vmatrix} \tag{9}$$

In the case of isotropic material, the corresponding classical moduli of elasticity presented in (9) have the form

$$C_{11}=C_{22}=C_{33}=\lambda+2\mu, \quad C_{12}=C_{13}=C_{23}=\lambda, \quad C_{44}=C_{55}=C_{66}=\mu \tag{10}$$

where λ and μ are Lamé constants of classical elasticity.

In the general case of inhomogeneous anisotropic body coefficients of the matrix of elastic properties depend on coordinates $C_{ij}(\mathbf{x})$ and orientation in the space. In the case of multilayer composite shells, it is possible to use layers of different materials, therefore the elastic moduli are piecewise constant over the thickness and depend only on the coordinate perpendicular to the middle surface x_3.

For the case of a multilayer composite shell consisting of K laminas the dependence of the elastic moduli on the stiffness coordinate can be mathematically expressed as follows

$$C_{ij}(x_3)=\sum_{k=1}^{K} C_{ij}^{k}\left(H(x_3-X_k-h_k)-H(x_3-X_k+h_{k+1})\right) \tag{11}$$

Here C_{ij}^{k} is the value of the elastic moduli of the k lamina, h_k is the coordinate of the lower surface of the k lamina and $H(x)$ is the Heaviside unit step function.

The individual layers generally are orthotropic with principal properties in orthogonal directions. Their mechanical properties depend on fibers orientation. The values of elastic moduli along the fibers much higher than in perpendicular direction. An orthotropic material behavior is characterized by three symmetry planes that are mutually orthogonal. Matrix of elastic moduli has the form

$$\mathbf{C}^{k}=\begin{vmatrix} C_{11}^{k} & C_{12}^{k} & C_{13}^{k} & 0 & 0 & 0 \\ C_{12}^{k} & C_{22}^{k} & C_{23}^{k} & 0 & 0 & 0 \\ C_{13}^{k} & C_{23}^{k} & C_{33}^{k} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{k} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^{k} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^{k} \end{vmatrix} \tag{12}$$

Laminate composite shells usually have layers with different fiber orientation, therefore, when setting problems for laminate structures, it is necessary to calculate mechanical properties with a change of the coordinate system. Let us consider the reference coordinate system $\mathbf{x}=(x_1,x_2,x_3)$ and rotate it around axis x_3 by an angle θ. Matrix operator of such transformation has the form

$$\mathbf{L}(\theta)=\begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{13}$$

The stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ vectors under such a transformation of coordinates are transformed as follows

$$\boldsymbol{\sigma}'=\mathbf{T}(\theta)\boldsymbol{\sigma},\quad \boldsymbol{\varepsilon}'=\mathbf{T}^{*}(\theta)\boldsymbol{\varepsilon} \tag{14}$$

Matrix operators $\mathbf{T}(\theta)$ and it transposed operator $\mathbf{T}^{*}(\theta)$ can be obtained from the operator $\mathbf{L}(\theta)$ by comparing the transformation rules of tensors and vectors, see [9,10] for details.

Taking the derivative of the potential energy density function with respect to the strain $\boldsymbol{\varepsilon}$ tensor and substituting the kinematic relations into the obtained result, the classical stress vector in the case of linear orthotropic elastic media can be presented in the following forms

$$\boldsymbol{\sigma}=\frac{\partial W}{\partial \boldsymbol{\varepsilon}}=\mathbf{C}\cdot\boldsymbol{\varepsilon}=\mathbf{C}\cdot\mathbf{D}\cdot\mathbf{u} \tag{15}$$

Substituting the equations for the matrix of material constants (9), and the stress vector (15), into the equations of motion (7) the differential in the form of displacements can be represented in a compact form as follows

$$\mathbf{L}\cdot\mathbf{u}-\mathbf{b}=\rho\ddot{\mathbf{u}} \tag{16}$$

In the same way, substituting the expressions for the stress vector from (15), one obtains the natural boundary conditions for the linear theory of elasticity in the form of a displacement vector

$$\mathbf{B}\cdot\mathbf{u}=\mathbf{p} \tag{17}$$

where $\mathbf{L}$ and $\mathbf{B}$ are the matrix differential operators, $\mathbf{u}$ is the vector of unknown functions and $\mathbf{b}$ and $\mathbf{p}$ are the vectors of external load and surface traction, respectively. They have the following form

$$\mathbf{L}=\begin{vmatrix} L_{u_1,u_1} & L_{u_1,u_2} & L_{u_1,u_3} \\ L_{u_2,u_1} & L_{u_2,u_2} & L_{u_2,u_3} \\ L_{u_3,u_1} & L_{u_3,u_2} & L_{u_3,u_3} \end{vmatrix},\quad \mathbf{u}=\begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}\quad \mathbf{b}=\begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix},\ \mathbf{B}=\begin{vmatrix} B_{u_1,u_1} & B_{u_1,u_2} & B_{u_1,u_3} \\ B_{u_2,u_1} & B_{u_2,u_2} & B_{u_2,u_3} \\ B_{u_3,u_1} & B_{u_3,u_2} & B_{u_3,u_3} \end{vmatrix},\qquad \mathbf{p}=\begin{vmatrix} p_1 \\ p_2 \\ p_3 \end{vmatrix} \tag{18}$$

In the case of free vibration with circular (natural) frequency ω vector of displacements can be presented in the form

$$\mathbf{u}(\mathbf{x},t)=\mathbf{U}(\mathbf{x})e^{i\omega t} \tag{19}$$

Then substituting (19) in the equation of motion (16) we obtain the equation of free vibration in the form

$$\mathbf{L}\cdot\mathbf{U}+\rho\omega^2\mathbf{U}=0 \tag{20}$$

This equation is used to model free vibration of elastic systems and analysis of the eigenvalue. For the purpose of the theory of shells of a higher order developed here, in the same way as in our previous publications, we introduce curvilinear coordinates $\mathbf{x}=(x_1,x_2,x_3)$ related to the middle surface of the shell. The coordinates (x_1,x_2) are related to the principal curvatures k_1 and k_2 whereas coordinate x_3 is perpendicular to the middle surface of the shell. Then position of any material point of the shell in the domain V may be presented by the position vector $\mathbf{R}(\mathbf{x})$ in the form

$$\mathbf{R}(\mathbf{x})=\mathbf{r}(x_1,x_2)+x_3\mathbf{n}(x_1,x_2) \tag{21}$$

where $\mathbf{r}(x_1,x_2)$ is the position vector of the points located in the middle surface of the shell, and $\mathbf{n}(x_1,x_2)$ is a unit vector perpendicular to the middle surface of the shell.
Since the introduced coordinates are orthogonal, the Lamé coefficients and their derivatives can be represented as

$$\begin{gathered} H_\alpha=A_\alpha(1+k_\alpha x_3)\ \text{ for } \alpha=1,2\ \text{ and }\ H_3=1, \\ \frac{\partial H_\beta}{\partial x_\alpha}=\frac{\partial A_\beta}{\partial x_\alpha}(1+k_\alpha x_3),\ \frac{\partial H_\beta}{\partial x_3}=k_\beta A_\beta,\ \frac{\partial H_3}{\partial x_i}=0 \end{gathered} \tag{22}$$

