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IMPROVED STOCHASTIC PERTURBATION ALGORITHM FOR GENERALIZED REPEATED EIGENVALUES PROBLEM BASED ON SURROGATE MODEL

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Abstract

In order to investigate the uncertainty and its propagation in the problems of generalized repeated eigenvalues, within the uncertainties of design variables taking into account, an improved method based on stochastic perturbation method and surrogate model approach is presented in this work. Initially repeated eigenvalue undergoing changes of design variables is expressed by the perturbation expansions, in which the first-order perturbation term of repeated eigenvalue could be calculated by solving a standard eigenvalue equation; then, to avoid solving the eigenvalue equation repeatedly in structural uncertainty analysis, a surrogate model which consists of polynomial chaos expansions (PCE) is established to approximate the first-order perturbation term of repeated eigenvalue; omitting the second and higher-order perturbation terms, perturbed repeated eigenvalue is expressed as the combination of original repeated eigenvalue and the surrogate model; lastly, the statistical quantities of perturbed repeated eigenvalue are calculated directly on the basis of the surrogate model. Via this proposed method, not only the uncertainty propagation analysis, but also the dynamic reanalysis, structural design optimization and importance measure for structures associated with repeated eigenvalues can be performed and accomplished expediently. The accuracy and efficiency of the proposed method have been validated thoroughly by three numerical examples.

Keywords: stochastic perturbation algorithm, repeated eigenvalues, surrogate model, polynomial chaos expansions, uncertainty propagation analysis

1. Introduction

The research purpose of this paper is to establish a novel uncertainty analysis method for the structures associated with repeated eigenvalues. By means of this method, considering the uncertainties in design variables or structural parameters, the uncertainty propagation of repeated eigenvalues, such as mean value and variance, can be acquired efficiently and accurately.

For complex engineering structures, such as stratospheric airship, commercial aircraft and large-scale rocket, due to the structural symmetry or sub-structural symmetry, the appearance of repeated eigenvalues or closely-spaced eigenvalues is widespread and practically inevitable [1-3]. If the impact of repeated eigenvalues is not taken into consideration when solving engineering problems, the mode shape of repeated eigenvalues will be more likely to be treated as distinct mode shape, which will result in the omission of modes and lay hidden perils in

structural safety and reliability [2]. Generally speaking, concerning the structures with repeated eigenvalues, owing to its universality in engineering practice, the relevant researches have attracted wide attention among the international scholars.

A fast algorithm of reanalysis problem for repeated frequencies is proposed by Zhao and Xu [4] based on relaxation combined approximations (CA) method. The generalized eigenvectors can be expressed by the combination of the basis vectors and the coefficients, and complex operations are simplified by avoidance of solving large scale equations. Palej and Krowiak [5] have completed the modal analysis of multiple degree of freedom systems with repeated frequencies in analytical approach. Vessel and Ram [6] et al. give two methods for calculating the sensitivity of a repeated eigenvalue of viscously damped vibrating systems with respect to an uncertain parameter. Wang and Zhang [7] have researched the tolerance of two closely-spaced but unequal natural frequencies when they are treated indiscriminately as double repeated frequencies. A mass-spring system is constructed by Chen and Jiao [8], by tuning the system physical parameters, any type of repeated natural frequency can be produced. Xia and Shi [9] et al. have presented a level set based shape and topology optimization method for maximizing the first repeated eigenvalue of structure vibration. Reimherr [10] has demonstrated that the commonly made assumption concerning unique eigenvalues is unnecessary. Fan and Xiang [11] et al. have presented a new method that redefines the stochastic problem into some sub-domains of random variables; the stochastic structural responses defined in the global domain can be explicitly reconstructed from the responses obtained in each sub-domain. This method is also able to deal with stochastic problems with repeated eigenvalues.

The demand for high-performance structural system calls for more refined analysis and design optimization method. Hence, the uncertainty analysis and optimization methods, which are adept at handling the propagation of various uncertain factors precisely and remarkably enhancing the structural performance under uncertainty environment, have become research hotspot in past decades [3,12,14,20,23]. Besides, structural system that possesses repeated eigenvalues is termed as the degenerate system. For degenerate system, the original repeated eigenvalues will easily convert to a set of clustered eigenvalues after the variations of design variables [1, 2]. Furthermore, on one aspect, the structures associated with repeated eigenvalues are very sensitive to the effect of the uncertainties; on the other aspect, real engineering structures always deviate from the original design due to the uncertainties (such as errors in manufacturing, differences in raw materials and variations in usage condition), which makes the impacts of uncertainties widely existed and unavoidable. In fact, when this type of structure encounters the impact of uncertainties, the distribution of its eigenvalues will be remarkably changed. Subsequently, conflicts between the design working condition and the perturbed eigenvalues may take place, which threatens the structural safety and performance. Accordingly, researches about uncertainty analysis and design optimization for the structures associated with repeated eigenvalues have attracted more and more attention [3,11,16,20].

The mainstream uncertainty analysis and dynamic reanalysis methods for structures with repeated eigenvalues can be roughly classified into statistical methods and non-statistical methods. The statistical methods, such as Latin hypercube sampling and Monte Carlo Simulation (MCS) [17], depend on the probabilistic distributions of random variables and a massive sample size. But for the real engineering structures, those preconditions are too costly to obtain. Furthermore, extensive computation is inevitable if we want to guarantee the precision of outcomes and to achieve the optimal solution, which is usually unacceptable for practical engineering structures.

Instead, serving as a sort of classic non-statistical method, perturbation method which permits quick sensitivity analysis and structural reanalysis is booming and is widely regarded as a vital tool for structural uncertainty analysis and optimization [1, 2]. High computational efficiency, less time consumption and easiness in execution are the conspicuous features of matrix

perturbation method [12-14].

Although the perturbation method has those advantages, it is still defective on two aspects. First, as for the Taylor series based perturbation method [6,15,16], the calculative results can only be guaranteed under small uncertainties. Furthermore, in terms of the structures with repeated eigenvalues, the derivative of repeated eigenvalue with respect to design variable may have more than one value [11], which is very difficult to handle via the Taylor series based perturbation method. Second, as for the classic matrix perturbation method, due to the fact that the perturbation expansions of repeated eigenvalue can't be written as the explicit function with respect to design variables, so the statistical quantities of perturbed repeated eigenvalue (e.g. mean value and variance) are almost unobtainable unless utilizing the statistical methods such as MCS, which not only hinders the applications of matrix perturbation method, but also impedes the dynamic reanalysis and variance-based importance measure.

To overcome these aforementioned obstacles, an improved stochastic perturbation method is proposed in this paper. In accordance with this proposed method, a surrogate model which consists of polynomial chaos expansions (PCE) is established to approximate the first-order perturbation term of repeated eigenvalue. Then, omitting the second and higher-order perturbation terms, the perturbed repeated eigenvalue is expressed as the combination of original repeated eigenvalue and the surrogate model. By virtue of this surrogate model, the statistical quantities of perturbed repeated eigenvalue are acquired directly.

In general, the contributions of this work are multifold: 1) by simply knowing the dimensionality and distribution of design variables, the surrogate model could be achieved after finite several calculations on collocation points, then the statistical characteristics of repeated eigenvalues under uncertainties could be computed efficiently and accurately; 2) the proposed method is available to cope with structures undergoing relatively large uncertainties, which is validated by the numerical examples; 3) via this new method, the uncertainty analysis, structural reanalysis and design optimization for structures with repeated eigenvalues can be performed and accomplished expediently.

The remainder of this paper is organized as follows. In Section 2, the perturbation analysis for repeated eigenvalues is systematically discussed. In Section 3, we introduce a surrogate model based on PCE method to approximate the first-order perturbation term of repeated eigenvalue. In Section 4, the expressions of statistical quantities for perturbed repeated eigenvalue are obtained. In Section 5, three numerical examples are utilized to verify the efficiency and accuracy of the proposed method. Lastly, the conclusions are drawn.

2. Perturbation Analysis for Repeated Eigenvalues

Assuming that λ_0^i are repeated eigenvalues with m multiplicities, which belongs to a degenerate system. Accordingly, there should be m eigenvectors $\boldsymbol{w}^i (i = 1,2,\cdots,m)$, which are pairwise orthogonal, subject to

$$\boldsymbol{K}_0 \boldsymbol{w}^i = \lambda_0^i \boldsymbol{M}_0 \boldsymbol{w}^i , \tag{1}$$

in which $\boldsymbol{K}_0$, $\boldsymbol{M}_0$ denote original stiffness matrix and original mass matrix of the system with repeated eigenvalues, respectively. Moreover, the linear combination of $\boldsymbol{w}^i$ is also the eigenvector of repeated eigenvalue λ_0^i, which can be expressed as

$$\begin{aligned} \boldsymbol{u}_0^i &= \alpha_1 \boldsymbol{w}^1 + \alpha_2 \boldsymbol{w}^2 + \cdots + \alpha_m \boldsymbol{w}^m \\ &= \left[\boldsymbol{w}^1, \boldsymbol{w}^2, \cdots, \boldsymbol{w}^m \right] \cdot \left[\alpha_1, \alpha_2, \cdots, \alpha_m \right]^{\mathrm{T}} , \\ &= \boldsymbol{W} \cdot \boldsymbol{\alpha} \end{aligned} \tag{2}$$

where $\boldsymbol{\alpha}$ is an undetermined vector.

The structural parameters of degenerate system will be changed under the external perturbation,

such as variations of serving environment and differences in raw materials, which finally results in the variations of mass matrix and stiffness matrix as follows:

$$\begin{cases} \boldsymbol{K} = \boldsymbol{K}_0 + \varepsilon \boldsymbol{K}_1 \\ \boldsymbol{M} = \boldsymbol{M}_0 + \varepsilon \boldsymbol{M}_1 \end{cases}, \tag{3}$$

in which ε is a minimum parameter, and the original system, known as the system corresponding to $\varepsilon = 0$. $\varepsilon \boldsymbol{M}_1$, $\varepsilon \boldsymbol{K}_1$ indicate the first-order perturbation term of $\boldsymbol{M}_0$, $\boldsymbol{K}_0$, respectively.

