



COUPLED THERMOELASTIC ANALYSIS OF BEAM STRUCTURES USING A REFINED 1D FINITE ELEMENT MODEL

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Abstract

In this paper, static and dynamic problems in the framework of coupled and uncoupled thermoelasticity are analyzed. A refined one-dimensional (1D) model, based on the Carrera Unified Formulation (CUF), is employed to provide accurate predictions for the displacement and temperature change fields within homogeneous isotropic structures under thermal loadings. This approach offers the distinct advantage of transforming the complex three-dimensional (3D) problem into a computationally efficient 1D model, ensuring a balance between precision and reduced computational costs. This work introduces generalized theories of thermoelasticity, specifically based on the Lord-Shulman and Green-Lindsay models. Other cases such as static, quasi-static and dynamic can be seen as particular cases of this generalized formulation. The study focuses on a simplified configuration, employing variable kinematics models, such as Lagrange polynomial and Taylor expansion functions. Numerical solutions and convergence studies are presented to demonstrate the accuracy of the formulation.

Keywords: Carrera Unified Formulation; coupled thermoelasticity; finite element method.

1. Introduction

In aeronautics, a large number of problems require the study of thermal stresses. Components are often subjected to very high temperatures or large temperature variations. Such problems can be solved according to different models.

In static uncoupled thermoelasticity, the stationary temperature distribution determines an effect on deformations. Quasi-static uncoupled thermoelasticity considers the time-dependent temperature distributions resulting from the transient thermal conduction equation, which leads to transient thermal stresses. In contrast, dynamic uncoupled thermoelasticity incorporates inertia effects when external thermomechanical loads vary rapidly over time. In uncoupled theories, temperature is obtained independently directly from the heat conduction equation, while displacements are partly due to the effect of temperature.

However, the uncoupled approach does not always make it possible to realistically evaluate the behaviour of a structure when, for example, it is subjected to rapid and high thermomechanical loads. In fact, in this case, the effects of deformations on temperature and inertial forces are no longer negligible. Hence, the need to use more sophisticated theories of coupled thermoelasticity.

Introduced by Duhamel [1] in 1837, the coupling effect in the equations of thermoelasticity was later studied by Biot, who in 1956 presented the theory of classical thermoelasticity [2]. According to the classical theory, thermal disturbances propagate with infinite speed through the body. Because of this nonphysical behaviour, several coupled thermoelasticity models have been developed that overcome the limitations of the classical model. The Lord-Shulman (LS) and Green-Lindsay (GL) models are among these [3–5].

The analytical solution of these coupled equations is usually available only for simple configurations and boundary conditions [6–8], which has led to an increasing reliance on numerical methods such

as the FEM. The FE formulation of the thermoelastic governing equations can be obtained through the Principle of Virtual Displacements (PVD) [9].

For complex configurations, 1D and 2D models do not always provide the accuracy the problem requires. The three-dimensional nature of the problem demands solid models, incurring high computational costs. A refined one-dimensional model in the framework of Carrera Unified Formulation [9–14] is used in this work. This approach was implemented first for the plate and shell [15] and later for the beam [16]. CUF allowed, compared to the 1D and 2D FE models usually used, to achieve accuracy comparable to solid models while maintaining a low computational cost. In addition, another strength of CUF is the ability to analyze multi-field problems (mechanical, thermal, electrical) [17–19] with high accuracy. Recently, CUF has also been used for thermoelastic analysis on beams and disks [20–22]. This paper aims to show the use of 1D FE-CUF models in the context of three-dimensional coupled thermoelasticity problems. In particular, some numerical results related to static and quasi-static problems are proposed.