Here $A_\alpha(x_1,x_2)=\sqrt{\dfrac{\partial\mathbf{r}(x_1,x_2)}{\partial x_\alpha}\cdot\dfrac{\partial\mathbf{r}(x_1,x_2)}{\partial x_\alpha}}$ are the coefficients of the first quadratic form of a surface, k_α are the principal curvatures and $\alpha=1,2$.
The CUF approach to the development of the theory of plates and shells of higher order consists in the following. The displacement $\mathbf{u}(x_1,x_2,x_3)$ vector and its variation $\delta\mathbf{u}(x_1,x_2,x_3)$, which are functions of curvilinear coordinates (x_1,x_2,x_3) are represented as series of functions of coordinate x_3 orthogonal to the middle surface of the shell, in the form

$$\mathbf{u}(x_1,x_2,x_3)=\mathbf{F}_{\mathbf{u},\tau}(x_3)\cdot\mathbf{u}_\tau(x_1,x_2),\quad \delta\mathbf{u}(x_1,x_2,x_3)=\mathbf{F}_{\mathbf{u},\tau}(x_3)\cdot\delta\mathbf{u}_\tau(x_1,x_2),\qquad \tau=1,2,\dots,M \tag{23}$$

Here, according to Einstein's notation, the repeated subscript τ indicates summation from 0 to M.
In the equations (23) the basic functions of the thickness coordinates $\mathbf{F}_{\mathbf{u},\tau}(x_3)$ and vector of displacement $\mathbf{u}_\tau(x_1,x_2)$ have the form

$$\mathbf{F}_{\mathbf{u},\tau}(x_3)=\begin{vmatrix} F_{u_1,\tau}(x_3) & 0 & 0 \\ 0 & F_{u_2,\tau}(x_3) & 0 \\ 0 & 0 & F_{u_3,\tau}(x_3) \end{vmatrix}, \quad \mathbf{u}_\tau(x_1,x_2)=\begin{vmatrix} u_{1,\tau}(x_1,x_2) \\ u_{2,\tau}(x_1,x_2) \\ u_{3,\tau}(x_1,x_2) \end{vmatrix} \tag{24}$$

In the general case, the choice of the number M and functions $\mathbf{F}_{\mathbf{u},\tau}(x_3)$ is arbitrary, i.e., to model the displacements field of shell over its thickness, one can consider various basis functions of any order. The final equation becomes simple if functions $\mathbf{F}_{\mathbf{u},\tau}$ are polynomials, especially orthogonal polynomials. The expansions coefficients $\mathbf{u}_\tau(x_1,x_2)$ are functions of the coordinates x_1 and x_2 which belong to the middle surface of the shell. The first subscript in the basis functions $\mathbf{F}_{\mathbf{u},\tau}$ indicates the displacement vector component, the second index indicates the number of the function in the series expansion.
Substituting the displacement vector represented by series expansion (23) to the kinematic Cauchy relations (7), one can obtain the strain vector in the form

$$\boldsymbol{\varepsilon}=\mathbf{D}_{u,\tau}\cdot\mathbf{u}_\tau \tag{25}$$

where $\mathbf{D}_{a,\tau}$ is a matrix operator of the form

$$\mathbf{D}^T_{\mathbf{u},\tau}=\begin{vmatrix} \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} & \frac{F_{u_2,\tau}}{A_1A_2}\frac{\partial A_2}{\partial x_1} & 0 & \frac{F_{u_1,\tau}}{A_2}\frac{\partial}{\partial x_2}-\frac{F_{u_2,\tau}}{A_1A_2}\frac{\partial A_1}{\partial x_2} & 0 & \frac{\partial F_{u_1,\tau}}{\partial x_3}-F_{u_3,\tau}k_1 \\ \frac{F_{u_1,\tau}}{A_1A_2}\frac{\partial A_1}{\partial x_2} & \frac{F_{u_2,\tau}}{A_2}\frac{\partial}{\partial x_2} & 0 & \frac{F_{u_2,\tau}}{A_1}\frac{\partial}{\partial x_1}-\frac{F_{u_1,\tau}}{A_1A_2}\frac{\partial A_2}{\partial x_1} & \frac{\partial F_{u_2,\tau}}{\partial x_3}-F_{u_3,\tau}k_2 & 0 \\ F_{u_1,\tau}k_1 & F_{u_2,\tau}k_2 & \frac{\partial F_{u_3,\tau}}{\partial x_3} & 0 & \frac{F_{u_3,\tau}}{A_2}\frac{\partial}{\partial x_2} & \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} \end{vmatrix} \tag{26}$$

Substituting kinematic Cauchy relations (25) into the generalized Hooke's law (15) the stress vector can be presented as

$$\boldsymbol{\sigma}=\mathbf{C}\cdot\mathbf{D}_{u,\tau}\cdot\mathbf{u}_\tau \tag{27}$$

Substituting the expressions for the strain vectors represented by equations (25), the stress vectors represented by equations (27) into (8), we obtain the variation of the potential energy density in the form

$$\delta W=\mathbf{D}^T_{u,\tau}\cdot\mathbf{F}_{u,\tau}(x_3)\cdot\delta\mathbf{u}_\tau(x_1,x_2)\cdot\mathbf{C}\cdot(\mathbf{D}_{u,s}\cdot\mathbf{F}_{u,s}(x_3)\cdot\mathbf{u}_s(x_1,x_2) \tag{28}$$

The variation of the work of the external body and surface forces can be represented as

$$\delta L_{ext}(\mathbf{b},\mathbf{p})=\int_V\mathbf{F}_{u,\tau}(x_3)\cdot\delta\mathbf{u}_\tau(x_1,x_2)\cdot\mathbf{F}_{u,\tau}(x_3)\cdot\mathbf{b}_{u,\tau}(x_1,x_2)dV+\int_{\partial V}\mathbf{F}_{u,\tau}(x_3)\cdot\delta\mathbf{u}_\tau(x_1,x_2)\cdot\mathbf{p}(x_1,x_2)dS \tag{29}$$