Repeated eigenvalues no longer exist in the system after perturbation, which means there should be m different eigenvalues $\lambda^1, \lambda^2, \cdots, \lambda^m$ corresponding to m different eigenvectors $\boldsymbol{u}^i \left(i = 1, 2, \cdots, m\right)$. It is similar to the circumstance of distinct eigenvalue. We expand the eigenvalue λ^i and eigenvector $\boldsymbol{u}^i$ as ε-series form

$$\lambda^i = \lambda_0^i + \varepsilon \lambda_1^i + \varepsilon^2 \lambda_2^i + \cdots, \tag{4}$$

$$\boldsymbol{u}^i = \boldsymbol{u}_0^i + \varepsilon \boldsymbol{u}_1^i + \varepsilon^2 \boldsymbol{u}_2^i + \cdots. \tag{5}$$

By virtue of the perturbation method, eqs.(3), (4), (5) are substituted into $\boldsymbol{K}\boldsymbol{u} = \lambda \boldsymbol{M}\boldsymbol{u}$, then we expand it with $O(\varepsilon^3)$ terms omitted. After that, comparing the coefficients of the same power terms of ε on each side of the equation, we have

$$\boldsymbol{K}_0 \boldsymbol{u}_1^i + \boldsymbol{K}_1 \boldsymbol{u}_0^i = \lambda_0^i \boldsymbol{M}_0 \boldsymbol{u}_1^i + \lambda_0^i \boldsymbol{M}_1 \boldsymbol{u}_0^i + \lambda_1^i \boldsymbol{M}_0 \boldsymbol{u}_0^i, \tag{6}$$

substituting eq.(2) into eq.(6), yielding

$$\boldsymbol{K}_0 \boldsymbol{u}_1^i + \boldsymbol{K}_1 \sum_{j=1}^{m} \alpha_j \boldsymbol{w}^j = \lambda_0^i \boldsymbol{M}_0 \boldsymbol{u}_1^i + \lambda_0^i \boldsymbol{M}_1 \sum_{j=1}^{m} \alpha_j \boldsymbol{w}^j + \lambda_1^i \boldsymbol{M}_0 \sum_{j=1}^{m} \alpha_j \boldsymbol{w}^j, \tag{7}$$

premultiplying eq.(7) with $[\boldsymbol{w}^k]^{\mathrm{T}}$ gives

$$[\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{K}_0 \boldsymbol{u}_1^i + \sum_{j=1}^{m} [\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{K}_1 \boldsymbol{w}^j \alpha_j = \lambda_0^i [\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{M}_0 \boldsymbol{u}_1^i + \lambda_0^i \sum_{j=1}^{m} [\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{M}_1 \boldsymbol{w}^j \alpha_j + \lambda_1^i \sum_{j=1}^{m} [\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{M}_0 \boldsymbol{w}^j \alpha_j, \tag{8}$$

taking the following conditions into consideration

$$[\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{K}_0 \boldsymbol{u}_1^i = [\boldsymbol{u}_1^i]^{\mathrm{T}} \boldsymbol{K}_0 \boldsymbol{w}^k = \lambda_0^i [\boldsymbol{u}_1^i]^{\mathrm{T}} \boldsymbol{M}_0 \boldsymbol{w}^k, \tag{9}$$

$$\lambda_0^i [\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{M}_0 \boldsymbol{u}_1^i = \lambda_0^i [\boldsymbol{u}_1^i]^{\mathrm{T}} \boldsymbol{M}_0 \boldsymbol{w}^k, \tag{10}$$

and the orthogonality

$$[\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{M}_0 \boldsymbol{w}^j = \delta_{kj}, \tag{11}$$

eq.(8) is rewritten as

$$\sum_{j=1}^{m} \left([\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{K}_1 \boldsymbol{w}^j - \lambda_0^i [\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{M}_1 \boldsymbol{w}^j \right) \alpha_j = \lambda_1^i \alpha_j \quad \left(k = 1, 2, \cdots, m\right). \tag{12}$$

Assuming that

$$\begin{cases} a_{kj} = [\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{M}_1 \boldsymbol{w}^j \\ b_{kj} = [\boldsymbol{w}^k]^{\mathrm{T}} \boldsymbol{K}_1 \boldsymbol{w}^j \end{cases}, \tag{13}$$

then eq.(12) is equivalent to

$$\sum_{j=1}^{m} \left(b_{jk} - \lambda_0^i a_{kj} \right) \alpha_j = \lambda_1^i \alpha_k, \quad k = 1, 2, \cdots, m, \tag{14}$$

the matrix form of eq.(14) can be represented as

$$\left(\boldsymbol{D} - \lambda_1^i \boldsymbol{I}^{m \times m} \right) \boldsymbol{\alpha} = 0, \tag{15}$$

where the element of matrix $\boldsymbol{D}$ is subject to

$$d_{kj} = b_{kj} - \lambda_0^i a_{kj}, \quad (k, j = 1, 2, \cdots, m). \tag{16}$$

Eq.(15) is a standard eigenvalue equation with m order, by solving this eigenvalue problem we will acquire the first-order perturbation term $\lambda_1^j \left(j = 1, 2, \cdots, m \right)$, which corresponds to the original repeated eigenvalue λ_0^i, and the undetermined coefficient α_j.

$\lambda_1^j \left(j = 1, 2, \cdots, m \right)$ are arranged in ascending order, if eq.(15) has no repeated eigenvalues, then both λ_1^j and α_j are unique. For simplicity, we assume that eq.(15) has no repeated eigenvalues.

Solving eq.(15) equals to solve the following equation

$$\det\left[\boldsymbol{D} - \lambda_1^j \boldsymbol{I}^{m \times m} \right] = 0, \tag{17}$$

then $\lambda_1^j \left(j = 1, 2, \cdots, m \right)$ is obtained, by substituting it back into eq.(15), coefficients $\alpha_j \left(j = 1, 2, \cdots, m \right)$ are acquired.

After that, according to eq.(2), we will get the eigenvector which correspond to the repeated eigenvalues λ_0^i as follows

$$u_0^j = \boldsymbol{W} \alpha_j \quad \left(j = i, i+1, \cdots, i+m-1 \right). \tag{18}$$

As for eq.(17), solving it directly, we will obtain $\lambda_1^j \left(j = 1, 2, \cdots, m \right)$. Nevertheless, by solving eq.(17), we cannot establish an explicit equation for $\lambda_1^j \left(j = 1, 2, \cdots, m \right)$ with respect to structural parameters $\boldsymbol{M}_1$ and $\boldsymbol{K}_1$. That being so, in order to perform the structural uncertainty analysis or structural dynamic reanalysis, every time the system undergoes perturbation in design variables, a brand new set of $\left(\boldsymbol{M}_1, \boldsymbol{K}_1 \right)$ will be generated; on top of that, we have to recalculate the eq.(17) all over again in order to acquire the correspondingly new set of $\lambda_1^j \left(j = 1, 2, \cdots, m \right)$. Indisputably, the repetitive computational processes are unacceptable and prohibitive for overwhelming majority engineering projects.

To sum up, owing to the limitation that an explicit expression for $\lambda_1^j \left(j = 1, 2, \cdots, m \right)$ with respect to structural parameters $\left(\boldsymbol{M}_1, \boldsymbol{K}_1 \right)$ or design variables is unattainable via the classic matrix perturbation method, the computational cost of structural reanalysis and uncertainty analysis will be far too much for engineering structure associated with repeated eigenvalues. Moreover, after the perturbations of design variables, the variation range and importance measure of repeated eigenvalues have absolutely no regularity to track except for time-consuming statistical methods. This not only causes side effect to the efficiency of dynamic reanalysis, but also hinders the structural design and optimization.

In this paper, we are willing to provide a surrogate model instead of solving eq.(17) repetitively. Hence, the explicit expression of $\lambda_1^j \left(j = 1, 2, \cdots, m \right)$ with respect to design variables could be approximated by the proposed surrogate model. Then the uncertainty analysis and dynamic reanalysis of repeated eigenvalues will be implemented via the surrogate model expediently.

3. Surrogate Model for Stochastic Repeated Eigenvalues Analysis

As Appendix part 1 states, by solving eq.(17), we will acquire the first-order perturbation term $\lambda_1^q \left(q = 1, 2, \cdots, m \right)$ of repeated eigenvalues. However, on one hand, solving eq.(17) requires extensive computation, which is usually unaffordable in engineering practice. On the other hand, even though $\lambda_1^q \left(q = 1, 2, \cdots, m \right)$ are obtained from eq.(17), one cannot establishes an explicit expression between design variables and λ_1^q, which means a general method for repeated

eigenvalues' uncertainty analysis is still unavailable.

In order to solve those problems, surrogate model is introduced. The random variables $\boldsymbol{\xi}=(\xi_1,\xi_2,\cdots,\xi_n)$ in system are chosen as independent variables, Hermite polynomials are taken as complete orthogonal basis, therefore the original response function in eq.(17) is approximated by the surrogate model $Y(\boldsymbol{\xi})$, which is presented particularly in Appendix part 1.

First of all, let the highest order of Hermite orthogonal polynomials be subject to $p=2$. Hence, concerning the system that corresponds to eq.(17), the surrogate model is expressed as

$$\lambda_1^q(\boldsymbol{\xi})=c_{0,q}+\sum_{i=1}^{n}c_{i,q}\xi_i+\sum_{i=1}^{n}c_{ii,q}\left(\xi_i^2-1\right)+\sum_{i=1}^{n-1}\sum_{j>i}^{n}c_{ij,q}\xi_i\xi_j,\quad (q=1,2,\cdots,m), \tag{19}$$

where $\boldsymbol{\xi}=(\xi_1,\xi_2,\cdots,\xi_n)$ denotes random variables which subject to Gaussian distribution, $c_{0,q}$, $c_{i,q}$, $c_{ii,q}$, $c_{ij,q}$,$\cdots$ stand for the undetermined coefficients of Hermite polynomial expansion, $q=1,2,\cdots,m$ stands for the multiplicity of repeated eigenvalues' first-order perturbation. According to eqs.(19), the number of undetermined coefficients is subject to $s=(n+2)(n+1)/2$.

Now we prepare for the collocation points. Considering that the highest order of polynomial chaos expansions in this paper subject to $p=2$. So the roots of third order Hermite polynomial, i.e. $x=\pm\sqrt{3}$, 0 are taken as collocation points. Assuming the random variables $\xi_i\,(i=1,2,\cdots,n)$ subject to Gaussian distribution $N(\mu_i,\sigma_i^2)$, $i=1,2,\cdots,n$, then there are three values for every collocation point, which is shown as $\xi_i=\mu_i-\sqrt{3}\sigma_i$, μ_i, $\mu_i+\sqrt{3}\sigma_i$. As for the n-dimensional random variables, the combinatorial number of random variables, which is also the number of collocation points, equals to 3^n. In addition, every set of collocation points is expressed as

$$\boldsymbol{\xi}_\beta=\left(\xi_{\beta1},\xi_{\beta2},\cdots,\xi_{\beta n}\right),\quad \beta=1,2,\cdots,3^n.$$

Substituting the previous collocation point $\boldsymbol{\xi}_\beta$ into eq.(12), in the meanwhile, re-writing the matrices $\boldsymbol{M}_1,\boldsymbol{K}_1$ in eq.(12) as

$$\begin{cases}\boldsymbol{M}_1=\boldsymbol{M}_{\beta1}=\sum_{i=1}^{n}\left[\boldsymbol{M}\left(\mu_1,\mu_2,\cdots,\mu_{i-1},\xi_{\beta i},\mu_{i+1},\cdots\mu_n\right)-\boldsymbol{M}\left(\mu_1,\mu_2,\cdots,\mu_{i-1},\mu_i,\mu_{i+1},\cdots,\mu_n\right)\right]\\ \boldsymbol{K}_1=\boldsymbol{K}_{\beta1}=\sum_{i=1}^{n}\left[\boldsymbol{K}\left(\mu_1,\mu_2,\cdots,\mu_{i-1},\xi_{\beta i},\mu_{i+1},\cdots\mu_n\right)-\boldsymbol{K}\left(\mu_1,\mu_2,\cdots,\mu_{i-1},\mu_i,\mu_{i+1},\cdots,\mu_n\right)\right]\end{cases} \tag{20}$$

in addition, as for $\xi_{\beta i}=\mu_i$, $(\beta=1,2,\cdots,3^n)$, then we let

$$\begin{cases}\left(\boldsymbol{M}\left(\mu_1,\mu_2,\cdots,\mu_{i-1},\xi_{\beta i},\mu_{i+1},\cdots\mu_n\right)-\boldsymbol{M}\left(\mu_1,\mu_2,\cdots,\mu_{i-1},\mu_i,\mu_{i+1},\cdots,\mu_n\right)\right)=0\\ \left(\boldsymbol{K}\left(\mu_1,\mu_2,\cdots,\mu_{i-1},\xi_{\beta i},\mu_{i+1},\cdots\mu_n\right)-\boldsymbol{K}\left(\mu_1,\mu_2,\cdots,\mu_{i-1},\mu_i,\mu_{i+1},\cdots,\mu_n\right)\right)=0\end{cases}.$$

Enlightened by the high dimensional model representation (HDMR) and stochastic perturbation method, we have deduced the expressions of $\boldsymbol{M}_1$ and $\boldsymbol{K}_1$ in eq.(20). The detailed derivation process of $\boldsymbol{M}_1,\boldsymbol{K}_1$ in eq.(20) can be seen in Appendix part 2 [18, 19].