2. Equations and models

2.1 Governing equations

The equation of motion in a three-dimensional domain is expressed as in [23, 24]:

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i + \zeta \dot{u}_i \quad (1)$$

where σ_{ij} is the stress component, u_i is the displacement component and X_i denote the volume forces. ρ and ζ are the density and damping coefficient, respectively. In the chosen notation, the derivative in time is denoted by the superscript $(\cdot)$ while the derivative in space by the subscript $(,)$. According to Hooke's law for a nonhomogeneous anisotropic material, the stress component is expressed as:

$$\sigma_{ij} = C_{ijpq} \varepsilon_{pq} - \beta_{ij} (T + t_1 \dot{T}) \quad (2)$$

where C_{ijpq} is the 4-order elasticity tensor, T is the temperature change with respect to the reference temperature T_0 and t_1 is one of the two relaxation times predicted by Green-Lindsay (GL) theory. Strains ε_{ij} can be expressed as a function of displacements:

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (3)$$

The thermo-stress module β_{ij} can be written as:

$$\beta_{ij} = C_{ijpq} \alpha_{pq} \quad (4)$$

where α_{pq} is the thermal expansion coefficient.

The energy equation can be expressed through the following relationship:

$$q_{i,i} = R - T_0 \dot{S} \quad (5)$$

where q_i is the component of heat flux, R denotes the internal heat per unit volume and time and S is the entropy per unit volume. S can be expressed as:

$$S = \frac{\rho c}{T_0} (T + t_2 \dot{T}) + \beta_{ij} \varepsilon_{ij} - \frac{1}{T_0} \tilde{c}_i T_{,i} \quad (6)$$

The energy equation can be rearranged and written as a function of displacements and temperature [24]:

$$\rho c (t_0 + t_2) \ddot{T} + \rho c \dot{T} - 2 \tilde{c}_i \dot{T}_{,i} - (\kappa_{ij} T_{,j})_{,i} + t_0 T_0 \beta_{ij} \ddot{u}_{i,j} + T_0 \beta_{ij} \dot{u}_{i,j} = R + t_0 \dot{R} \quad (7)$$

Parameter c is the specific heat, t_0 and t_2 are the relaxation times relative to LS theory and GL theory, respectively, and $\tilde{c}$ is a vector of new constants of the material. The thermal conductivity tensor is indicated by the parameter κ_{ij} .

The equations written above are in the most general form possible. In the LS theory case only the relaxation time t_0 is nonzero ($t_1 = t_2 = \tilde{c}_i = 0$), while in the GL case only $t_0 = 0$. Since they are coupled, the governing equations must be solved simultaneously.

2.2 Principle of Virtual Displacements

To derive the finite element formulation, the principle of virtual displacements (PVD) is used. For the coupled thermoelastic case, the PVD has the following form [9]:

$$\int_V (\delta \boldsymbol{\varepsilon}_p \boldsymbol{\sigma}_p + \delta \boldsymbol{\varepsilon}_n \boldsymbol{\sigma}_n + \delta TS) dV = \delta L_{ext} - \delta L_{ine} \quad (8)$$

By convention, subscript p is used for in-plane components, while subscript n denotes out-of-plane components. The vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ represent the components of strain and stress. S is the entropy per unit volume. δL_{ext} and δL_{ine} are the external and inertial virtual works, respectively.

2.3 CUF form of governing equations

The 1D FE model involves discretizing the structure along the y -axis of the beam into a number of finite elements. According to the FEM, the displacement and temperature change fields can be written as:

$$\begin{aligned} \mathbf{u} &= N_m \mathbf{u}^m \\ T &= N_m T^m \end{aligned} \quad (9)$$

where N_m are the shape functions, $\mathbf{u}^m$ and T^m are the vector of displacements and temperature change in the m -th node in the element. Depending on the number of nodes in the beam element, one can have a linear (2-nodes), quadratic (3-nodes) and cubic (4-nodes) interpolation function of temperature and displacements along y -axis.

The displacements and temperature change in the individual node of the beam can be expressed by the distribution of the displacements and temperature over the cross section:

$$\begin{aligned} \mathbf{u}^m &= F_\tau \mathbf{U}^{m\tau} \\ T^m &= F_\tau \Theta^{m\tau} \end{aligned} \quad (10)$$

where F_τ are the generic expansion functions, $\mathbf{U}^{m\tau}$ is the generalized vector of displacements, and $\Theta^{m\tau}$ is the generalized temperature change. The parameter τ denotes the number of terms of the expansion. Taylor expansion (TE) and Lagrange expansion (LE) functions can be used. Lagrange expansions involve discretizing the section using different types of elements for example a biquadratic nine-node (1L9) or bicubic 16-node (1L16) elements. According to the Taylor expansions, the number of terms depends on the order of the model.