Substituting representations (23) for vector of the displacements and its variation, the variation of the potential energy density (28) and the variation of the work of the external body and surface forces (29) in general variation principle (2) and using the matrix analogy of Gauss-Ostrogradsky divergence theorem in the form

$$\begin{aligned}&\int_V\mathbf{D}^y_{u,\tau}\cdot\mathbf{C}\cdot\mathbf{D}_{u,s}\cdot\mathbf{F}_{u,s}(x_3)\cdot\mathbf{u}_s(x_1,x_2)\cdot\delta\mathbf{u}_\tau(x_1,x_2)dV=\\ &\qquad=\int_{\partial V}\mathbf{D}^{u,T}_{n,\tau}\cdot\mathbf{C}\cdot\mathbf{D}_{u,s}\cdot\mathbf{F}_{u,s}(x_3)\cdot\mathbf{u}_s(x_1,x_2)\cdot\delta\mathbf{u}_\tau(x_1,x_2)dS\end{aligned} \tag{30}$$

where $\mathbf{D}^{u,T}_{n,\tau}$ is the matrix analogy of the vector normal to the boundary, which has the form

$$\mathbf{D}_{n,\tau}^{u,T} = \begin{vmatrix} n_1 F_{u_1,\tau} & 0 & n_2 F_{u_1,\tau} & 0 & 0 & 0 \\ 0 & n_2 F_{u_2,\tau} & n_1 F_{u_2,\tau} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_2 F_{u_3,\tau} & n_1 F_{u_3,\tau} \end{vmatrix} \tag{31}$$

we obtain differential equations of motion for high-order elastic shells in the form of displacements. They can be represented in matrix form

$$\mathbf{L}_M^G \cdot \mathbf{u}_M^G - \mathbf{b}_M^G = \mathbf{M}_M^G \ddot{\mathbf{u}}_M^G \tag{32}$$

where the global matrix operator $\mathbf{L}_n^G$, the vectors of unknown functions $\mathbf{u}_M^G$ and the right hand $\mathbf{b}_M^G$ side have the form

$$\mathbf{L}_M^G = \begin{vmatrix} \mathbf{L}_{1,1}^{loc} & \cdots & \mathbf{L}_{1,M}^{loc} \\ \vdots & \ddots & \vdots \\ \mathbf{L}_{M,1}^{loc} & \cdots & \mathbf{L}_{M,M}^{loc} \end{vmatrix}, \quad \mathbf{M}_M^G = \begin{vmatrix} \mathbf{M}_{1,1}^{loc} & \cdots & \mathbf{M}_{1,M}^{loc} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{M,1}^{loc} & \cdots & \mathbf{M}_{M,M}^{loc} \end{vmatrix}, \quad \mathbf{u}_M^G = \begin{vmatrix} \mathbf{u}_1^{loc} \\ \vdots \\ \mathbf{u}_M^{loc} \end{vmatrix}, \quad \mathbf{b}_M^G = \begin{vmatrix} \mathbf{b}_1^{loc} \\ \vdots \\ \mathbf{b}_M^{loc} \end{vmatrix} \tag{33}$$

Matrices $\mathbf{L}_{\tau,s}^{loc}$ are the fundamental nuclei of the differential equations of equilibrium of elastic shells of higher orders. They, as well as the vectors of local unknown functions $\mathbf{u}_s^{loc}$ and local expression for external body and surface loads $\mathbf{b}_s^{loc}$ have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_1,u_1}^{\tau,s} & L_{u_1,u_2}^{\tau,s} & L_{u_1,u_3}^{\tau,s} \\ L_{u_2,u_1}^{\tau,s} & L_{u_2,u_2}^{\tau,s} & L_{u_2,u_3}^{\tau,s} \\ L_{u_3,u_1}^{\tau,s} & L_{u_3,u_2}^{\tau,s} & L_{u_3,u_3}^{\tau,s} \end{vmatrix}, \mathbf{M}_{\tau,s}^{loc} = \begin{vmatrix} M_{u_1,u_1}^{\tau,s} & 0 & 0 \\ 0 & M_{u_2,u_2}^{\tau,s} & 0 \\ 0 & 0 & M_{u_3,u_3}^{\tau,s} \end{vmatrix}, \mathbf{u}_s^{loc} = \begin{vmatrix} u_{1,s} \\ u_{2,s} \\ u_{3,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_1,\tau} \\ \tilde{b}_{u_2,\tau} \\ \tilde{b}_{u_3,\tau} \end{vmatrix} \tag{34}$$

For free vibration and eigenvalue analysis similarly to (20) from (32) we obtain

$$\mathbf{L}_M^G \cdot \mathbf{U}_M^G + \omega^2 \mathbf{M}_M^G \mathbf{U}_M^G = 0 \tag{35}$$

When developing differential equations of motion for high-order elastic shells in the form of displacements, we take into account that the integrals over volume and surface in the equation (3) can be represented as

$$\int_V (\cdot) dV = \int_\Omega \int_{-h}^{h} (\cdot) dx_3 d\Omega, \quad \int_{\partial V} (\cdot) dV = \int_{\partial\Omega} \int_{-h}^{h} (\cdot) dx_3 dS \tag{36}$$

Similarly, by performing the mathematical transformations described in the article Carrera and Zozulya [9, 10], the natural boundary conditions for higher order elastic shells can be represented in the matrix form

$$\mathbf{B}_M^{N,G} \cdot \mathbf{u}_M^G = \mathbf{p}_M^G \tag{37}$$

where the global matrix operator $\mathbf{B}_M^{N,G}$, the vectors of unknown functions $\mathbf{u}_M^G$ and the right-hand side $\mathbf{p}_M^G$ have the form

$$\mathbf{B}_M^{N,G} = \begin{vmatrix} \mathbf{B}_{1,1}^{loc} & \cdots & \mathbf{B}_{1,M}^{loc} \\ \vdots & \ddots & \vdots \\ \mathbf{B}_{M,1}^{loc} & \cdots & \mathbf{B}_{M,M}^{loc} \end{vmatrix}, \quad \mathbf{p}_M^G = \begin{vmatrix} \mathbf{p}_1^{loc} \\ \vdots \\ \mathbf{p}_M^{loc} \end{vmatrix} \tag{38}$$

The matrices $\mathbf{B}_{\tau,s}^{loc}$ are the fundamental nuclei for the natural boundary conditions for higher order elastic shells and $\mathbf{p}_s^{loc}$ represent an expression for the vectors of the external load applied to the ends of the shells. They can be presented as

$$\mathbf{B}_{\tau,s}^{loc} = \begin{vmatrix} B^{\tau,s}{}_{u_1,u_1} & B^{\tau,s}{}_{u_1,u_2} & B^{\tau,s}{}_{u_1,u_3} \\ B^{\tau,s}{}_{u_2,u_1} & B^{\tau,s}{}_{u_2,u_2} & B^{\tau,s}{}_{u_2,u_3} \\ B^{\tau,s}{}_{u_3,u_1} & B^{\tau,s}{}_{u_3,u_2} & B^{\tau,s}{}_{u_3,u_3} \end{vmatrix}, \quad \mathbf{p}_s^{loc} = \begin{vmatrix} J_{\tau,s}^{u_1,u_1} P_{u_1,s} \\ J_{\tau,s}^{u_2,u_2} P_{u_2,s} \\ J_{\tau,s}^{u_3,u_3} P_{u_3,s} \end{vmatrix} \tag{39}$$

The essential boundary conditions for higher order elastic shells can be represented in matrix form

$$\mathbf{B}_M^{E,G} \cdot \mathbf{u}_M^G \Big|_0^L = \mathbf{u}_M^{0,G} \tag{40}$$

Here the global matrix operator $\mathbf{B}_M^{E,G}$, the vectors of the right-hand side have the form

$$\mathbf{B}_M^{E,G} = \begin{vmatrix} \mathbf{I} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{I} \end{vmatrix}, \quad \mathbf{u}_M^{0,G} = \begin{vmatrix} \mathbf{u}_1^{0,loc} \\ \vdots \\ \mathbf{u}_M^{0,loc} \end{vmatrix} \tag{41}$$

where $\mathbf{I}$ is the identity matrix and therefore the global matrix operator $\mathbf{B}_M^{E,G}$ is the identity matrix operator.