After the preceding procedure, solving eq.(12) as follows:

$$\sum_{q=1}^{m}\left([\boldsymbol{w}^k]^{\mathrm{T}}\boldsymbol{K}_{\beta1}\boldsymbol{w}^q-\lambda_0^q[\boldsymbol{w}^k]^{\mathrm{T}}\boldsymbol{M}_{\beta1}\boldsymbol{w}^q\right)\alpha_q=\lambda_{\beta1}^q\alpha_q\quad (k=1,2,\cdots,m), \tag{21}$$

re-arranging all the roots of eq.(21) in ascending order, one will obtain a set of solutions

$$\boldsymbol{\lambda}_{\beta1}^q=\left[\lambda_{\beta1}^1,\lambda_{\beta1}^2,\lambda_{\beta1}^3,\cdots,\lambda_{\beta1}^m\right]^{\mathrm{T}}.$$

Repeating the procedure above, substituting every collocation point into eq.(21) one by one, then the solution vector is acquired as

$$\begin{bmatrix} \boldsymbol{\lambda}_{11}^{q} & \boldsymbol{\lambda}_{21}^{q} & \boldsymbol{\lambda}_{31}^{q} & \cdots & \boldsymbol{\lambda}_{N1}^{q} \end{bmatrix} = \begin{bmatrix} \lambda_{11}^{1} & \lambda_{21}^{1} & \lambda_{31}^{1} & \cdots & \lambda_{N1}^{1} \\ \lambda_{11}^{2} & \lambda_{21}^{2} & \lambda_{31}^{2} & \cdots & \lambda_{N1}^{2} \\ \lambda_{11}^{3} & \lambda_{21}^{3} & \lambda_{31}^{3} & \cdots & \lambda_{N1}^{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \underline{\lambda_{11}^{\gamma}} & \underline{\lambda_{21}^{\gamma}} & \underline{\lambda_{31}^{\gamma}} & \cdots & \underline{\lambda_{N1}^{\gamma}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{11}^{m} & \lambda_{21}^{m} & \lambda_{31}^{m} & \cdots & \lambda_{N1}^{m} \end{bmatrix}, \tag{22}$$

in which $N = 3^n$, considering the solution with arbitrary order γ , we have $\boldsymbol{\lambda}_1^{\gamma} = \begin{bmatrix} \lambda_{11}^{\gamma} & \lambda_{21}^{\gamma} & \lambda_{31}^{\gamma} & \cdots & \lambda_{N1}^{\gamma} \end{bmatrix}$, $\gamma = 1, 2, \cdots, m$, from eq.(22), we can form the equations as follows:

$$\begin{bmatrix} \Gamma_0(\boldsymbol{\xi}_0) & \Gamma_1(\boldsymbol{\xi}_0) & \cdots & \Gamma_{s-1}(\boldsymbol{\xi}_0) \\ \Gamma_0(\boldsymbol{\xi}_1) & \Gamma_1(\boldsymbol{\xi}_1) & \cdots & \Gamma_{s-1}(\boldsymbol{\xi}_1) \\ \vdots & \vdots & \cdots & \vdots \\ \Gamma_0(\boldsymbol{\xi}_N) & \Gamma_1(\boldsymbol{\xi}_N) & \cdots & \Gamma_{s-1}(\boldsymbol{\xi}_N) \end{bmatrix} \begin{bmatrix} c_0^{\gamma} \\ c_1^{\gamma} \\ \vdots \\ c_{s-1}^{\gamma} \end{bmatrix} = \begin{bmatrix} \lambda_{11}^{\gamma} \\ \lambda_{21}^{\gamma} \\ \vdots \\ \lambda_{N1}^{\gamma} \end{bmatrix}, \tag{23}$$

It is worth mentioning that eq.(23) consists of 3^n equations, and contains s undetermined coefficients. As for PCE method, normally we have $3^n > s$, therefore via regression analysis based on the least square method, we will determine a set of coefficients $\begin{bmatrix} c_0^{\gamma} & c_1^{\gamma} & \cdots & c_{s-1}^{\gamma} \end{bmatrix}$.

For clarity, the constructing procedure of surrogate model based on PCE method is illustrated in Figure 1,

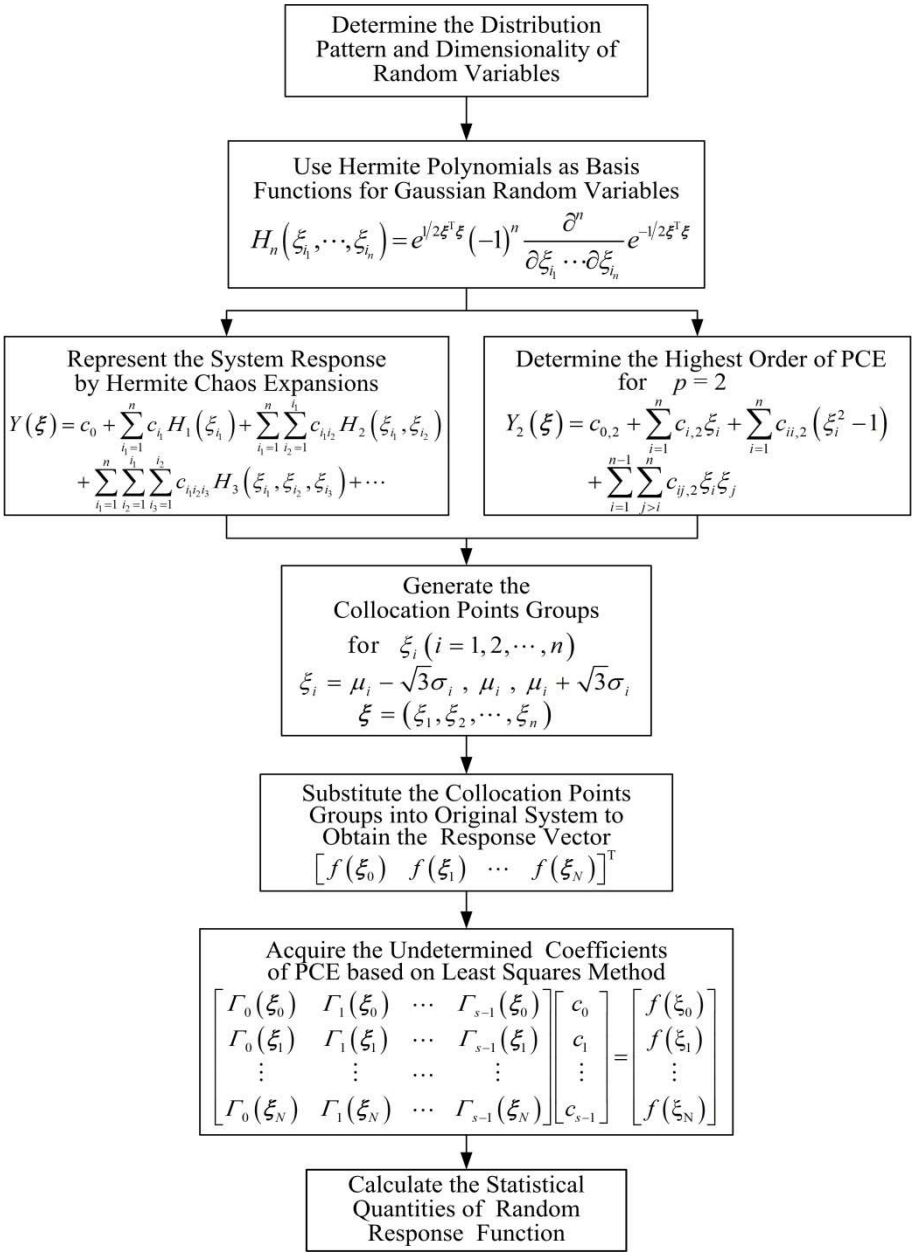


Figure 1 – Constructing procedure of surrogate model based on PCE method

4. Statistical Quantities of Random Response Function in Surrogate Model

As illustrated in Section 3 and Appendix part 1, the first-order perturbation term of the γth repeated eigenvalue can be demonstrated as

$$\lambda_1^{\gamma}\left(\boldsymbol{\xi}\right)=c_{0,\gamma}+\sum_{i=1}^{n}c_{i,\gamma}\xi_i+\sum_{i=1}^{n}c_{ii,\gamma}\left(\xi_i^2-1\right)+\sum_{i=1}^{n-1}\sum_{j>i}^{n}c_{ij,\gamma}\xi_i\xi_j\ , \tag{24}$$

where $\boldsymbol{\xi}=\left(\xi_1,\xi_2,\cdots,\xi_n\right)$ denotes random variables that are subject to Gaussian distribution $\mathrm{N}(\mu_i,\sigma_i^2),\ i=1,2,\cdots,n$. Now all the coefficients $c_{0,\gamma}$, $c_{i,\gamma}$, $c_{ii,\gamma}$, $c_{ij,\gamma}$ have been known via the process presented in Section 3 and Appendix part 1.

According to eq.(4), omitting the terms higher than second order, one yields

$$\lambda^{\gamma}=\lambda_0^{\gamma}+\varepsilon\lambda_1^{\gamma}=\lambda_0^{\gamma}+\varepsilon\left(c_{0,\gamma}+\sum_{i=1}^{n}c_{i,\gamma}\xi_i+\sum_{i=1}^{n}c_{ii,\gamma}\left(\xi_i^2-1\right)+\sum_{i=1}^{n-1}\sum_{j>i}^{n}c_{ij,\gamma}\xi_i\xi_j\right), \tag{25}$$

for clarity, both superscript γ and subscript γ are removed from eq.(25) in the derivation hereinafter, i.e.,

$$\lambda = \lambda_0 + \varepsilon\lambda_1 = \lambda_0 + \varepsilon\left(c_0 + \sum_{i=1}^{n} c_i\xi_i + \sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right). \tag{26}$$