Using the geometric and Hooke's equations (Eqs. 2 and 3) and substituting Eqs. 9 and 10 within the Principle of Virtual Displacements (8), the following system written in matrix form is obtained [24]:

$$\begin{aligned} & \begin{bmatrix} \mathbf{M}_{UU}^{lm\tau s} & 0 \\ \mathbf{M}_{\theta U}^{lm\tau s} & \mathbf{M}_{\theta\theta}^{lm\tau s} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{U}}^{ls} \\ \ddot{\Theta}^{ls} \end{Bmatrix} + \begin{bmatrix} \mathbf{G}_{UU}^{lm\tau s} & \mathbf{G}_{U\theta}^{lm\tau s} \\ \mathbf{G}_{\theta U}^{lm\tau s} & \mathbf{G}_{\theta\theta}^{lm\tau s} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{U}}^{ls} \\ \dot{\Theta}^{ls} \end{Bmatrix} \\ & + \begin{bmatrix} \mathbf{K}_{UU}^{lm\tau s} & \mathbf{K}_{U\theta}^{lm\tau s} \\ 0 & \mathbf{G}_{\theta\theta}^{lm\tau s} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{ls} \\ \Theta^{ls} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^{ls} \\ \mathcal{Q}^{ls} \end{Bmatrix} \end{aligned} \quad (11)$$

where the terms of the matrices are expressed through fundamental nucleus, a condensed notation that does not depend on model order or expansion type:

$$\begin{aligned} [\mathbf{M}_{UU}^{lm\tau s}]_{3 \times 3} &= \int_{L^e} \int_{A^e} (\rho N_m N_l \mathbf{I} F_\tau F_s) dA dL \\ [\mathbf{M}_{\theta U}^{lm\tau s}]_{1 \times 3} &= \int_{L^e} \int_{A^e} (t_0 T_0 N_m N_l [\boldsymbol{\beta}_p^T (\mathbf{D}_p F_s) + \boldsymbol{\beta}_n^T (\mathbf{D}_{np} F_s)] F_\tau) dA dL \\ &+ \int_{L^e} \int_{A^e} (t_0 T_0 [\boldsymbol{\beta}_n^T N_m (\mathbf{D}_{ny} N_l) F_s F_\tau]) dA dL \\ [\mathbf{M}_{\theta\theta}^{lm\tau s}]_{1 \times 1} &= \int_{L^e} \int_{A^e} (\rho c t_0 N_m N_l F_\tau F_s) dA dL \\ &+ \int_{L^e} \int_{A^e} (\rho c t_2 N_m N_l F_\tau F_s) dA dL \end{aligned} \quad (12)$$