Equations (40)-(41) are valid for the theory of shells of any order. The classical theory of shells based on Timoshenko-Mindlin hypothesis can be obtained from these equations as a special case if we take in the equations (32) the basic functions of the thickness coordinates $\mathbf{F}_{\mathbf{u},\tau}(x_3)$ and vector of displacement $\mathbf{u}_\tau(x_1,x_2)$ (24) in the form

$$\mathbf{F}_{\mathbf{u},1}(x_3) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mathbf{u}_1(x_1,x_2) = \begin{vmatrix} u_{1,1}(x_1,x_2) \\ u_{2,1}(x_1,x_2) \\ u_{3,1}(x_1,x_2) \end{vmatrix}, \quad \mathbf{F}_{\mathbf{u},2}(x_3) = \begin{vmatrix} x_3 & 0 & 0 \\ 0 & x_3 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \mathbf{u}_2(x_1,x_2) = \begin{vmatrix} u_{1,2}(x_1,x_2) \\ u_{2,0}(x_1,x_2) \\ 0 \end{vmatrix} \tag{42}$$

In the following sections, based on the equations presented here, we consider in more detail shells of revolution of various geometries and solve problems free vibration numerically using build-in functions **NDEigensystem** and **NDEigenvalues** of the computer algebra software **Mathematica**.

3. Analysis of eigenvalues and natural frequencies of shells of revolution

In this section, based on the approach developed here, we consider in more detail composite laminate axisymmetric shells of revolution of various geometries and solve free vibration problems and eigenvalue analysis. For that purpose we will use system of differential equations (35) and the corresponding essential boundary conditions (40).

Usually, such problems are solved using the FEM or other numerical methods. Many commercial and free open software programs can be used to solve considered here problems. In this study, we use the computer algebra software **Mathematica**, namely the build-in functions **NDEigensystem** and **NDEigenvalues**. As an introduction to the Mathematica and its programming language, one can refer the books by Wolfram [20] and the Wolfram Language documentation may also be useful.

Here we consider as special cases of shells of revolution a laminate composite axisymmetric circular plate, cylindrical, and conical shells with three layers and angle-ply layup $(0/90/0)$ under uniform symmetrical loading applied to the upper surface of the shell. The layers have the same thickness $h_1 = 0.33h$, $h_2 = 0.33h$ and $h_3 = 0.33h$. The shells are fixed at the ends and the mechanical properties of the laminas are taken as in [9, 10]. The following mechanical properties of lamina are used:

$$E_1 = 25,\ E_2 = 1, G_{12} = G_{13} = G_{23} = 0.5, \nu_{12} = \nu_{13} = \nu_{23} = 0.25, \rho = 1 \tag{43}$$

Here E_1, E_2 are Young's moduli, G_{12}, G_{13}, G_{23} are the shear moduli, ν_{12}, ν_{13} are Poisson's ratios and ρ is a density of material.

Due to the fact that we consider axisymmetric shells of revolution, all functions describing the stress-strain state of the shell do not depend on rotational coordinate, all derivatives with respect to it vanish, and the displacement in the circumferential direction is zero. Therefore, tensors of stress $\boldsymbol{\sigma}(x,r)$ and strain $\boldsymbol{\varepsilon}(x,r)$ tensors, as well as the displacements $\mathbf{u}(x,r)$, forces vectors introduced in (1) the form

$$\boldsymbol{\sigma} = \left|\sigma_{xx}, \sigma_{\varphi\varphi}, \sigma_{rr}, \sigma_{xr}\right|^T, \quad \boldsymbol{\varepsilon} = \left|\varepsilon_{xx}, \varepsilon_{\varphi\varphi}, \varepsilon_{rr}, \varepsilon_{xr}\right|^T, \quad \mathbf{u} = \left|u_x, u_r\right|^T \tag{44}$$

The resulting differential equations can be easily obtained from the general case presented above, using the values of the coefficients of the first quadratic form of the surface and the principal curvatures. More details for the case of the homogeneous shells can be found in Carrera and Zozulya [6-10].

3.1 Circular axisymmetric plate

Let us consider circular plate as special case of the shell of revolution. In order to obtain equations that describe mechanical behaviour of the higher order composite multilayer circular plate, let us introduce polar coordinates, where $x_1 = \rho$, $x_2 = \varphi$ and $x_3 = z$, $z \in [-h, h]$. The middle surface of the circular plate is the circle of radius R_1 with hole of radius R_2 shown in the Fig. 1. Coefficients of the first quadratic form of a surface and principal curvatures are equal to $A_1 = 1, A_2 = \rho,$ and $k_1 = 0,\ k_2 = 0$, respectively.

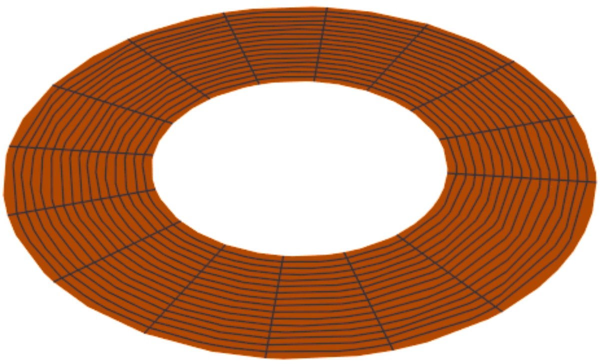

Fig.1. Circular plate as special case of the shell of revolution

Here we present some results of numerical simulation of the higher order three layers composite laminate circular plate based on CUF. Calculations are made for the following data: mechanical loading is axisymmetric, uniform and applied to the upper surface of the plate, $L = L_2 - L_1$, in the axial direction x_3. Here L_2 and L_1 are outer and inner radius of the plate, $L_2 / L_1 = 2$ and $h / L = 0.1$

Since the external load is applied axisymmentricaly, the situation becomes simpler. In this case, all functions describing the stress-strain state of the plate are one-dimensional, and the resulting equations are much simpler.

The resulting differential equations have the same structure as (35), but the local matrices $\mathbf{L}^{loc}_{\tau,s}$ of the fundamental nuclei of differential equations of equilibrium for the higher order axisymmetric spherical elastic shells, as well as the vectors of local unknown functions $\mathbf{U}^{loc}_s$ have the form

$$\mathbf{L}^{loc}_{\tau,s} = \begin{vmatrix} L^{\tau,s}_{u_\rho,u_\rho} & L^{\tau,s}_{u_\rho,u_z} \\ L^{\tau,s}_{u_z,u_\rho} & L^{\tau,s}_{u_z,u_z} \end{vmatrix}, \quad \mathbf{U}^{loc}_s = \begin{vmatrix} U_{\rho,s} \\ U_{z,s} \end{vmatrix} \tag{45}$$

The coefficients of the fundamental nuclei $\mathbf{L}_{\tau,s}^{loc}$ can be easily calculated using the equations presented in our previous publications [6-10].