Using the expectation operator on eq.(26), we have

$$\begin{aligned} \mathrm{E}[\lambda] = \mathrm{E}[\lambda_0 + \varepsilon\lambda_1] &= \mathrm{E}\left[\lambda_0 + \varepsilon\left(c_0 + \sum_{i=1}^{n} c_i\xi_i + \sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right) \right] \\ &= \lambda_0 + \varepsilon\mathrm{E}\left[c_0 + \sum_{i=1}^{n} c_i\xi_i + \sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right] \\ &= \lambda_0 + \varepsilon c_0 \end{aligned}, \tag{27}$$

utilizing the variance operator on eq.(26), and taking the orthogonality of Hermite polynomial into account, one yields

$$\begin{aligned} \mathrm{Var}[\lambda] = \mathrm{Var}[\lambda_0 + \varepsilon\lambda_1] &= \mathrm{Var}\left[\lambda_0 + \varepsilon\left(c_0 + \sum_{i=1}^{n} c_i\xi_i + \sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right) \right] \\ &= \varepsilon^2\mathrm{Var}\left[\left(c_0 + \sum_{i=1}^{n} c_i\xi_i + \sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right) \right] \\ &= \varepsilon^2\sum_{i=1}^{n} c_i^2 + \varepsilon^2\mathrm{Var}\left[\sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) \right] + \varepsilon^2\mathrm{Var}\left[\sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right] \end{aligned}, \tag{28}$$

in conjunction with

$$\begin{cases} \varepsilon^2\mathrm{Var}\left[\sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) \right] = \varepsilon^2\left(\mathrm{E}\left[\left(\sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) \right)^2 \right] - \left(\mathrm{E}\left[\sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) \right] \right)^2 \right) \\ \qquad\qquad = \varepsilon^2\left(\sum_{i=1}^{n} c_{ii}^2\left\langle \Gamma_{ii}^2 \right\rangle \right) \\ \varepsilon^2\mathrm{Var}\left[\sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right] = \varepsilon^2\left(\mathrm{E}\left[\left(\sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right)^2 \right] - \left(\mathrm{E}\left[\sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right] \right)^2 \right) \\ \qquad\qquad = \varepsilon^2\left(\sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}^2\left\langle \Gamma_{ij}^2 \right\rangle \right) \end{cases} \tag{29}$$

taking eqs.(28) and (29) into consideration, we have

$$\begin{aligned} \mathrm{Var}[\lambda] = \mathrm{Var}[\lambda_0 + \varepsilon\lambda_1] &= \mathrm{Var}\left[\lambda_0 + \varepsilon\left(c_0 + \sum_{i=1}^{n} c_i\xi_i + \sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right) \right] \\ &= \varepsilon^2\mathrm{Var}\left[\left(c_0 + \sum_{i=1}^{n} c_i\xi_i + \sum_{i=1}^{n} c_{ii}\left(\xi_i^2 - 1\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}\xi_i\xi_j \right) \right] \\ &= \varepsilon^2\sum_{i=1}^{n} c_i^2 + \varepsilon^2\left(\sum_{i=1}^{n} c_{ii}^2\left\langle \Gamma_{ii}^2 \right\rangle \right) + \varepsilon^2\left(\sum_{i=1}^{n-1}\sum_{j>i}^{n} c_{ij}^2\left\langle \Gamma_{ij}^2 \right\rangle \right) \end{aligned}. \tag{30}$$

5. Numerical Examples

In this section, a 4-DOF mass-spring system is first utilized to verify the feasibility of this proposed method. Then, a dome structure with fifty two bars and an airship structure with repeated eigenvalues are used to demonstrate the capability of the proposed method in coping with large and complex engineering structures.

5.1 Four-DOF Mass-Spring System

As demonstrated in Figure 2, the mass-spring system is in a closed-loop form, where 4 mass blocks are symmetrically organized on a circumference with a radius of R. A ring transgresses the central holes of mass blocks and limits the block's track along the ring. Mass blocks are interconnected by springs and the friction in this example is ignored.

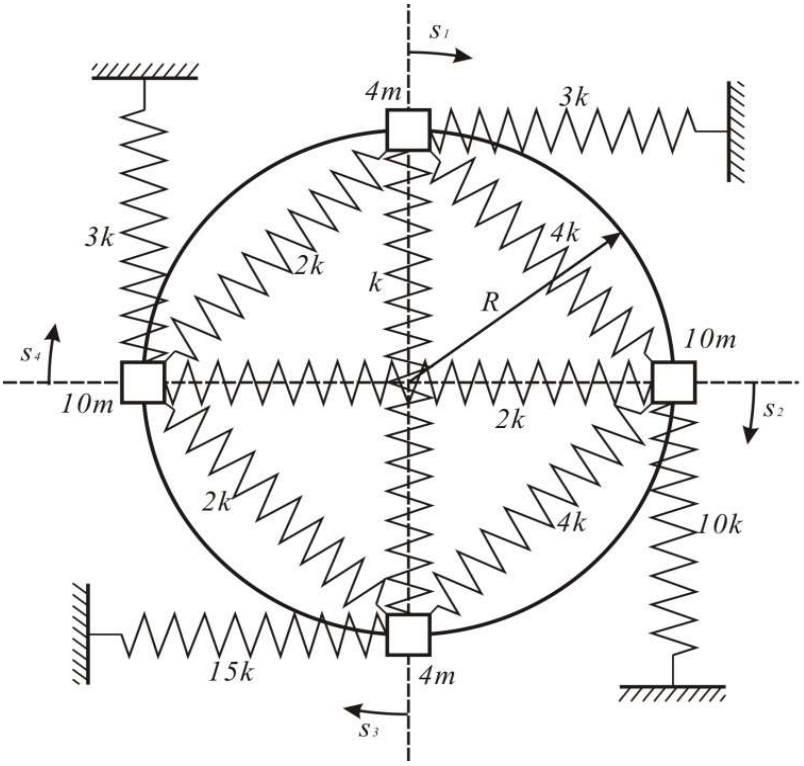


Figure 2 – 4-DOF mass-spring system

In this case, the original stiffness matrix $\boldsymbol{K}_0$ and original mass matrix $\boldsymbol{M}_0$ are expressed as

$$\boldsymbol{K}_0 = \begin{bmatrix} 7k & -2k & -k & -k \\ -2k & 16k & -2k & -2k \\ -k & -2k & 19k & -k \\ -k & -2k & -k & 7k \end{bmatrix}, \boldsymbol{M}_0 = \begin{bmatrix} 4m & 0 & 0 & 0 \\ 0 & 10m & 0 & 0 \\ 0 & 0 & 10m & 0 \\ 0 & 0 & 0 & 4m \end{bmatrix}$$

according to eq.(1), the eigenvalue matrix $\boldsymbol{\Lambda}_0$ and eigenvector matrix $\boldsymbol{U}_0$ are obtained as follows:

$$\boldsymbol{\Lambda}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \boldsymbol{U}_0 = \begin{bmatrix} -0.25 & 0.111 & 0.391 & -0.1494 \\ -0.2 & 0.0902 & -0.1971 & -0.1141 \\ -0.1 & 0.1283 & 0.0032 & 0.2712 \\ -0.25 & -0.4197 & 0 & 0.1064 \end{bmatrix}.$$

Therefore, the distinct eigenvalue is $\lambda_{01} = 1$, repeated eigenvalues are $\lambda_{02} = \lambda_{03} = \lambda_{04} = 2$, as for the eigenvectors, the distinct eigenvector is $w_{01} = [-0.25, -0.2, -0.1, -0.25]^{\mathrm{T}}$, the corresponding repeated eigenvectors are

$$\begin{cases} w_{02} = [0.111,\ 0.0902,\ 0.1283,\ -0.4197]^{\mathrm{T}} \\ w_{03} = [0.391,\ -0.1971,\ 0.0032,\ 0]^{\mathrm{T}} \\ w_{04} = [-0.1494,\ -0.1141,\ 0.2712,\ 0.1064]^{\mathrm{T}} \end{cases},$$

assuming that k is subject to Gaussian distribution $\mathrm{N}(1, 0.1^2)$, m is subject to Gaussian

distribution $\mathrm{N}\left(1,0.2^2\right)$. According to the proposed method, the collocation points of (k,m) are assigned in Figure 3.

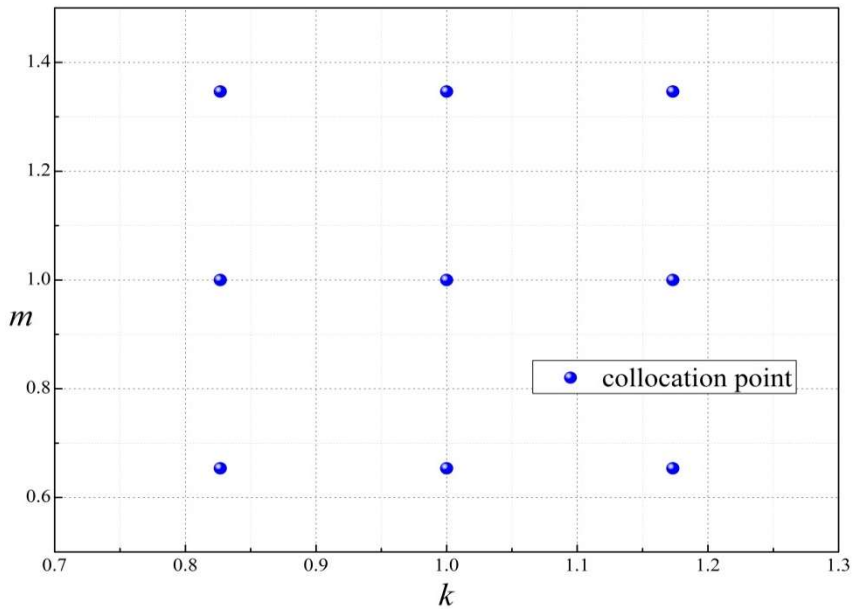


Figure 3 – Distribution of collocation points

Utilizing the proposed method in this work, the first-order perturbation term of the original repeated eigenvalues, with ranking first and second (similar discussions could be applied on λ_1^3), are represented in the form of polynomial chaos expansions as follows:

$$\begin{cases}\lambda_1^1 = 1.0003\times10^{-3} + 0.2310k - 0.4619m - 1.1112\times10^{-5}\left(k^2-1\right) - 4.4447\times10^{-5}\left(m^2-1\right) + 3.3335\times10^{-5}km \\ \lambda_1^2 = 1.0006\times10^{-3} + 0.2313k - 0.4623m + 1.1115\times10^{-5}\left(k^2-1\right) + 4.4452\times10^{-5}\left(m^2-1\right) - 3.3377\times10^{-5}km\end{cases}.$$

Firstly, as shown in Table , different sample size has been tested in order to determine the sample size utilized in Monte-Carlo Simulation (MCS).

Table 1 – Determine the sample size of Monte-Carlo Simulation

Repeated Eigenvalues (Rank)	Statistic Quantities	Sample Size			
		10^4	10^5	10^6	3×10^6
λ_1	Mean Value	2.0907	2.0938	2.0927	2.0925
	Variance	0.2876	0.2934	0.2908	0.2907

As illustrated in Table , with the increase of sample size, the fluctuations of $\mathrm{Var}(\lambda_1)$ and $\mathrm{E}(\lambda_1)$ calculated by MCS tend to be stable. Noting that when the sample size is larger than 10^6, statistic quantities of repeated eigenvalue computed by MCS almost converge to a certain value. Hence, for the sake of time expense and calculative accuracy, 10^6 is chosen as the sample size of MCS for exact solutions.

Secondly, according to eq.(30), the statistic quantities of repeated eigenvalues under perturbations are obtained in Table . As can be seen in Table , comparing to MCS, whose sample size is 10^6, calculative accuracy of the proposed method is acceptable for engineering application.

Table 2 – Comparison of variance of repeated eigenvalues calculated by two methods

Repeated Eigenvalues	Methods	Variance	Relative Error	Mean Value	Relative Error
λ_1	Proposed Method	0.2880	0.96%	2.0128	3.82%
	Monte-Carlo Sample Size 10^6	0.2908		2.0927	
λ_2	Proposed Method	0.2883	0.995%	2.0243	3.27%
	Monte-Carlo Sample Size 10^6	0.2912		2.0928	

After the preceding discussion, we have investigated the results of proposed method concerning the random variables with fixed coefficient of variation (i.e. 0.1 for k, 0.2 for m). Hereinafter, in order to further study the feasibility and applicability of this new method, we take different coefficient of variation into account. Considering the coefficient of variation varies from 0.01 to 0.2 for all the random variables, we intend to research the calculative capability of the proposed method under large uncertainty. Complementarily, in this part, coefficient of variation varies step by step simultaneously for all the random variables.