$$\begin{aligned}
 [\mathbf{G}_{UU}^{lm\tau s}]_{3 \times 3} &= \int_{L^e} \int_{A^e} (\zeta N_m N_l \mathbf{I} F_\tau F_s) dAdL \\
 [\mathbf{G}_{U\Theta}^{lm\tau s}]_{3 \times 1} &= - \int_{L^e} \int_{A^e} (t_1 N_m N_l [(\mathbf{D}_p^T F_\tau \mathbf{I}) \boldsymbol{\beta}_p + (\mathbf{D}_{np}^T F_\tau \mathbf{I}) \boldsymbol{\beta}_n] F_s) dAdL \\
 &\quad - \int_{L^e} \int_{A^e} (t_1 (\mathbf{D}_{ny}^T N_m) N_l [F_\tau \boldsymbol{\beta}_n F_s]) dAdL \\
 [\mathbf{G}_{\Theta U}^{lm\tau s}]_{1 \times 3} &= \int_{L^e} \int_{A^e} (T_0 N_m N_l [\boldsymbol{\beta}_p^T (\mathbf{D}_p F_s) + \boldsymbol{\beta}_n^T (\mathbf{D}_{np} F_s)] F_\tau) dAdL \\
 &\quad + \int_{L^e} \int_{A^e} (T_0 [\boldsymbol{\beta}_n^T N_m (\mathbf{D}_{ny} N_l) F_s F_\tau]) dAdL \\
 [\mathbf{G}_{\Theta\Theta}^{lm\tau s}]_{1 \times 1} &= \int_{L^e} \int_{A^e} (\rho c N_m N_l F_\tau F_s) dAdL \\
 &\quad - \int_{L^e} \int_{A^e} (2\tilde{\mathbf{c}}^T N_m [\nabla_n N_l] F_\tau F_s) dAdL \\
 &\quad - \int_{L^e} \int_{A^e} (2\tilde{\mathbf{c}}^T N_m N_l F_\tau [\nabla_p F_s]) dAdL
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 [\mathbf{K}_{UU}^{lm\tau s}]_{3 \times 3} &= \int_{L^e} \int_{A^e} \left(N_m N_l [(\mathbf{D}_{np}^T F_\tau \mathbf{I}) [\mathbf{C}_{nn} (\mathbf{D}_{np} F_s \mathbf{I}) + \mathbf{C}_{np} (\mathbf{D}_p F_s \mathbf{I})] \right. \\
 &\quad \left. + (\mathbf{D}_p^T F_\tau \mathbf{I}) [\mathbf{C}_{pp} (\mathbf{D}_p F_s \mathbf{I}) + \mathbf{C}_{pn} (\mathbf{D}_{np} F_s \mathbf{I})] \right) dAdL \\
 &\quad + \int_{L^e} \int_{A^e} \left(N_m (\mathbf{D}_{ny} N_l) [(\mathbf{D}_{np}^T F_\tau \mathbf{I}) \mathbf{C}_{nn} + (\mathbf{D}_p^T F_\tau \mathbf{I}) \mathbf{C}_{pn}] F_s \right) dAdL \\
 &\quad + \int_{L^e} \int_{A^e} \left((\mathbf{D}_{ny}^T N_m) N_l F_\tau [\mathbf{C}_{np} (\mathbf{D}_p F_s \mathbf{I}) + \mathbf{C}_{nn} (\mathbf{D}_{np} F_s \mathbf{I})] \right) dAdL \\
 &\quad + \int_{L^e} \int_{A^e} \left((\mathbf{D}_{ny}^T N_m) (\mathbf{D}_{ny} N_l) F_\tau \mathbf{C}_{nn} F_s \right) dAdL \\
 [\mathbf{K}_{U\Theta}^{lm\tau s}]_{3 \times 1} &= - \int_{L^e} \int_{A^e} (N_m N_l [(\mathbf{D}_p^T F_\tau) \boldsymbol{\beta}_p + (\mathbf{D}_{np}^T F_\tau) \boldsymbol{\beta}_n] F_s) dAdL \\
 &\quad - \int_{L^e} \int_{A^e} ((\mathbf{D}_{ny}^T N_m) N_l [F_\tau \boldsymbol{\beta}_n F_s]) dAdL
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 [\mathbf{K}_{\Theta\Theta}^{lm\tau s}]_{1 \times 1} &= \int_{L^e} \int_{A^e} \left(N_m N_l (\nabla_p^T F_s) \boldsymbol{\kappa} (\nabla_p F_\tau) \right) dAdL \\
 &\quad + \int_{L^e} \int_{A^e} \left((\nabla_n^T N_l) (N_m) \boldsymbol{\kappa} (\nabla_p F_\tau) F_s \right) dAdL \\
 &\quad + \int_{L^e} \int_{A^e} \left(N_l (\nabla_n N_m) \boldsymbol{\kappa} (\nabla_p^T F_s) F_\tau \right) dAdL \\
 &\quad + \int_{L^e} \int_{A^e} \left((\nabla_n^T N_l) (\nabla_n N_m) F_\tau \boldsymbol{\kappa} F_s \right) dAdL \\
 [\mathbf{F}^{m\tau}]_{3 \times 1} &= \int_{L^e} \int_{A^e} (\mathbf{X} N_m F_\tau) dAdL + \int_{S^e} (\mathbf{t} N_m F_\tau) dS \\
 [\mathbf{Q}^{m\tau}]_{1 \times 1} &= - \int_{S^e} (\mathbf{q}^T \mathbf{n} N_m F_\tau) dS + \int_{L^e} \int_{A^e} (R N_m F_\tau) dAdL \\
 &\quad - \int_{L^e} \int_{A^e} (t_0 \dot{R} N_m F_\tau) dAdL
 \end{aligned} \tag{15}$$

