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An Alternative but Equivalent Realization of Finite-Horizon Suboptimal Control for Vehicle Landing

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Abstract

This paper is mainly a tutorial and experimental research on the finite-horizon state-dependent Riccati equation method, in which a differential Riccati equation arises instead of an algebraic one. With this method, the finite-horizon sub-optimal control law may be obtained for the non-linear system with affine inputs. As the foundation, the analytic solution of the differential Riccati equation, derived in the finite-horizon linear quadratic regulator problem, is reviewed. The solution has two presentations while only one is numerically applicable. For comparison, an alternative but equivalent way of direct numerical integration is considered. A reusable launch vehicle landing example is solved to demonstrate the good performance and high efficiency in both ways. Furthermore, three approaches to improve the performance of the solution are investigated, including reformulating the linear-like form, the reduced-horizon control strategy, and the iterative control strategy. It is shown that they have the potential to generate a better solution.

Keywords: optimal control; non-linear affine system; state-dependent Riccati equation method; analytic solution of differential Riccati equation

1. Introduction

Optimal control theory aims at determining the inputs to a dynamic system, which optimize a specified performance index while satisfying constraints on the motion of the system. It is closely related to the engineering and has been widely studied. Because of the complexity, except the linear quadratic Optimal Control Problems (OCPs) [1] and some special cases [2], generally OCPs for nonlinear systems are solved numerically. The numerical methods, often categorized as the direct and indirect methods [3], are prosperously developed and their efficiency has been greatly enhanced. For example, the Pseudo-spectral (PS) method is widely studied in recent decades. It possesses good properties including insensitiveness to the initial guess and exponential convergence rate [4-6]. Notably some studies were carried out for its practical application [7-10] or even on-line optimal control [11-13]. Recently, a new Variation Evolving Method (VEM) is proposed for the optimal control computation [14, 15], and due to its on-line approximation to the optimal solution, it may provide a possible way for the real-time optimal control.

A different strategy for the optimal control, which may bring a closed-form solution, is the State-dependent Riccati Equation (SDRE) method [16, 17]. Upon the quadratic performance index, Cloutier *et al.* proposed this method for the infinite-horizon optimal control of non-linear affine systems. A state-dependent algebraic Riccati equation is solved to give the approximate optimal solution and properties including the optimality, the stability, and the robustness are investigated [18]. Compared with the infinite-horizon case, the finite-horizon optimal control of nonlinear systems is challenging due to the time dependency of the associated Hamilton–Jacobi–Bellman (HJB) partial differential equation. Heydari and Balakrishnan [19] introduced the state-dependent differential Riccati equation for the finite-horizon optimal control problem, and presented an approximate closed-form solution, which is the analytic solution for the time-invariant case. Note that the solution has two forms while one form is not practically usable, because the involved matrix inverse is easy to become singular for problems with large horizon. This will lead to the oscillation of control commands or even failure to obtain a solution.

In this paper, the finite-horizon suboptimal control for the non-linear input-affine system, following the SDRE principle, is studied. Since the differential Riccati equation is stable in reverse-time [20], an alternative way of direct integration, upon the efficient numerical computation capacity, is investigated. We will not systematically investigate the SDRE control law under various uncertainties and disturbances here, whereas just show its ability to generate the control command rapidly and investigate strategies to improve the performance of the solution. For this aim, an example of Reusable Launch Vehicle (RLV) landing path planning with ideal model is considered. The rest of this paper is organized as follows. In Section 2, the finite-horizon SDRE controller for the non-linear affine plant is developed. Section 3 applies it to the RLV landing problem. The vehicle landing scenario, with the approximate analytic method and the numerical integration method respectively, is demonstrated by the closed-loop simulation in Section 4. In Section 5, three approaches, including reformulating the linear-like form, the reduced-horizon control strategy, and the iterative control strategy, are investigated to improve the performance of the control. The remarks are presented at the end.

2. Development of SDRE Controller

2.1 Preliminaries of Finite-Horizon LQR Problem

The finite-horizon Linear Quadratic Regulator (LQR) problem is reviewed first. Such problem has the following quadratic performance index as

$$J = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(t_f)\boldsymbol{F}\boldsymbol{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{u})\mathrm{d}t \tag{1}$$

subject to the linear dynamic equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{2}$$

with initial conditions

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{3}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector and $\boldsymbol{u} \in \mathbb{R}^m$ is the control vector. $\boldsymbol{Q}$ and $\boldsymbol{F}$ are right-dimensional positive semi-definite matrixes and $\boldsymbol{R}$ is a right-dimensional positive definite matrix. The initial time t_0 and the terminal time t_f are all fixed. "T" denotes the transpose operator. Note that the coefficient matrixes $\boldsymbol{A}$, $\boldsymbol{B}$ and the weight matrixes $\boldsymbol{Q}$, $\boldsymbol{R}$ may be time-varying. According to the optimal control theory, this problem has an optimal control solution as follows.

$$\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} \tag{4}$$

where $\boldsymbol{P}$ is the Riccati matrix that conforms to the differential equation

$$-\dot{\boldsymbol{P}} = \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} \tag{5}$$

and

$$\boldsymbol{P}(t_f) = \boldsymbol{F} \tag{6}$$

For the time-invariant case where the coefficient matrixes and the weight matrixes are all constant, the differential Riccati equation remains the form. In particular, the Riccati matrix for the time-invariant case may have a solution in explicit analytic form under the following assumption.