As can be seen in Figure 4, the outcomes of the proposed method are in agreement with those of the MCS. When the coefficient of variation is small in value (less than 0.1), those two results match very well. As for the coefficient of variation varies from 0.1 to 0.2, the deviation in results between MCS and proposed method enlarge gradually. It is worth mentioning that even if the coefficient of variation is assigned as 0.2, which corresponds to large uncertainty in random variables, the relative error between proposed method and MCS is just 15.9%. With regard to the condition that the variation coefficient is zero, which indicates all the structural parameters are deterministic without uncertainty, the results of these two methods both turn to zero value.

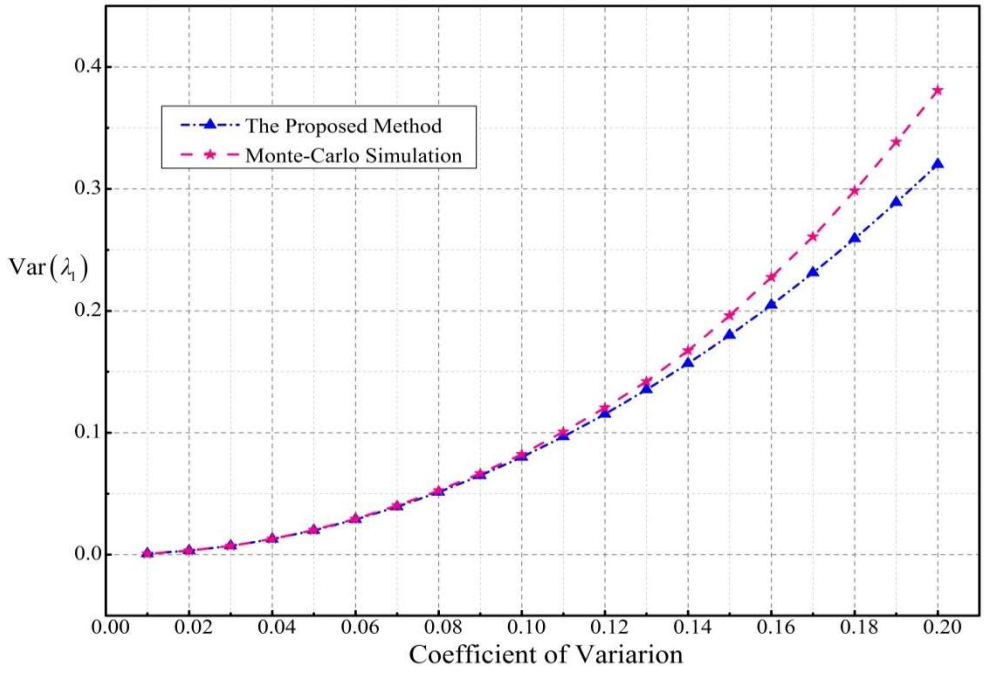


Figure 4 – Comparison of $\mathrm{Var}(\lambda_1)$ obtained by two methods versus different coefficient of variation

Without loss of generality, in the next discussion, each random variable is assumed varying from 0.01 to 0.2 independently. The 3D-distributions of $\mathrm{Var}(\lambda_1)$ which are obtained by MCS and the proposed method respectively are demonstrated in Figure 5. As Figure 5 states, the upper-layer surface stands for $\mathrm{Var}(\lambda_1)$ computed by MCS, while the lower-layer surface denotes $\mathrm{Var}(\lambda_1)$ calculated by the proposed method.

Three main conclusions could be achieved via Figure 5: Firstly, $\mathrm{Var}(\lambda_1)$ calculated by the proposed method is consistent with that of MCS concerning the random variables varying independently; Secondly, $\mathrm{Var}(\lambda_1)$ obtained by the proposed method are relatively smaller in value than that of MCS, which is similar to Figure 4; Thirdly, even if the variation coefficients are large (close to 0.2), the relative errors between the proposed method and exact solutions are still acceptable, which means this new method is applicable to the problems with large uncertainty.

For simplicity and clarity, only the first repeated-eigenvalue λ_1 is discussed in this part, similar discussions can be performed on λ_2 and λ_3.

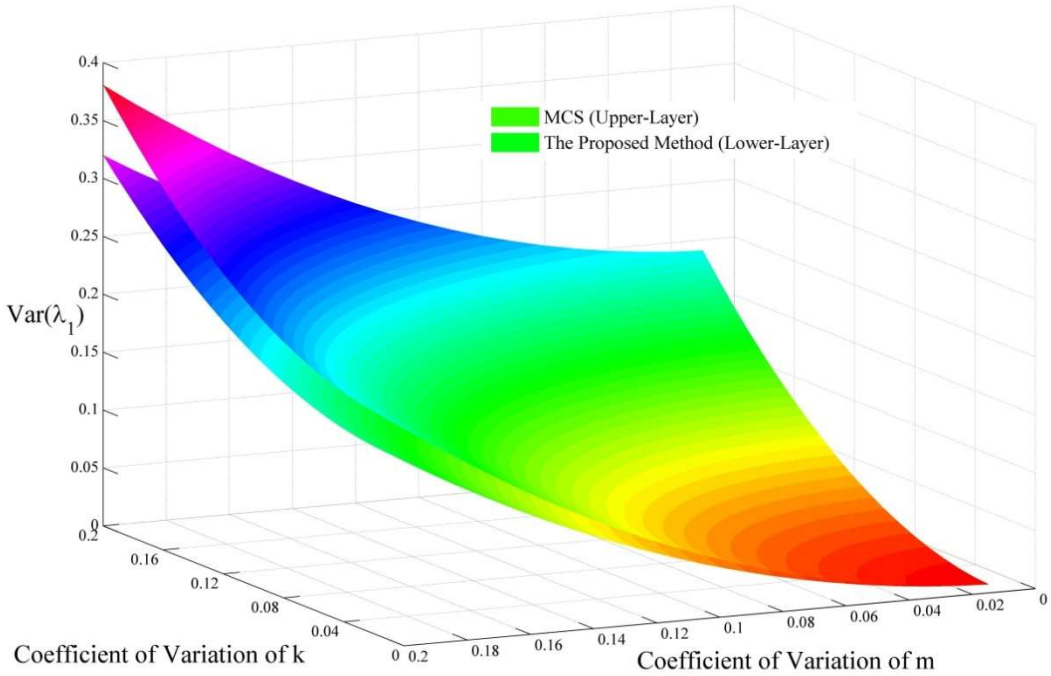


Figure 5 – 3D distribution of $\mathrm{Var}(\lambda_1)$ obtained by two methods with random variables varying independently

Evidently, the proposed method in this paper has considerable advantages on computational efficiency, which is fully demonstrated in Table . Note that in Table , even for the sample size of 1.0×10^6, the MCS consumes two orders of magnitude more time than the proposed method. As for the engineering structure, the advantage of computational efficiency for the proposed method will be unfolded more obviously.

Table 3 – Calculative time consumption for two methods

Methods		Calculative Time Consumption (s)
		λ_1
Monte-Carlo Simulation (Sample	1.0×10^6	16.912
	2.0×10^6	33.609

Size)	3.0×10^6	50.718
	4.0×10^6	66.696
	5.0×10^6	84.014
	6.0×10^6	100.321
The Proposed Method		0.0962

Both the proposed method and MCS in this case were operated on an Intel Core i7-2600@3.40GHz computer.

5.2 Dome Structure with 52bars

As Figure 6 shows, the dome structure with 52 bars and 21 nodes is investigated for its variance of repeated eigenvalues under the perturbations of random variables. The nominal value of cross-section areas of the bars is assigned as $10\,\mathrm{cm}^2$. The dome is symmetric with respect to *Y*-axis and *X*-axis. Besides, the nominal values of Young's modulus, material density and Poisson ratio are assigned as $E=210\,\mathrm{GPa}$, $\rho=7850\ \mathrm{kg/m}^3$, $\mu=0.3$ respectively. The representative nodes of the dome structure are listed in Table for their coordinates.

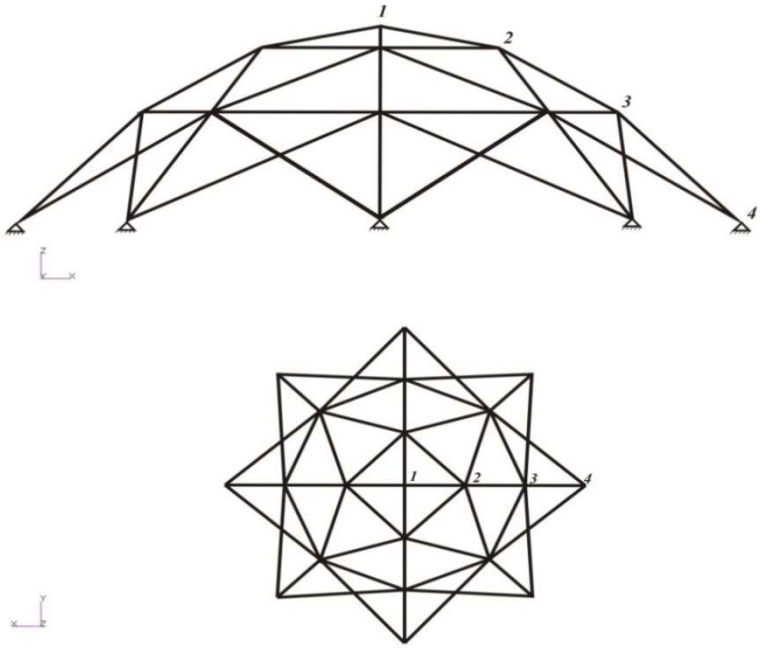
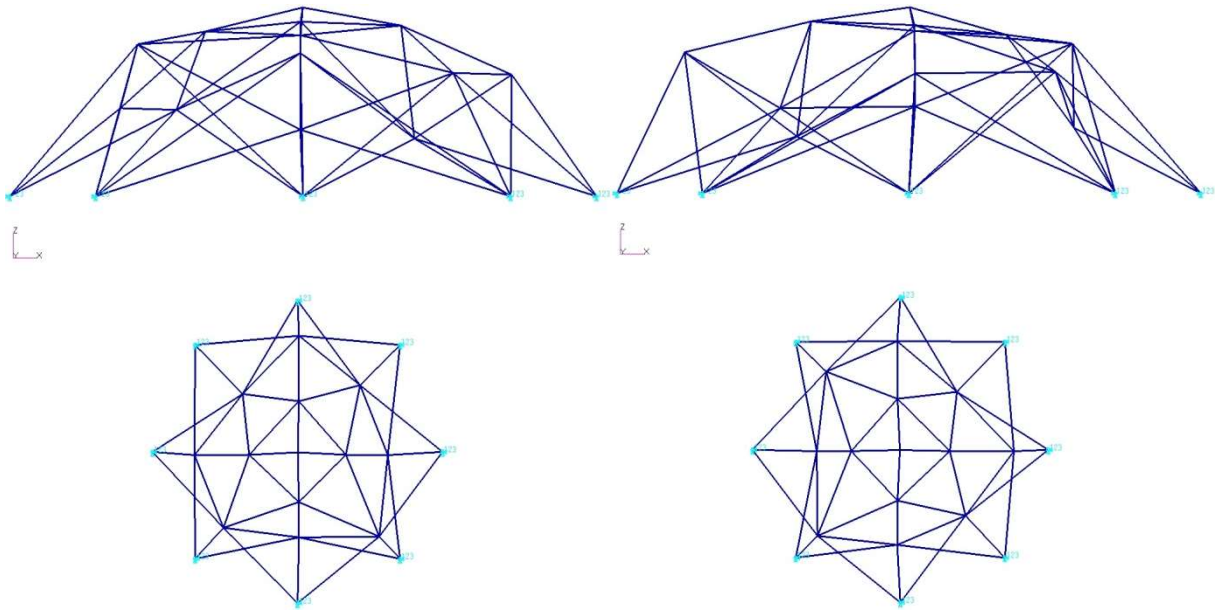


Figure 6 – Dome structure with repeated eigenvalues

Table 4 – Coordinates of the representative nodes in dome structure

Node	Coordinates (*m*)		
	X	Y	Z
1	0.0	0.0	9.25
2	5.0	0.0	8.22
3	10.0	0.0	5.14
4	15.0	0.0	0.0

It is worth mentioning the second-order and third-order vibration modes share the same natural frequency of 24.043Hz. The corresponding mode shapes are shown in Figure 7.