In Table 1, the lowest eight frequency parameters $\Omega = \omega\sqrt{\rho / L^{0,0}_{u_\rho,u_\rho}}$ for the there-layered composite axisymmetric circular fixed at the ends are presented for the Timoshenko's shear deformation, the first, second, third, fourth and fifth order models.

Table 1. Frequency parameter $\Omega = \omega\sqrt{\rho / L^{0,0}_{u_\rho,u_\rho}}$ of the axisymmetric circular plate

Models, h/L=0.1	Eigenmode 1	Eigenmode 2	Eigenmode 3	Eigenmode 4	Eigenmode 5	Eigenmode 6	Eigenmode 7	Eigenmode 8
Timoshenko	1.8518	3.67503	5.66747	7.62113	9.58899	10.8162	11.5651	12.5857
First order	1.8518	3.67503	5.66747	7.62113	9.58899	10.8162	11.5651	13.4973
Second order	1.85109	3.67079	5.65776	7.60014	9.54887	10.7954	11.4945	13.359
Third order	1.6599	3.37149	5.25001	7.16241	9.11536	9.67998	11.0734	13.0143
Fourth order	1.6591	3.3705	5.24915	7.16035	9.10677	9.67162	11.0398	12.8773
Fifth order	1.6471	3.34141	5.21079	7.11638	8.88799	9.05891	10.9812	12.7899

Fig. 2 shows graphs of the distribution of the first eight axisymmetric eigenmodes in the radial direction for the fifth-order model $\tau = 5$ for a there-layered composite axisymmetric circular plate fixed at the ends.

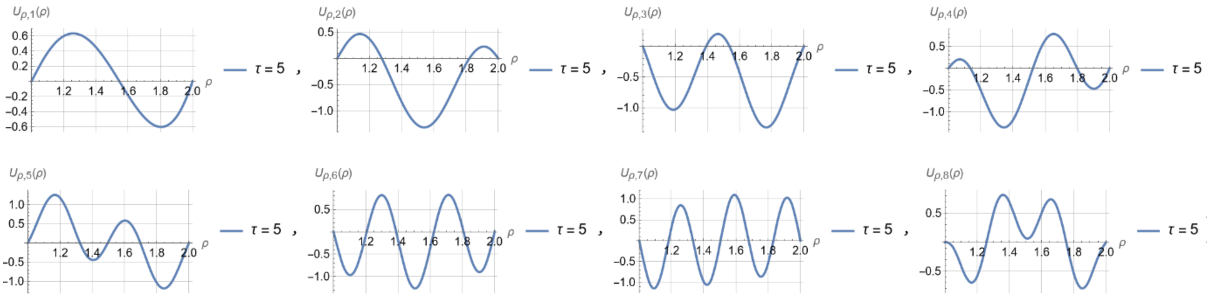

Fig. 2. First eight axisymmetric eigenmodes for circular plate in axial direction for fifth order model.

Fig. 3 shows graphs of the distribution of the first eight axisymmetric eigenmodes in the axial direction for the fifth-order model $\tau = 5$ for a there-layered composite axisymmetric circular plate fixed at the ends.

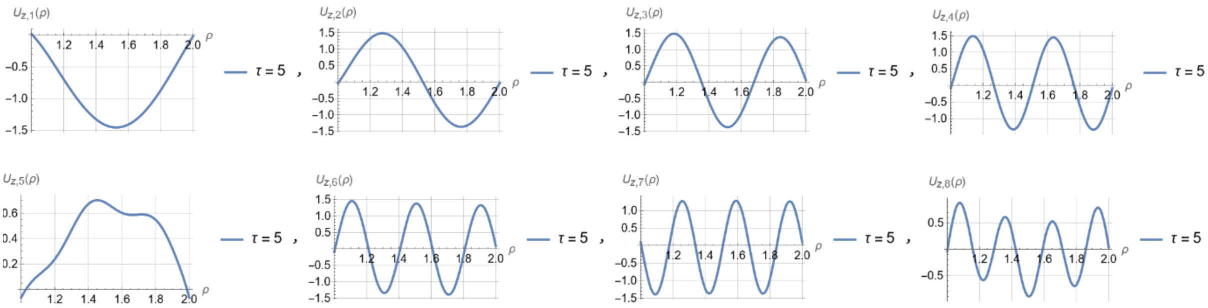

Fig. 3. First eight axisymmetric eigenmodes for circular plate in radial direction for fifth order model.

The data presented in Table 1 and Fig 2, 3 gives qualitative and quantitative information about the behavior of the first eight axisymmetric eigenvalues and eigenvectors of the the fixed at the ends circular plate within the framework of the CUF for the first, second, third, fourth and fifth order $\tau = 1,\ldots,5$ models and a comparison with the classical Timoshenko model. The reported data are in good agreement; indeed, for models of the third order and higher, the results are the same. Table 1 follows that the third order and higher models give more accurate results and better models the free vibration of circular axisymmetric plates. These results can be used as benchmark examples for finite element analysis of elastic composite multilayers circular plates in the case of axisymmetric loading.

3.2. Axisymmetric cylindrical shell

Let us consider a cylindrical shell formed by rotation around an axis x_3 of a straight line that parallel to it and located at a distance R from it. The middle surface of the shell is a cylinder, the analytical representation of which in Cartesian coordinates x_1, x_2, x_3 is given by the equation

$$x_1^2 + x_2^2 = R^2 \tag{46}$$

We introduce cylindrical coordinates, such that $x_1 = x$, $x_2 = \varphi$ and $x_3 = r$, $r \in [R-h, R+h]$. Then parametric equations of the surface of revolution (50) can be represented in the following vector form

$$\mathbf{r}(x,\varphi) = R\cos(\varphi)\mathbf{e}_1 + R\sin(\varphi)\mathbf{e}_2 + x\mathbf{e}_3 \tag{47}$$

If the cylindrical coordinates x and φ belong to the intervals $x \in [0, H]$, $\varphi \in [0, 2\pi]$, then we have a closed in Fig. 4.

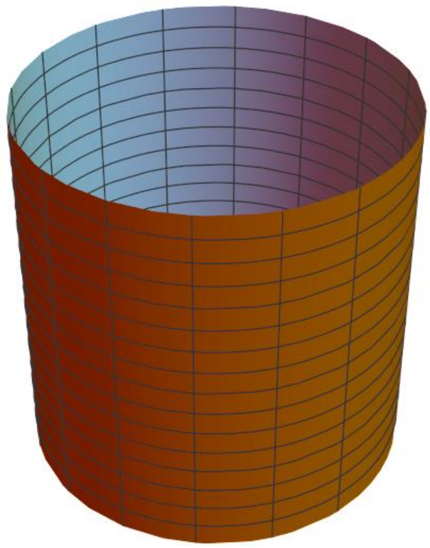

Fig. 4. Straight circular cylinder.