in which the matrices $\mathbf{D}_p$, $\mathbf{D}_{np}$ and $\mathbf{D}_{ny}$ are:

$$\mathbf{D}_p = \begin{bmatrix} 0 & 0 & \partial_z \\ \partial_x & 0 & 0 \\ \partial_z & 0 & \partial_x \end{bmatrix}, \quad \mathbf{D}_{np} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & \partial_x & 0 \end{bmatrix}, \quad \mathbf{D}_{ny} = \begin{bmatrix} 0 & \partial_y & 0 \\ 0 & 0 & \partial_y \\ \partial_y & 0 & 0 \end{bmatrix}, \tag{16}$$

and the vectors ∇_p and ∇_n are:

$$\nabla_p = \{\partial_x \quad 0 \quad \partial_z\}^T, \quad \nabla_n = \{0 \quad \partial_y \quad 0\}^T, \tag{17}$$

2.4 Newmark method

The system 11 can be rewritten in compact form as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{R} \quad (18)$$

where $\mathbf{q}$ is the vector containing the unknowns of displacements and temperature, $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively, and finally $\mathbf{R}$ is the generalized force vector. To solve this system, Newmark's method of integration is adopted [25].

Eq. 19 at time step $t + \Delta t$ is:

$$\mathbf{M}\ddot{\mathbf{q}}_{t+\Delta t} + \mathbf{D}\dot{\mathbf{q}}_{t+\Delta t} + \mathbf{K}\mathbf{q}_{t+\Delta t} = \mathbf{R}_{t+\Delta t} \quad (19)$$

The solution of the equation is obtained by assuming that:

$$\begin{aligned} \dot{\mathbf{q}}_{t+\Delta t} &= \dot{\mathbf{q}}_t + [(1 - \delta)\ddot{\mathbf{q}}_t + \delta\ddot{\mathbf{q}}_{t+\Delta t}]\Delta t \\ \mathbf{q}_{t+\Delta t} &= \mathbf{q}_t + \dot{\mathbf{q}}_t\Delta t + \left[\left(\frac{1}{2} - \alpha \right) \ddot{\mathbf{q}}_t - \alpha\ddot{\mathbf{q}}_{t+\Delta t} \right] \Delta t^2 \end{aligned} \quad (20)$$

The parameters α and δ are for stability and accuracy of integration and are equal to $\alpha = \frac{1}{6}$ and $\delta = \frac{1}{2}$. To find $\mathbf{q}_{t+\Delta t}$, Newmark's method involves solving the system:

$$\hat{\mathbf{K}}\mathbf{q}_{t+\Delta t} = \hat{\mathbf{R}}_{t+\Delta t} \quad (21)$$

where $\hat{\mathbf{K}}$ is the effective stiffness matrix:

$$\hat{\mathbf{K}} = \mathbf{K} + \frac{1}{\alpha(\Delta t)^2}\mathbf{M} + \frac{\delta}{\alpha\Delta t}\mathbf{C} \quad (22)$$

and $\hat{\mathbf{R}}_{t+\Delta t}$ are the effective loads:

$$\begin{aligned} \hat{\mathbf{R}}_{t+\Delta t} &= \mathbf{R}_{t+\Delta t} + \mathbf{M} \left(\frac{1}{\alpha(\Delta t)^2}\mathbf{q}_t + \frac{1}{\alpha\Delta t}\dot{\mathbf{q}}_t + \left(\frac{1}{2\alpha} - 1 \right) \ddot{\mathbf{q}}_t \right) \\ &\quad + \mathbf{C} \left(\frac{\delta}{\alpha\Delta t}\mathbf{q}_t + \left(\frac{\delta}{\alpha} - 1 \right) \dot{\mathbf{q}}_t + \frac{\Delta t}{2} \left(\frac{\delta}{\alpha} - 2 \right) \ddot{\mathbf{q}}_t \right) \end{aligned} \quad (23)$$

After finding the solution $\mathbf{q}_{t+\Delta t}$, one can also calculate the derivatives in time $\dot{\mathbf{q}}_{t+\Delta t}$ and $\ddot{\mathbf{q}}_{t+\Delta t}$ via Eq. 20. The same procedure is repeated for each time-step. More details can be found in [25].