(a) second-order mode **(b) third-order mode**

Figure 7 – The second/third order mode shapes share the same natural frequency

In this numerical example, Young's modulus E, material density ρ are treated as random variables, which are subject to the Gaussian distributions $N(210,10.5^2)$, $N(7850,392.5^2)$ respectively. By implementing the whole procedure that was proposed in this paper, the mean value and variance of the first repeated eigenvalues λ_2 are obtained in Table .

Table 5 – Comparison of the outcomes calculated by two methods

First Repeated Eigenvalues	Methods	Mean Value	Relative Error	Variance	Relative Error
λ_2	The Proposed Method	22883	0.27%	2695507.24	4.76%
	Monte-Carlo Sample Size 10^5	22946		2830133.29	

Aiming at engineering problem such as this dome structure, computational accuracy of the proposed method is satisfactory. Furthermore, by simply collocating finite several points in the original system, the statistical quantities of repeated eigenvalues can be obtained by this proposed method. As for the MCS, on the contrary, in order to obtain the statistic quantities of the repeated eigenvalues, the full process of the FEM analysis has to be implemented for each and every sampling point, hence the total time cost is extremely tremendous. In Table , the calculative time consumption for MCS and the proposed method are compared, and the new method possesses an overwhelming advantage on computational efficiency. Both the proposed method and MCS were operated on an Intel Core i7-2600@3.40GHz computer in this case. In addition, with the increase of sample size, outcomes determined via the MCS tend to be stable. Particularly, once the sample size is larger than 1×10^5, the results evaluated by MCS are basically converged.

Table 6 – Contrast of calculative time consumption for two methods

Methods		Calculative Time Consumption (s)
		λ_2
Monte-Carlo	1×10^5	4689

simulation (Sample Size)	2×10^5	9837
	3×10^5	16041
The Proposed Method		3.67

5.3 High Altitude Airship Structure

In order to validate the proposed method in dealing with the large and complex engineering structure, a practical high altitude airship structure is utilized as the numerical study object. This airship is 16.5m in length and has a slenderness ratio of 3.9, as shown in Figure 8 and Figure 9. 2D membrane element is used in numerical simulation, which is implemented on MSC.Patran. We use the isotropic material with the nominal values of Young's modulus, material density and Poisson ratio are assigned as $E=1096\,\mathrm{MPa}$, $\rho=727\,\mathrm{kg/m^3}$, $\mu=0.3$ respectively.

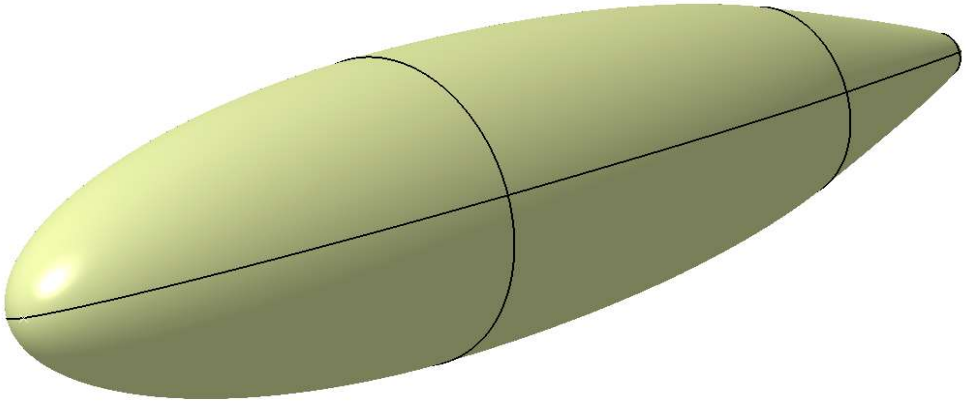


Figure 8 – Geometric model of the high altitude airship structure

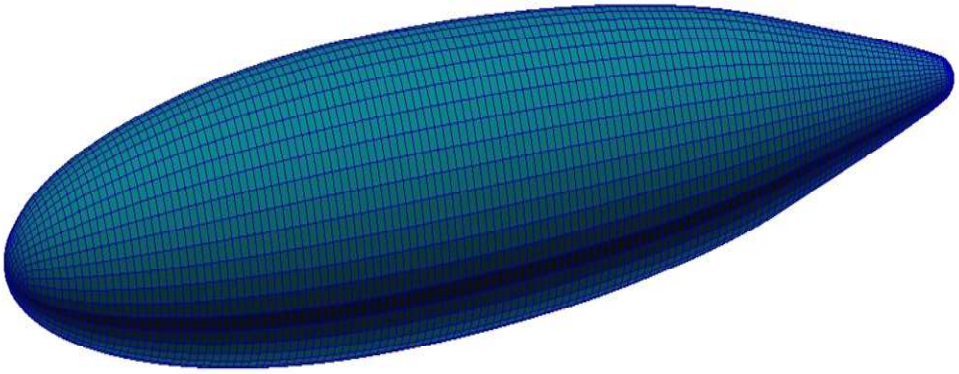
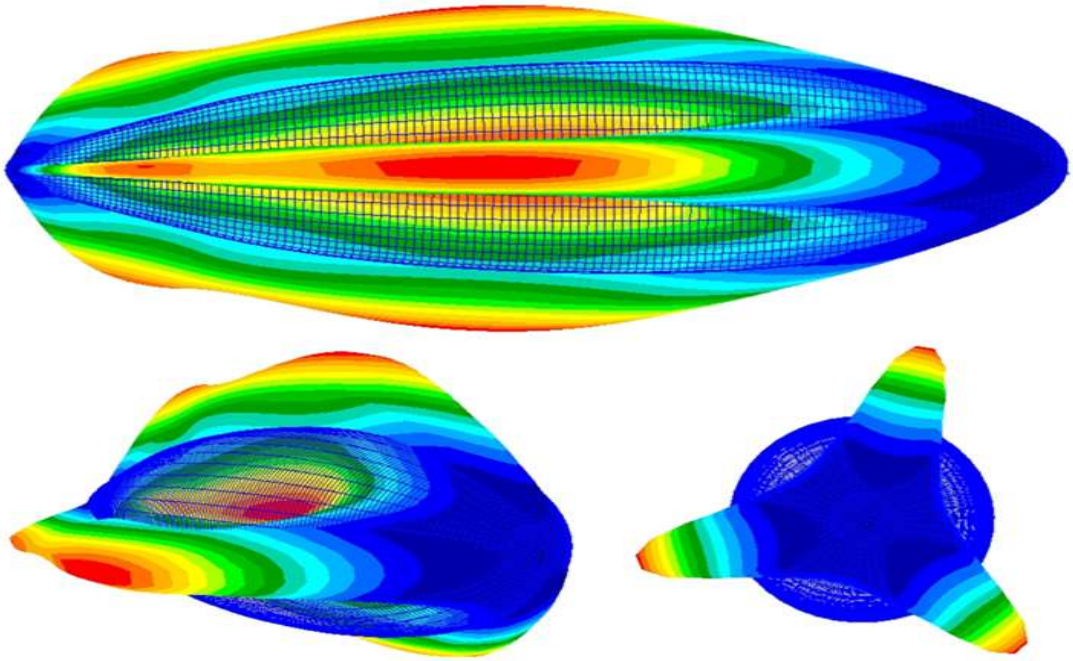


Figure 9 – FEM model of the high altitude airship structure

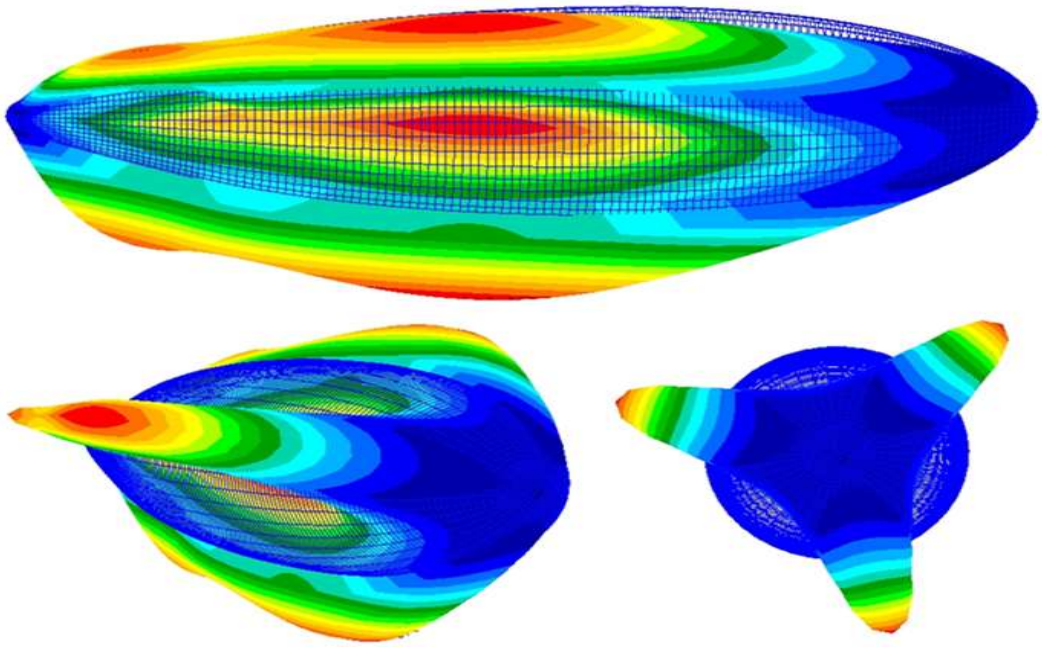
Table 7 – Repeated eigenvalues captured in the airship structure

Order	Eigenvalue	Natural Frequency
6th 7th	3844.8	9.8687Hz

The prestressed mode analysis is implemented, in which the internal pressure of the airship is set as 500Pa. As can be seen in Figure 10 and Table for details, clustered eigenvalues of the airship structure are acquired, in which 6th and 7th order mode shapes share the natural frequency of 9.8687Hz.



(a) 6th-order mode shape with natural frequency of 9.8687Hz



(b) 7th-order mode shape with natural frequency of 9.8687Hz

Figure 10 – Mode shapes of repeated eigenvalues of the airship structure

Firstly, the Young's modulus E and material density ρ are handled as random variables, which are subject to Gaussian distributions $\mathrm{N}(1096, 54.8^2)$ and $\mathrm{N}(727, 36.35^2)$ respectively. Then, by implementing the proposed method, mean value and variance of the sixth-order eigenvalue λ_6 and seventh-order eigenvalue λ_7 are obtained in Table .