The coefficients of the first quadratic form of a conical surface and the principal curvatures are

$$A_1 = 1, \quad A_2 = R \quad \kappa_1 = 0, \quad \kappa_2 = \frac{1}{R} \tag{48}$$

respectively.

Here we present some numerical simulation results for the higher order CUF-based composite three layers cylindrical laminate. Calculations are made for the following data: the thickness to length ratio is $R/h = 10$ and the length to radius ratio is $H/R = 3$, mechanical load is axisymmetric, uniform and applied to the upper surface of the shell, in the radial direction r.

Since the external load is applied axisymmentricaly, the situation becomes simpler. In this case, all functions describing the stress-strain state of the shell are one-dimensional, and the resulting equations are much simpler.

The resulting differential equations have the same structure as (35), but the local matrices $\mathbf{L}_{\tau,s}^{loc}$ of the fundamental nuclei of differential equations of equilibrium for the higher order axisymmetric spherical elastic shells, as well as the vectors of local unknown functions $\mathbf{U}_s^{loc}$ have the form

$$\mathbf{L}^{loc}_{\tau,s}=\begin{vmatrix} L^{\tau,s}{}_{u_x,u_x} & L^{\tau,s}{}_{u_x,u_r} \\ L^{\tau,s}{}_{u_r,u_x} & L^{\tau,s}{}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{U}^{loc}_{s}=\begin{vmatrix} U_{x,s} \\ U_{r,s} \end{vmatrix} \tag{49}$$

The coefficients of the fundamental nuclei $\mathbf{L}^{loc}_{\tau,s}$ can be easily calculated using the equations presented in our previous publications [6-10].

In Table 2, the lowest eight frequency parameters $\Omega=\omega\sqrt{\rho / L^{0,0}{}_{u_x,u_x}}$ for the there-layered composite cylindrical axisymmetric shell fixed at the ends are presented for the Timoshenko's shear deformation, the first, second, third, fourth and fifth order models.

Table 2. Frequency parameter $\Omega=\omega\sqrt{\rho / L^{0,0}{}_{u_x,u_x}}$ of the axisymmetric cylindrical shell

Models, h/L=0.1	Eigenmode 1	Eigenmode 2	Eigenmode 3	Eigenmode 4	Eigenmode 5	Eigenmode 6	Eigenmode 7	Eigenmode 8
Timoshenko	3.32075	3.43172	3.7414	4.08202	4.13005	4.33649	4.54281	5.02811
First order	3.30118	3.41463	3.72242	4.06603	4.10299	4.33284	4.52184	5.00347
Second order	3.26682	3.39041	3.70004	3.98507	4.05172	4.32076	4.50789	4.98911
Third order	3.25193	3.343	3.6002	3.70578	3.93018	3.97619	4.31543	4.67986
Fourth order	3.21479	3.30968	3.5617	3.69857	3.71752	3.89378	4.28017	4.39163
Fifth order	3.2123	3.2993	3.48052	3.55386	3.71728	3.87652	4.27397	4.38675

In Fig. 5 shows graphs of the distribution of the first eight axisymmetric eigenmodes in the axial direction for the fifth-order model for a there-layered composite axisymmetric cylindrical shell fixed at the ends.

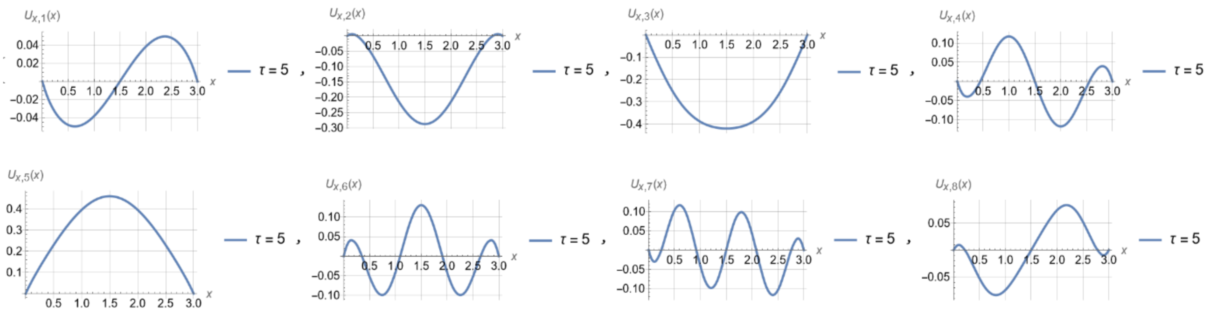

Fig. 5. First eight axisymmetric eigenmodes for cylindrical shell in axial direction for fifth order model.

In Fig. 5 shows graphs of the distribution of the first eight axisymmetric eigenmodes in the radial direction for the fifth-order model for a there-layered composite axisymmetric cylindrical shell fixed at the ends.

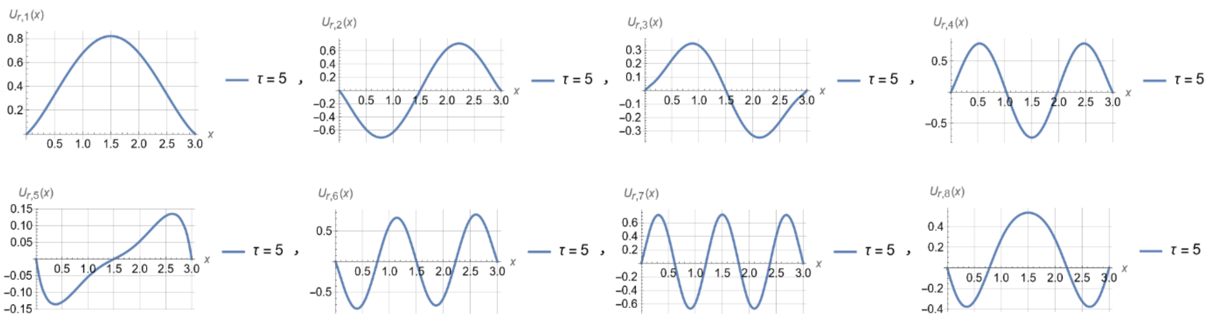

Fig. 6. First eight axisymmetric eigenmodes for cylindrical shell in radial direction for fifth order model.

The data presented in Table 2 and Fig 5, 6 gives qualitative and quantitative information about the behavior of the first eight axisymmetric eigenvalues and eigenvectors of the the fixed at the ends cylindrical shell within the framework of the CUF for the first, second, third, fourth and fifth order $\tau = 1,...,5$ models and a comparison with the classical Timoshenko model. The reported data are in good agreement; indeed, for models of the third order and higher, the results are the same. Table 2 follows that the third order and higher models give more accurate results and better models the free vibration of axisymmetric cylindrical shell. These results can be used as benchmark examples for finite element analysis of elastic composite multilayers cylindrical shell in the case of axisymmetric loading.