3. Numerical results

In this section, some numerical results are presented for a simple isotropic beam [20]. For representative purposes, first a coupled thermoelastic static analysis and then, a transient quasi-static analysis are proposed.

3.1 Static coupled thermoelastic analysis

The case examined [20] is a cantilever beam with a square cross section, whose area is $A = 20 \text{ cm}^2$ and length is $L = 50 \text{ cm}$. The material of the beam is aluminum and it has the following characteristics: Young's modulus $E = 73.1 \text{ GPa}$, Poisson's coefficient $\nu = 0.33$, coefficient of thermal expansion $\alpha = 23.1 \times 10^{-6} \text{ K}^{-1}$ and thermal conductivity $\kappa = 237 \text{ W/(m K)}$. A heat flux of $q = 100 \text{ W}$ is applied at the clamped edge, while the free edge is left at room temperature.

Displacements along y and temperature changes are calculated at different sections along the beam. The number of 4-node beam elements for two different lagrange expansions (one element L9 and one L16 on the cross-section) is varied to evaluate the convergence of the results (Tables 1 and 2). Table 3 then shows the results obtained using second-, third- and fourth-order Taylor models in the case where the beam is discretized with 30 B4 elements.

The results obtained are in obvious agreement with those obtained in [20]. It can be seen from the results that it takes only a few elements to already achieve good accuracy of results.

The Fig. 1 shows how displacements and temperature along the beam vary for the 30 B4/1L9 model.

		Location along y-axis [m]						DOF
Number of FEs		0.0	0.1	0.2	0.3	0.4	0.5	
5 B4	u_y [mm]	0.0	0.242	0.411	0.532	0.606	0.630	576
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
10 B4	u_y [mm]	0.0	0.235	0.405	0.527	0.600	0.625	1116
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
20 B4	u_y [mm]	0.0	0.234	0.404	0.526	0.599	0.624	2196
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
30 B4	u_y [mm]	0.0	0.234	0.404	0.526	0.599	0.624	3276
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
Ref. [20] (20 B4)	u_y [mm]	0.0	0.232	0.403	0.525	0.599	0.622	—
	T [°C]	105.5	84.38	63.28	42.19	21.09	0.0	

Table 1 – Displacement in y-direction and temperature change along the beam axis as the number of finite elements changes for a 1L9 model.

		Location along y-axis [m]						DOF
Number of FEs		0.0	0.1	0.2	0.3	0.4	0.5	
5 B4	u_y [mm]	0.0	0.242	0.411	0.532	0.605	0.630	1024
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
10 B4	u_y [mm]	0.0	0.234	0.405	0.527	0.600	0.624	1984
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
20 B4	u_y [mm]	0.0	0.233	0.403	0.525	0.598	0.622	3904
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
30 B4	u_y [mm]	0.0	0.232	0.403	0.525	0.598	0.622	5824
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
Ref. [20] (20 B4)	u_y [mm]	0.0	0.231	0.402	0.524	0.597	0.621	—
	T [°C]	105.5	84.38	63.28	42.19	21.09	0.0	

Table 2 – Displacement in y-direction and temperature change along the beam axis as the number of finite elements changes for a 1L16 model.

Model		Location along y-axis [m]						DOF
		0.0	0.1	0.2	0.3	0.4	0.5	
TE2	u_y [mm]	0.0	0.234	0.404	0.526	0.599	0.624	2184
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
TE3	u_y [mm]	0.0	0.232	0.403	0.525	0.598	0.622	3640
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
TE4	u_y [mm]	0.0	0.232	0.403	0.524	0.598	0.622	5460
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
1L9	u_y [mm]	0.0	0.234	0.404	0.526	0.599	0.624	3276
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	
1L16	u_y [mm]	0.0	0.232	0.403	0.525	0.598	0.622	5824
	T [°C]	105.5	84.4	63.3	42.2	21.1	0.0	

Table 3 – Displacement in the y-direction and temperature change along the beam axis as the Taylor model and Lagrange model changes for the 30 B4 case.