Table 8 – Comparison of calculative outcomes by two methods in numerical case 3

Repeated Eigenvalues	Methods	Mean Value	Relative Error	Variance	Relative Error

λ_6	The Proposed Method	3844.95	0.041%	56655.71	3.79%
	Monte-Carlo Sample Size 5×10^6	3846.53		54507.77	
λ_7	The Proposed Method	3854.35	0.052%	52585.09	2.79%
	Monte-Carlo Sample Size 5×10^6	3856.35		51118.68	

According to Table , under the circumstance that all the coefficient of variation of random variables are assigned as 0.05, the statistic quantities of the original repeated eigenvalues (i.e. λ_7 and λ_6) calculated by the proposed method are very close to the exact solutions, which has roughly disclosed the computational feasibility and accuracy of the proposed method.

After the preceding discussion, we have investigated the outcomes that are computed concerning the random variables with fixed coefficient of variation (i.e. 0.05). In order to ulteriorly study the computational capacity of this proposed method, now we take different coefficient of variation into account. Hereinafter, for all the random variables, the coefficient of variation varies from 0 to 0.1 step by step simultaneously.

By adopting both the proposed method and MCS, the variance of repeated eigenvalues ($\mathrm{Var}(\lambda_6)$ and $\mathrm{Var}(\lambda_7)$) with respect to different coefficient of variation, are demonstrated and contrasted in Figure 11. As Figure 11 states, variance of repeated eigenvalues determined by the proposed method increases along with the increase of coefficient of variation. Outcomes of the MCS have the similar variation trends as that of the proposed method, and are relatively smaller in value than that of the proposed method (both for $\mathrm{Var}(\lambda_6)$ and $\mathrm{Var}(\lambda_7)$). As shown in Figure 11, when the coefficient of variation is small (less than 0.05), results of the proposed method are basically consistent with those of MCS; when the coefficient of variation approaches 0.1, deviation of the results between these two methods increases gradually. Particularly, as for the situation when coefficient of variation is zero in value, which means structural parameters (E and ρ) are all deterministic without uncertainties, results obtained by the previous two methods converge to zero.

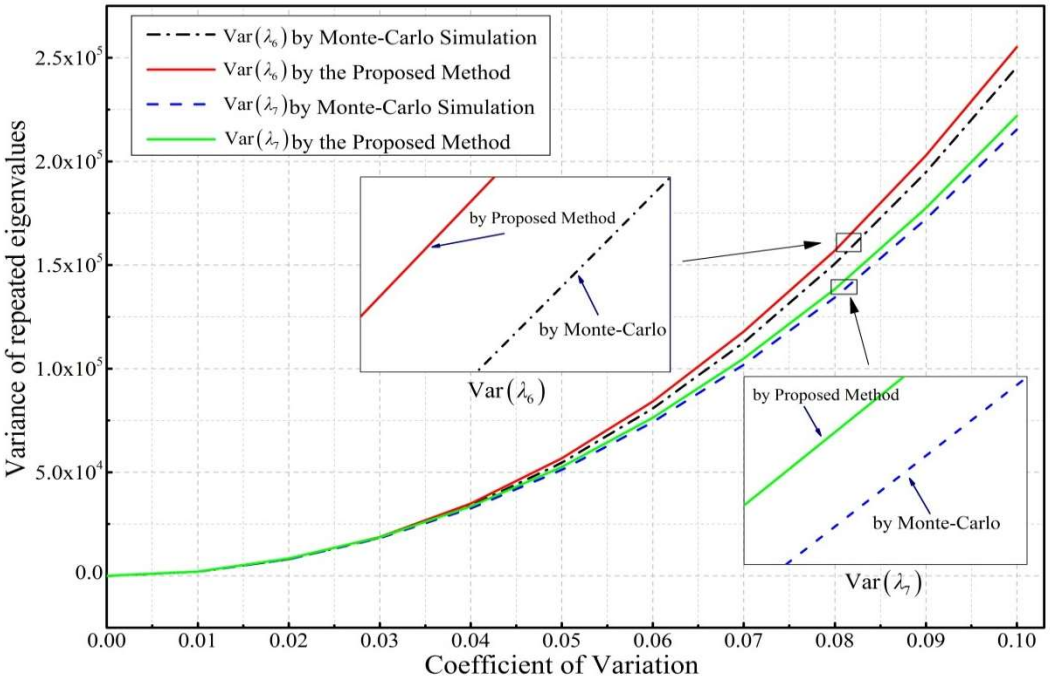


Figure 11 – The tendencies of variance of repeated eigenvalues with respect to different coefficient of variation calculated by the proposed method and Monte-Carlo simulation

For better clarification, we extract $\mathrm{Var}(\lambda_6)$ and $\mathrm{Var}(\lambda_7)$ acquired by the previous two methods to form the Figure 12 and Figure 13 respectively. Actually, the similar conclusions would be achieved as those for the Figure 11.

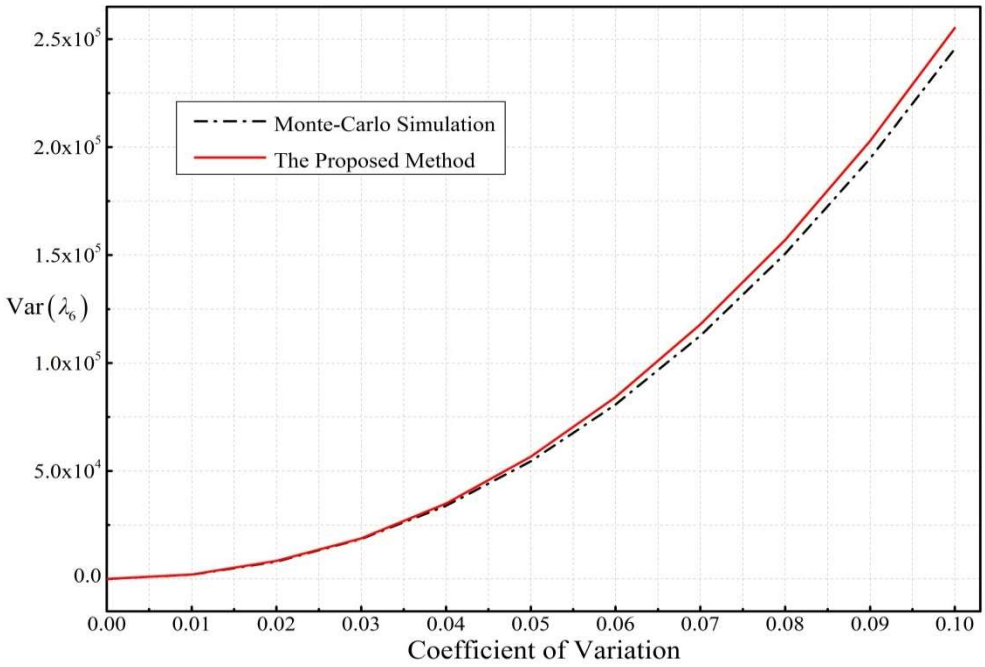


Figure 12 – $\mathrm{Var}(\lambda_6)$ calculated by two different methods

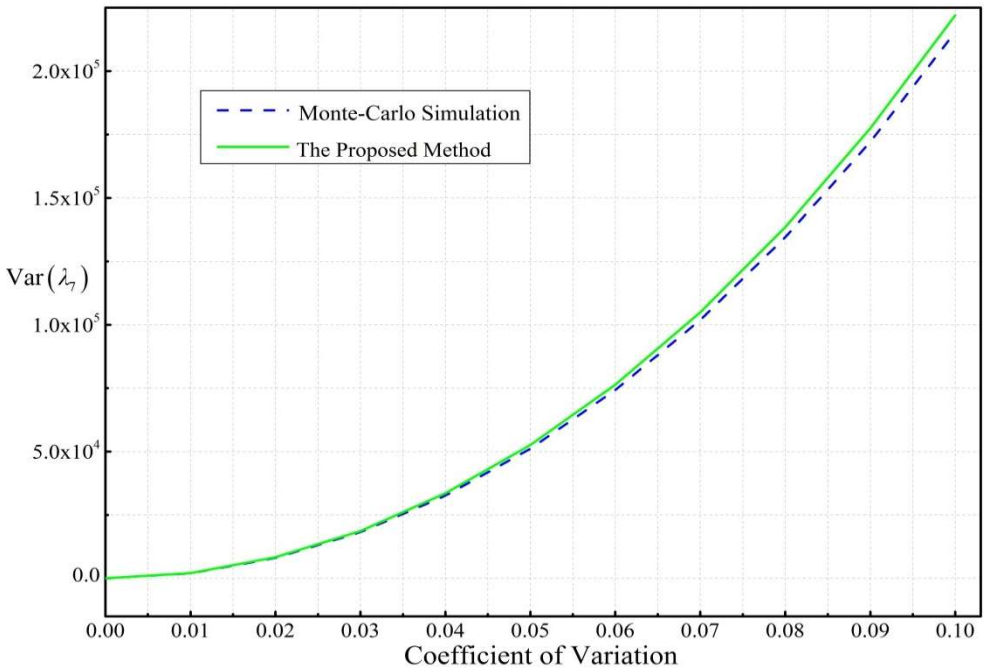


Figure 13 – $\mathrm{Var}(\lambda_7)$ **calculated by two different methods**

In terms of Figure 11, Figure 12 and Figure 13, considering the deviation in results between these two methods comprehensively, the main reasons are summed up as follows:

Firstly, the first-order perturbation method is utilized during the deducing process, so the higher-order perturbation terms of the repeated eigenvalue are omitted, which cause some deviation in computational results; Secondly, surrogate model constructed by polynomial chaos expansions represents merely an approximation of the original system, in this paper we truncate the Hermite orthogonal polynomials to second-order terms, which also lead to some deviation from the exact solution.

Note that even if the coefficient of variation is assigned as 0.1, relative error between the proposed method and MCS is just approximately 4% (for $\mathrm{Var}(\lambda_6)$). But the calculative time consumption for the proposed method shrinks at least by two orders of magnitude comparing with MCS. Due to the fact that time consumption of MCS is absolutely unaffordable in practical engineering. If we are willing to enhance the efficiency of structural reanalysis and to acquire accurate statistical quantities of perturbed repeated eigenvalue, the proposed method is a very good tradeoff between calculative accuracy and efficiency.

It should be clarified complementarily that the sample size of the MCS in this case is chosen as 3×10^5. To determine the sample size for MCS, the same procedure is adopted as we have used in case 1. Moreover, both of the two methods utilized in this case are implemented on an Intel Xeon(R) CPU E5-2680 v3@2.5GHz(dual) computer with RAM 128GB.

According to the preceding three numerical examples, the significance and effects of uncertain design variables on the statistical quantities of repeated eigenvalues have been completely realized and investigated. Besides, the new method has comprehensively demonstrated its computational capability in efficiency and accuracy. Especially when handling the structures undergoing large uncertainties, the outcomes of proposed method still meet the demand of engineering application. Furthermore, based on the second and third numerical cases, this proposed method has disclosed its application potential on two aspects. The first aspect is that the secondary development of commercial CAE/FEM software based on this proposed method is expedient to be achieved; the second aspect is this method has the ability to deal with large and complex engineering structures.

6. Conclusions

This paper is oriented to studying the uncertainty and its propagation analysis for the structures with repeated eigenvalues. Combining the stochastic perturbation method with surrogate model approach, an improved stochastic perturbation algorithm is developed in this paper. In light of this present method, statistical quantities of the perturbed repeated eigenvalue, such as variance and mean value, can be acquired conveniently and accurately. Numerical results have indicated that the proposed method guarantees the calculative precision and enhances the computational efficiency during the uncertainty propagation analysis. Furthermore, through numerical examples, this new method also shows the ability in handling the structures with relatively large uncertainties, which used to be a bottleneck for conventional perturbation method. Accordingly, the proposed method also paves the way for dynamic reanalysis and design optimization for large and complex engineering structures with repeated eigenvalues.