3.3 Conical axisymmetric shell

Let us consider a truncated conical shell formed by rotation around an axis x_3 of a straight line that formed constant angle ψ with it. The middle surface of the shell is a cone, the analytical representation of which in Cartesian coordinates x_1, x_2, x_3 is given by the equation

$$x_1^2 + x_2^2 = x_3^2 \cos(\psi)^2 \tag{50}$$

We introduce cylindrical coordinates, such that $x_1 = x$, $x_2 = \varphi$ and $x_3 = r$, $r \in [x-h, x\sin(\psi)+h]$. Then parametric equations of the surface of revolution (50) can be represented in the following vector form

$$\mathbf{r}(x,\varphi) = x\sin(\psi)\cos(\varphi)\mathbf{e}_1 + x\sin(\psi)\sin(\varphi)\mathbf{e}_2 + x\cos(\psi)\mathbf{e}_3 \tag{51}$$

If the cylindrical coordinates x and φ belong to the intervals $x \in [H_1, H_2]$, $\varphi \in [0, 2\pi]$, then we have a closed in Fig. 3.

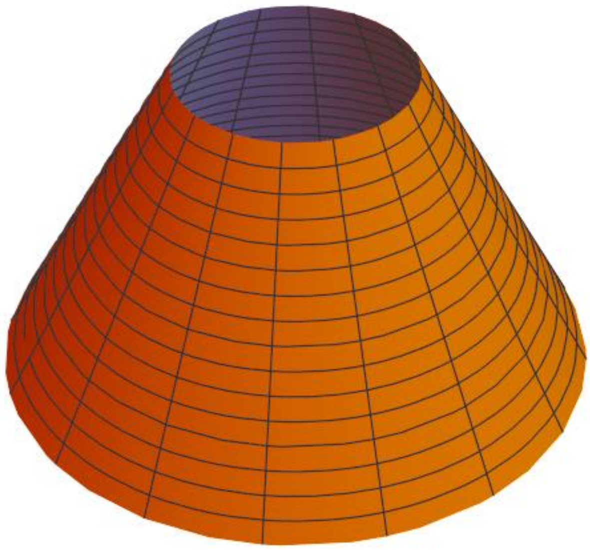

Fig. 6. Truncated circular cone.

The coefficients of the first quadratic form of a conical surface and the principal curvatures are

$$A_1 = 1, \quad A_2 = x\sin(\psi), \quad \kappa_1 = 0, \quad \kappa_2 = \frac{\cot(\psi)}{x} \tag{52}$$

respectively.

Here we present some numerical simulation results of a higher order CUF-based conical composite laminate. Calculations are made for the following data: angle $\psi = \pi/6$, mechanical load is axisymmetric, uniform and applied to the upper surface of the shell, in the radial direction r.

Since the external load is applied axisymmentricaly, the situation becomes simpler. In this

case, all functions describing the stress-strain state of the shell are one-dimensional, and the resulting equations are much simpler.
The resulting differential equations have the same structure as (35), but the local matrices $\mathbf{L}_{\tau,s}^{loc}$ of the fundamental nuclei of differential equations of equilibrium for the higher order axisymmetric spherical elastic shells, as well as the vectors of local unknown functions $\mathbf{U}_{s}^{loc}$ have the form (53).
The coefficients of the fundamental nuclei $\mathbf{L}_{\tau,s}^{loc}$ can be easily calculated using the equations presented in our previous publications [6-10].
In Table 3, the lowest eight frequency parameters $\Omega = \omega\sqrt{\rho / L^{0,0}_{u_x,u_x}}$ for the there-layered composite cylindrical axisymmetric shell fixed at the ends are presented for the Timoshenko's shear deformation, the first, second, third, fourth and fifth order models.

Table 3. Frequency parameter $\Omega = \omega\sqrt{\rho / L^{0,0}_{u_x,u_x}}$ of the axisymmetric cylindrical shell

Models, h/L=0.1	Eigenmode 1	Eigenmode 2	Eigenmode 3	Eigenmode 4	Eigenmode 5	Eigenmode 6	Eigenmode 7	Eigenmode 8
Timoshenko	1.90761	3.40916	5.14379	6.86953	8.61787	10.0406	10.403	11.2587
First order	1.90571	3.40802	5.1429	6.86875	8.61709	10.0403	10.3797	10.4025
Second order	1.90267	3.40407	5.13582	6.8544	8.5904	9.88442	10.0187	10.2147
Third order	1.74996	3.13178	4.74505	6.41658	8.14213	8.93217	9.11417	9.1914
Fourth order	1.7458	3.12859	4.74288	6.41481	8.13853	8.44831	8.92537	8.93736
Fifth order	1.74007	3.12118	4.73208	6.4041	8.1225	8.29038	8.44015	8.91213

In Fig. 7 shows graphs of the distribution of the first eight axisymmetric eigenmodes in the axial direction for the fifth-order model for a there-layered composite axisymmetric conical shell fixed at the ends.

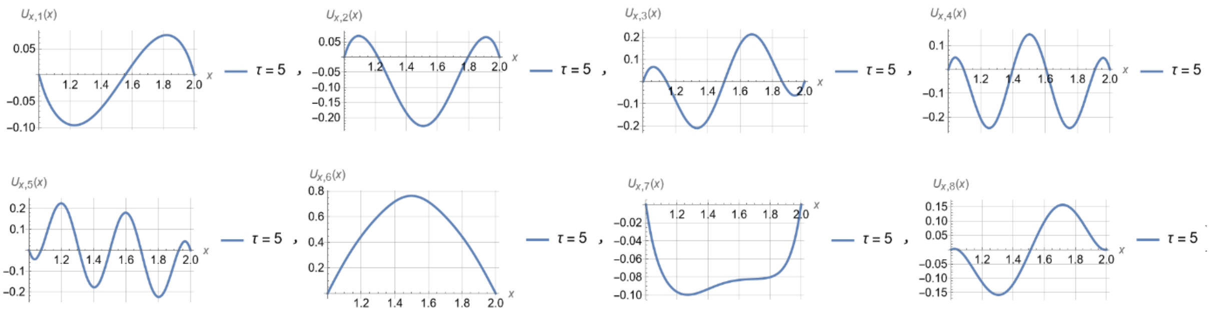

Fig. 7. First eight axisymmetric eigenmodes for conical shell in axial direction for fifth order model.

In Fig. 8 shows graphs of the distribution of the first eight axisymmetric eigenmodes in the radial direction for the fifth-order model for a there-layered composite axisymmetric conical shell fixed at the ends.