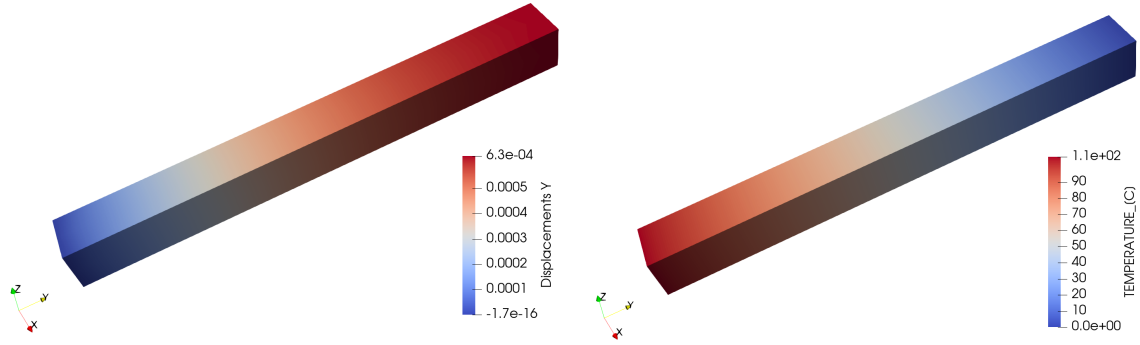


Figure 1 – Displacement and temperature variation along the beam for the 30 B4/1L9 model.

3.2 Transient quasi-static analysis

The same configuration is considered as in the static case [20]. Inertial effects are not considered in this analysis. In addition, the coefficients of the governing equations t_0 , t_1 , t_2 and $\tilde{c}$ are null. The heat capacity is equal to $c = 903 \text{ J}/(\text{kg K})$.

We consider a model with 5 B4 finite elements and a Lagrange expansion on cross-section 1L9. Fig. 2 show the displacements and temperature changes over time for different positions along the y-axis. After a short transient, the stationary solution obtained in the previous paragraph (Table 1) is reached. The Fig. 3 shows the temperature trend along the beam for different times.

4. Conclusion

This paper presented the study of coupled thermoelasticity of isotropic and homogeneous beams. The static and quasi-static problems are analyzed starting from the generalized governing equations. The results show that the use of refined 1D models based on the Carrera Unified Formulation allows the accurate evaluation of displacements and temperature changes for a structure subjected to heat flow and with lower computational costs than 3D models. In addition, it was possible to compare the results for different polynomial expansions of Lagrange and Taylor types. The convergence rate is excellent for Lagrange models. It takes only five beam elements to find the correct temperature value and ten elements to predict displacements. In addition, the results show that the use of Taylor models also allow to obtain a good accuracy even with low orders ($N \geq 3$) and with computational costs usually lower than Lagrange models.

In future work, it will be possible to include the study of thermal shock phenomena using coupled thermoelasticity.