Assumption 1: $(\boldsymbol{A},\boldsymbol{B})$ is stabilizable and $(\boldsymbol{A},\boldsymbol{Q}^{1/2})$ has no unobservable modes on the imaginary axis, where $\boldsymbol{Q} = \boldsymbol{Q}^{1/2}\boldsymbol{Q}^{1/2}$.

Theorem 1 [21]: For the differential Riccati equation (5), where the matrixes $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$ are time-invariant, then under Assumption 1, the analytic solution of the Riccati matrix $\boldsymbol{P}(t)$ is

$$\boldsymbol{P}(t) = \left[e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})(t-t_f)}\left((\boldsymbol{F}-\boldsymbol{P}_{ss})^{-1} - \boldsymbol{D}\right)e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}(t-t_f)} + \boldsymbol{D}\right]^{-1} + \boldsymbol{P}_{ss} \tag{7}$$

where $\boldsymbol{P}_{ss}$ is the solution of the following algebraic Riccati equation

$$\boldsymbol{P}_{ss}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{ss} - \boldsymbol{P}_{ss}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss} + \boldsymbol{Q} = \boldsymbol{0} \tag{8}$$

and $\boldsymbol{D}$ satisfies the following Lyapunov equation

$$(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})\boldsymbol{D}+\boldsymbol{D}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}=\boldsymbol{0} \tag{9}$$

Proof. At first, Assumption 1 guarantees the existence of $\boldsymbol{P}_{ss}$ for the algebraic Riccati equation (8) [22]. Furthermore, all the eigenvalues of $\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss}$ will either fall in the open left-half complex plane or the open right-half complex plane, which implies that the Lyapunov equation (9) has a unique solution [22]. Now the proof is direct. Obviously when $t=t_f$, we have $\boldsymbol{P}(t_f)=\boldsymbol{F}$ from Eq. (7). On the other hand, with the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{M}^{-1}(t)=-\boldsymbol{M}^{-1}\frac{\mathrm{d}\boldsymbol{M}(t)}{\mathrm{d}t}\boldsymbol{M}^{-1} \tag{10}$$

where $\boldsymbol{M}$ is a matrix function of t, Eq. (7) may be differentiated to give

$$\begin{aligned}-\dot{\boldsymbol{P}}&=(\boldsymbol{P}-\boldsymbol{P}_{ss})\begin{pmatrix}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})(t-t_f)}\left((\boldsymbol{F}-\boldsymbol{P}_{ss})^{-1}-\boldsymbol{D}\right)e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}(t-t_f)}\\+e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})(t-t_f)}\left((\boldsymbol{F}-\boldsymbol{P}_{ss})^{-1}-\boldsymbol{D}\right)e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}(t-t_f)}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}\end{pmatrix}(\boldsymbol{P}-\boldsymbol{P}_{ss})\\&=(\boldsymbol{P}-\boldsymbol{P}_{ss})\left((\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})\left((\boldsymbol{P}-\boldsymbol{P}_{ss})^{-1}-\boldsymbol{D}\right)+\left((\boldsymbol{P}-\boldsymbol{P}_{ss})^{-1}-\boldsymbol{D}\right)(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}\right)(\boldsymbol{P}-\boldsymbol{P}_{ss})\end{aligned} \tag{11}$$

Expanding the terms in the right-hand of Eq. (11) renders

$$\begin{aligned}-\dot{\boldsymbol{P}}&=(\boldsymbol{P}-\boldsymbol{P}_{ss})(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})+(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}(\boldsymbol{P}-\boldsymbol{P}_{ss})\\&\quad-(\boldsymbol{P}-\boldsymbol{P}_{ss})\left((\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})\boldsymbol{D}+\boldsymbol{D}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}\right)(\boldsymbol{P}-\boldsymbol{P}_{ss})\end{aligned} \tag{12}$$

with Eq. (9), there is

$$-\dot{\boldsymbol{P}}=(\boldsymbol{P}-\boldsymbol{P}_{ss})(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})+(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss})^{\mathrm{T}}(\boldsymbol{P}-\boldsymbol{P}_{ss})-(\boldsymbol{P}-\boldsymbol{P}_{ss})\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{P}-\boldsymbol{P}_{ss}) \tag{13}$$