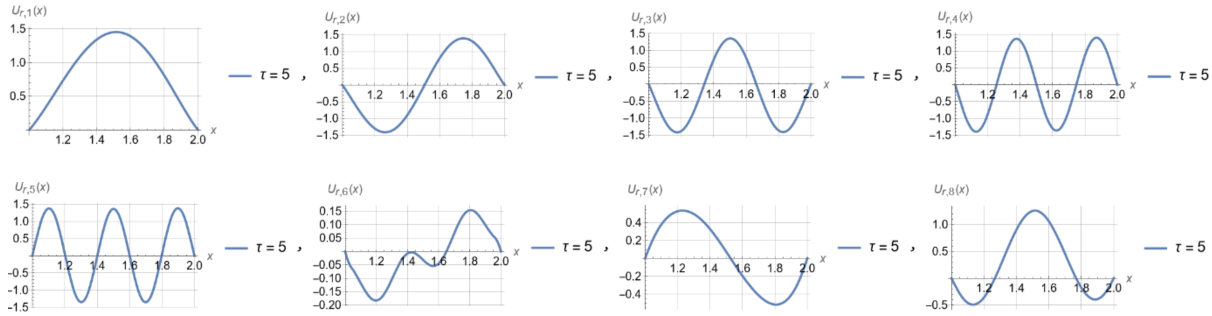

Fig. 8. First eight axisymmetric eigenmodes for conical shell in radial direction for fifth order model.

The data presented in Table 3 and Fig 7, 8 gives qualitative and quantitative information about the behavior of the first eight axisymmetric eigenvalues and eigenvectors of the fixed at the ends conical shell within the framework of the CUF for the first, second, third, fourth and fifth order $\tau = 1,...,5$ models and a comparison with the classical Timoshenko model. The reported data are in good agreement; indeed, for models of the third order and higher, the results are the same. Table 3 follows that the third order and higher models give more accurate results and better models the free vibration of axisymmetric conical shell. These results can be used as benchmark examples for finite element analysis of elastic composite multilayers conical shell in the case of axisymmetric loading.

4. Conclusion

Higher-order theories for composite multilayered elastic shells of revolution have been developed here using the CUF approach which is based on the series expansion of general 3-D equations of linear theory of elasticity into a series expansion with respect to shell thickness. The 2-D higher order theory of composite multilayered shells of revolution is developed from general 3-D equations of linear anisotropic theory of elasticity using the principle of virtual power. All the functions that determinate the stress-strain state of the shell, such as stress and strain tensors, vectors of displacements and body forces are expressed in terms of the coefficients of the series expansion with respect to the thickness coordinate of the shell. Thus, all equations of the linear theory of elasticity, including generalized Hooke's law, were transformed to the corresponding equations for the coefficients of the series expansion in accordance with the (CUF) approach. A system of the equations of motion in terms of the series expansion of displacement vectors coefficients and essential boundary conditions is obtained.
The equations of 2-D models of higher orders of shells of revolution are developed and presented here, for the cases the middle surfaces of which can be represented analytically. More specifically, we represent here a higher order theory for plates in polar coordinates, cylindrical, conical shells. The First eight axisymmetric eigen vectors and eigen values are calculated numerically using the built-in functions **NDEigensystem** and **NDEigenvalues**. Of the computer algebra software **Mathematica**.
The resulting equations can be used for theoretical analysis and calculation of the stress-strain state, as well as for modeling thin-walled structures used in science, engineering, and technology. The results of calculation can be used as benchmark examples for finite element analysis of the higher order elastic shells.

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References

[1] Carrera E. Theories and finite elements for multilayered plates and shells, *Archives of Computational Methods in Engineering*, 2002, **9**(2), 87-140.

[2] Carrera E. Historical review of Zig-Zag theories for multilayered plates and shells, *Applied Mechanics Review*, 2003, **56**(3), 287–308.

[3] Carrera E. Theories and Finite Elements for Multilayered Plates and Shells: A Unified Compact Formulation with Numerical Assessment and Benchmarking, *Archives of Computational Methods in Engineering*, 2003, **10**(3), 215-296.

[4] Carrera E., Cinefra M., Petrolo M., Zappino E. *Finite Element Analysis of Structures through Unified Formulation*, John Wiley & Sons, Ltd., New Delhi, 2014, 385 p.

[5] Carrera E., Fazzolari F.A., Giunta G. *Thermal Stress Analysis of Composite Beams, Plates and Shells. Computational Modelling and Applications*, Academic Press, Amsterdam, 2017, 416 p.

[6] Carrera E., Zozulya V.V. Carrera Unified Formulation (CUF) for the Shells of Revolution. I. Higher Order Theory. *Acta Mechanica*, 2023, **234**, 109-136.

[7] Carrera E., Zozulya V.V. Carrera Unified Formulation (CUF) for the Shells of Revolution. II. Navier close form solutions. *Acta Mechanica*, 2023, **234**, 137-161.

[8] Carrera E., Zozulya V.V. Carrera Unified Formulation (CUF) for the Shells of Revolution. Numerical Evaluation. *Mechanics of Advanced Materials and Structures*, 2024, **31**(7), 1597-1619.

[9] Carrera E., Zozulya V.V. Carrera Unified Formulation (CUF) for the Composite Shells of Revolution. Equivalent single layer models. *Mechanics of Advanced Materials and Structures*, 2024, **31**(1), 22-44.

[10] Carrera E., Zozulya V.V. Carrera Unified Formulation (CUF) for the Composite Plates and Shells of Revolution. Layer-Wise Models. *Composite Structures*, 2024, **334**, 117936, 1-39.

[11] Czekanski A., Zozulya V.V. Dynamics of vibrating beams using first-order theory based on Legendre polynomial expansion, *Archive of Applied Mechanics*, 2020, **90**, 789–814.

[12] Gurtin M. E. Variational Principles for Linear Elastodynamics, *Archive for Rational Mechanics and Analysis*, 1964, **16**(1), pp. 34–50.

[13] Jin G., Ye T., Su Z. *Structural Vibration*. *A Uniform Accurate Solution for Laminated Beams, Plates and Shells with General Boundary Conditions*, Springer-Verlag, Berlin, 2015, 318 p.

[14] Khoma I. Y. *Generalized Theory of Anisotropic Shells*. Naukova dumka, Kyiv, 1987, 172 p.

[15] Kil'chevskiy N. A. *Fundamentals of the Analytical Mechanics of Shells*, NASA TT, F-292, Washington, D.C., 1965, 361 p.

[16] Pelekh B.L., Lazko V.A. *Laminated anisotropic plates and shells with stress concentrators*, Naukova dumka, Kyiv, 1982. 296 p.

[17] Qatu M.S. *Vibration of Laminated Shells and Plates*, Elsevier, New York, 2004, 426 p.

[18] Reddy J. N. *Mechanics of laminated composite plates and shells: theory and analysis*, 2nd edition, CRC Press LLC, 2004, 855 p.

[19] Timoshenko S., Woinowsky-Krieger S. *Theory of Plates and Shells*. 2nd edition, McGraw-Hill Book Company, Paris, 1959, 611 p.

[20] Wolfram S. *An Elementary Introduction to the Wolfram Language*, 3rd edition, Wolfram Media, Champaign, 2023, 376 p.

[21] Zozulya V.V. Laminated shells with debonding between laminas in temperature field, *International Applied Mechanics*, 2006, **42**(7), 842-848.

[22] Zozulya V.V. A higher order theory for shells, plates, and rods. *International Journal of Mechanical Sciences*, 2015, **103**, 40-54.