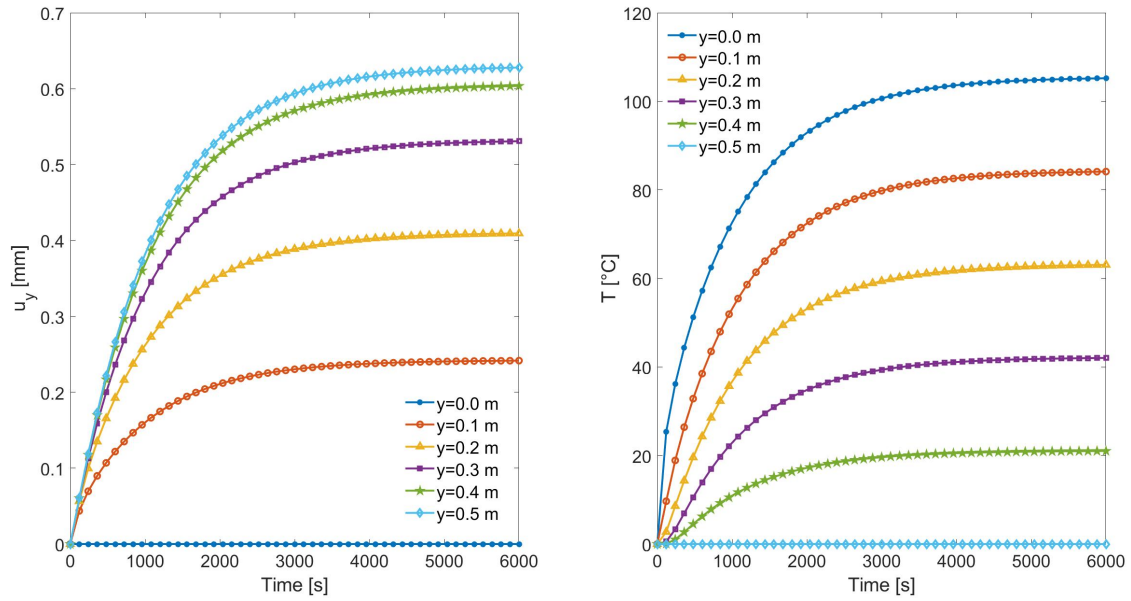


Figure 2 – Displacements and temperature changes over time for different positions along the y -axis.

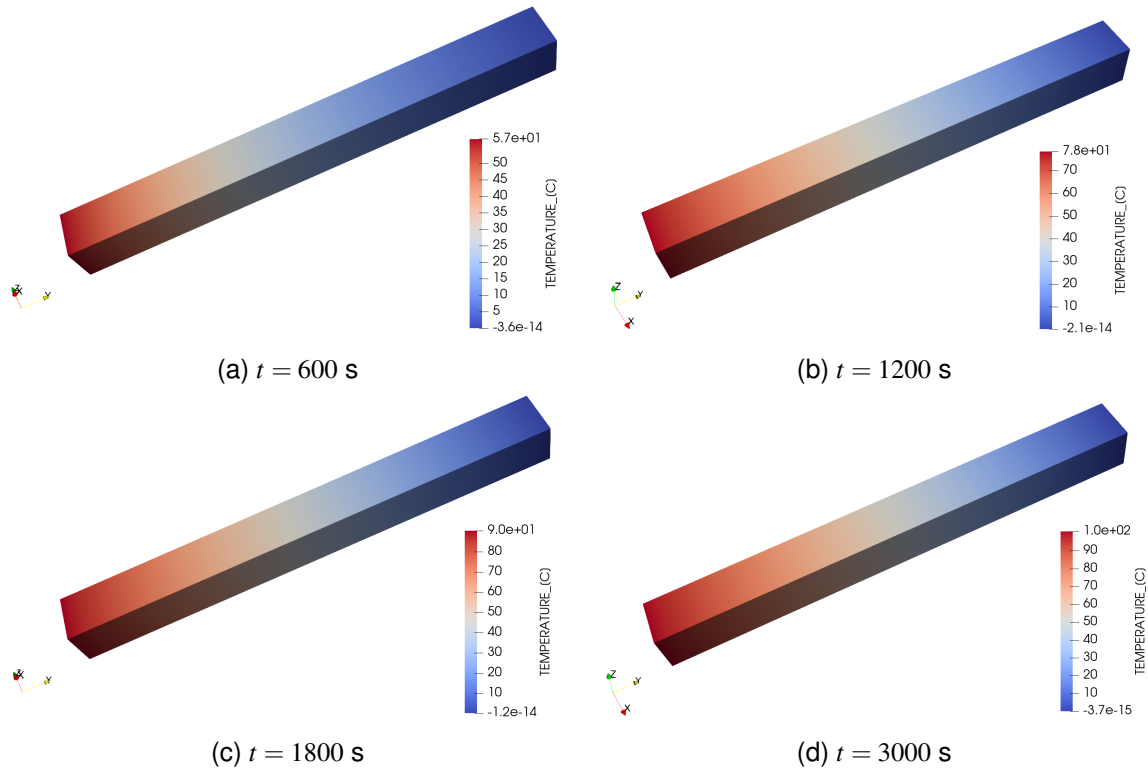


Figure 3 – Temperature variation along the beam at different times.

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