Acknowledgement

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Appendix

1. Construction of Surrogate Model Based on PCE Method

Essentially speaking, it is the uncertainty of structural parameters, such as statistical variability of material property and geometry condition, that leads to the perturbation of mass matrix $\boldsymbol{M}_1$ and perturbation of stiffness matrix $\boldsymbol{K}_1$, i.e.,

$$\boldsymbol{M}_1 = \boldsymbol{M}_1(\boldsymbol{\xi}) = \boldsymbol{M}_1(\xi_1, \xi_2, \cdots, \xi_n), \quad \boldsymbol{K}_1 = \boldsymbol{K}_1(\boldsymbol{\xi}) = \boldsymbol{K}_1(\xi_1, \xi_2, \cdots, \xi_n), \tag{1}$$

where $\boldsymbol{\xi} = \xi_1, \xi_2, \cdots, \xi_n$ denote the random variables in system. It is worth mentioning that, for notational convenience and avoiding ambiguity in derivation, we will use superscript q standing for the multiplicity of repeated eigenvalues' first-order perturbation. Assuming the original system as

$$\lambda_1^q = f(\boldsymbol{M}_1, \boldsymbol{K}_1) = f(\xi_1, \xi_2, \cdots, \xi_n). \tag{2}$$

Let the probability space of original system be smooth adequately, in addition $f(\xi_1, \xi_2, \cdots, \xi_n)$ cannot be expressed explicitly. But it's available to obtain the corresponding responses of original system by means of numerical approximation.

The polynomial chaos expansion (PCE) method is introduced in this paper. In another word, the surrogate model of original system is established by PCE method hereinafter.

Homogeneous Hermite polynomial expansion is employed in terms of Gaussian random process [21, 22], and the random response $Y(\boldsymbol{\xi})$ is expressed as

$$Y(\boldsymbol{\xi}) = c_0 + \sum_{i_1=1}^{n} c_{i_1} H_1(\xi_{i_1}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1} c_{i_1 i_2} H_2(\xi_{i_1}, \xi_{i_2}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} c_{i_1 i_2 i_3} H_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) + \cdots, \tag{3}$$

in which $\boldsymbol{c} = \left(c_0, c_{i_1}, \cdots, c_{i_1 i_2}, \cdots, c_{i_1 i_2 i_3}, \cdots\right)$ is a vector of undetermined coefficients, $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ denotes the Gaussian random variables, $H_n(\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_n})$ stands for the n th order multidimensional Hermite polynomial.

Eq.(3) is legitimate to be truncated by finite terms, assuming the remaining number of terms is s, then eq.(3) will be simplified as

$$Y(\boldsymbol{\xi}) \approx \sum_{j=0}^{s-1} \hat{\boldsymbol{c}}_j \Gamma_j(\boldsymbol{\xi}), \tag{4}$$

where $\hat{\boldsymbol{c}}_j$ stands for the undetermined coefficients, $\Gamma_j(\boldsymbol{\xi}) = \Gamma_j(\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_n})$ denotes the j th order generalized Wiener-Askey polynomial chaos. Simultaneously, there is a one-to-one correspondence between the functions $H_n(\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_n})$ and $\Gamma_j(\boldsymbol{\xi})$, and their coefficients $c_0, c_{i_1}, \cdots, c_{i_1 i_2}, \cdots, c_{i_1 i_2 i_3}, \cdots$ and $\hat{\boldsymbol{c}}_j$.

For clarity, the n-dimensional Hermite polynomial is shown as

$$H_n(\xi_{i_1}, \cdots, \xi_{i_n}) = e^{1/2\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}} (-1)^n \frac{\partial^n}{\partial\xi_{i_1} \cdots \partial\xi_{i_n}} e^{-1/2\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}},$$

and one-dimensional Hermite polynomials are demonstrated as

$$H_0 = 1,\ H_1(\xi) = \xi,\ H_2(\xi) = \xi^2 - 1,\ H_3(\xi) = \xi^3 - 3\xi, \cdots$$
$$H_{i+1}(\xi) = \xi H_i(\xi) - i H_{i-1}(\xi), \cdots$$

The polynomial chaos $\Gamma_j(\xi)$ forms a complete orthogonal basis in the L_2 space of the

Gaussian random variables, i.e.,

$$\left\langle \Gamma_i, \Gamma_j \right\rangle = \left\langle \Gamma_i^2 \right\rangle \delta_{ij},$$

where δ_{ij} is the Kronecker Delta operator and $\langle \cdot, \cdot \rangle$ denotes the ensemble average which is also the inner product in the Hilbert space of Gaussian random variables $\boldsymbol{\xi}$.

$$\left\langle f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) \right\rangle = \int f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) W(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

where the weighting function is

$$W(\boldsymbol{\xi}) = \frac{1}{\sqrt{(2\pi)^n}} e^{-1/2 \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{\xi}},$$

in this way, a complete set of orthogonal basis which is square-integrable is established with respect to n-dimensional Hermite polynomial. Moreover, the polynomials in eq.(3) are convergence in mean square.

If the surrogate model is formed by second order Hermite polynomial chaos expansion, then the random response $Y(\boldsymbol{\xi})$ can be expressed as

$$Y_2(\boldsymbol{\xi}) = c_{0,2} + \sum_{i=1}^{n} c_{i,2} \xi_i + \sum_{i=1}^{n} c_{ii,2} \left(\xi_i^2 - 1 \right) + \sum_{i=1}^{n-1} \sum_{j>i}^{n} c_{ij,2} \xi_i \xi_j, \tag{5}$$

where n denotes the dimensionality of random variables, $c_{0,2}$, $c_{i,2}$, $c_{ii,2}$, $c_{ij,2}$ stand for the undetermined coefficients of Hermite polynomial expansion. By virtue of eq.(5), one can induce the number of undetermined coefficients in second order Hermite polynomial expansion for approximate random response $Y_2(\boldsymbol{\xi})$ as $s = (n+p)!/(n!p!)$, in which p stands for the highest order in polynomial expansion.

Note that in eq.(5), the key of constructing the surrogate model, which is formed by PCE method, is the calculation of undetermined coefficients $c_{0,2}$, $c_{i,2}$, $c_{ii,2}$, $c_{ij,2}, \cdots$. Concerning the PCE surrogate model which consists of n-dimensional random variables, the corresponding response functions will be generated upon the specified collocation points groups, and each collocation points group corresponds to a set of sample $\tilde{\boldsymbol{\xi}} = \left(\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_n \right)$, which belongs to random variable $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$.

As mentioned before, all the variables in this paper are subject to Gaussian distribution. Besides that, Hermite polynomials are chosen as the orthogonal basis. It is a routine that if the highest order of Hermite polynomial is p, then roots of the $(p+1)$th order Hermite polynomial are adopted as collocation points. Specifically, the highest order of Hermite polynomial utilized in this paper is set as two. So the roots of $H_3(x) = \left(1/\sqrt{6}\right) \times \left(x^3 - 3x\right)$, which are $x = \pm\sqrt{3}, 0$, should be chosen as the collocation points for every random variable.

After preparing all the collocation points, one substitutes the previous collocation points into the original system one-by-one. The corresponding response function will be generated sequently, and the following formula is established as

$$\begin{bmatrix} \Gamma_0(\boldsymbol{\xi}_0) & \Gamma_1(\boldsymbol{\xi}_0) & \cdots & \Gamma_{s-1}(\boldsymbol{\xi}_0) \\ \Gamma_0(\boldsymbol{\xi}_1) & \Gamma_1(\boldsymbol{\xi}_1) & \cdots & \Gamma_{s-1}(\boldsymbol{\xi}_1) \\ \vdots & \vdots & \cdots & \vdots \\ \Gamma_0(\boldsymbol{\xi}_N) & \Gamma_1(\boldsymbol{\xi}_N) & \cdots & \Gamma_{s-1}(\boldsymbol{\xi}_N) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{s-1} \end{bmatrix} = \begin{bmatrix} f(\boldsymbol{\xi}_0) \\ f(\boldsymbol{\xi}_1) \\ \vdots \\ f(\boldsymbol{\xi}_N) \end{bmatrix}, \tag{6}$$

where $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_N$ denote the sampling points, N stands for the number of sampling points, s is the number of undetermined coefficients. By means of regression analysis based on the

least squares method, the undetermined coefficients of polynomial chaos expansion could be calculated from eq.(6).

2. Construction of $\boldsymbol{M}_1, \boldsymbol{K}_1$ in Section 3

According to the stochastic perturbation method, which is truncated to first-order perturbation term in this paper, the system response $g(\boldsymbol{x}) = g(x_1, x_2, \cdots, x_n)$ can be expressed as

$$g(\boldsymbol{x}) = g(\bar{\boldsymbol{x}}) + g_1(\boldsymbol{x}), \tag{1}$$

viz. $$g(x_1, x_2, \cdots, x_n) = g(\mu_1, \mu_2, \cdots, \mu_n) + g_1(x_1, x_2, \cdots, x_n),$$

where $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$ denotes the random variables in structural system, $\bar{\boldsymbol{x}} = [\mu_1, \mu_2, \cdots, \mu_n]$ stands for the mean values of random variables, $g_1(\boldsymbol{x})$ is the first-order perturbation term of system response.

On the other aspect, in the light of high dimensional model representation (HDMR), the first-order approximation of system response also can be denoted as

$$g(\boldsymbol{x}) \equiv g(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} g(\mu_1, \cdots, \mu_{i-1}, x_i, \mu_{i+1}, \cdots, \mu_n) - (n-1) g(\bar{\boldsymbol{x}}), \tag{2}$$

Taking eqs.(1) and (2) into consideration, the first-order perturbation term $g_1(\boldsymbol{x})$ is obtained as

$$\begin{aligned} g_1(\boldsymbol{x}) &= \sum_{i=1}^{n} g(\mu_1, \cdots, \mu_{i-1}, x_i, \mu_{i+1}, \cdots, \mu_n) - n g(\bar{\boldsymbol{x}}) \\ &= \sum_{i=1}^{n} \left[g(\mu_1, \cdots, \mu_{i-1}, x_i, \mu_{i+1}, \cdots, \mu_n) - g(\bar{\boldsymbol{x}}) \right] \end{aligned}, \tag{3}$$

on the basis of the preceding derivation, the first-order perturbation term of mass matrix and stiffness matrix can be expressed as follows, respectively

$$\begin{cases} \boldsymbol{M}_1 = \boldsymbol{M}_{\beta 1} = \sum_{i=1}^{n} \left[\boldsymbol{M}(\mu_1, \mu_2, \cdots, \mu_{i-1}, \xi_{\beta i}, \mu_{i+1}, \cdots \mu_n) - \boldsymbol{M}(\mu_1, \mu_2, \cdots, \mu_{i-1}, \mu_i, \mu_{i+1}, \cdots, \mu_n) \right] \\ \boldsymbol{K}_1 = \boldsymbol{K}_{\beta 1} = \sum_{i=1}^{n} \left[\boldsymbol{K}(\mu_1, \mu_2, \cdots, \mu_{i-1}, \xi_{\beta i}, \mu_{i+1}, \cdots \mu_n) - \boldsymbol{K}(\mu_1, \mu_2, \cdots, \mu_{i-1}, \mu_i, \mu_{i+1}, \cdots, \mu_n) \right] \end{cases}.$$