Simplify Eq. (13) and add a zero term $\boldsymbol{Q}-\boldsymbol{Q}=\boldsymbol{0}$ in the right-hand side; we have

$$-\dot{\boldsymbol{P}}=\boldsymbol{P}\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}-\boldsymbol{P}_{ss}\boldsymbol{A}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{ss}+\boldsymbol{P}_{ss}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss}-\boldsymbol{Q} \tag{14}$$

With Eq. (8), we then see that $\boldsymbol{P}(t)$ given by Eq. (7) satisfies the differential Riccati equation (5). □

Note that there are two solutions for the algebraic Riccati equation (8); one is positive semi-definite, denoted by $\boldsymbol{P}_{ss}^{+}$, and the other is negative semi-definite, denoted by $\boldsymbol{P}_{ss}^{-}$. It may be verified that $\boldsymbol{P}_{ss}^{-}=-\boldsymbol{P}_t$, where $\boldsymbol{P}_t$ is the positive semi-definite solution of the following algebraic Riccati equation

$$-\boldsymbol{P}_t\boldsymbol{A}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_t-\boldsymbol{P}_t\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_t+\boldsymbol{Q}=\boldsymbol{0} \tag{15}$$

The solution given by Eq. (7) will not be changed for either $\boldsymbol{P}_{ss}^{+}$ or $\boldsymbol{P}_{ss}^{-}$ theoretically. However, the numerical computation results may be quite different in practice. With $\boldsymbol{P}_{ss}^{+}$, the matrix $\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss}^{+}$ will have all its eigenvalues in the open left-half complex plane, and the term $e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss}^{+})(t-t_f)}$ will be fairly large when the horizon t_f-t is large. This may lead to the numerical singularity when calculating the matrix inverse. On the contrary, with $\boldsymbol{P}_{ss}^{-}$, the matrix $\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss}^{-}$ will have all its eigenvalues in the open right-half complex plane, and the term $e^{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_{ss}^{-})(t-t_f)}$ will be within a reasonable range despite the value of t_f-t.

2.2 Suboptimal Controller for Non-Linear Affine System

The finite-horizon optimal control for general non-linear input-affine systems with quadratic performance index is now formulated. The system considered takes the following form

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} \tag{16}$$

where $\boldsymbol{x}\in\mathbb{R}^n$ and $\boldsymbol{u}\in\mathbb{R}^m$ are the state and control vectors, respectively. The functions $\boldsymbol{f}:\mathbb{R}^n\to\mathbb{R}^n$ and $\boldsymbol{g}:\mathbb{R}^n\to\mathbb{R}^{n\times m}$ represent the continuous dynamics of the system. The performance index is given by Eq. (1). Note that here the matrixes $\boldsymbol{Q}$ and $\boldsymbol{R}$ may be state-dependent, i.e., $\boldsymbol{Q}(\boldsymbol{x})$ and $\boldsymbol{R}(\boldsymbol{x})$. The problem is to find the controller that minimizes the cost functional (1) subject to the state equation given by Eq. (16).

To use the SDRE method, the dynamic equation (16) is transformed to the linear-like form, where the coefficient matrixes are state-dependent, that is

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} \tag{17}$$

where $\boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$. The factorization of the dynamics to create $\boldsymbol{A}(\boldsymbol{x})$ is not unique, and techniques suggested in [23] may be employed. Note that the linear-like form should satisfy the following assumption.

Assumption 2: $\left(\boldsymbol{A}(\boldsymbol{x}), \boldsymbol{B}(\boldsymbol{x})\right)$ is pointwise stabilizable and $\left(\boldsymbol{A}(\boldsymbol{x}), \boldsymbol{Q}^{1/2}(\boldsymbol{x})\right)$ always has no unobservable modes on the imaginary axis for all $\boldsymbol{x} \in \mathbb{R}^n$, where $\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{Q}^{1/2}(\boldsymbol{x})\boldsymbol{Q}^{1/2}(\boldsymbol{x})$.

Following the principle of the SDRE method, the state-dependent differential Riccati equation for the finite-horizon problem is

$$-\dot{\boldsymbol{P}}(\boldsymbol{x},t) = \boldsymbol{P}(\boldsymbol{x},t)\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x},t) - \boldsymbol{P}(\boldsymbol{x},t)\boldsymbol{B}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x},t) + \boldsymbol{Q}(\boldsymbol{x}) \tag{18}$$

with the terminal condition

$$\boldsymbol{P}(\boldsymbol{x},t_f) = \boldsymbol{F} \tag{19}$$

where $\dot{\boldsymbol{P}}(\boldsymbol{x},t) = \dfrac{\partial \boldsymbol{P}(\boldsymbol{x},t)}{\partial t} + \dfrac{\partial \boldsymbol{P}(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ is the total time derivative of the Riccati matrix $\boldsymbol{P}(\boldsymbol{x},t)$. Then the control is analogously calculated as

$$\boldsymbol{u} = -\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x},t)\boldsymbol{x} \tag{20}$$

In Ref. [19], Heydari and Balakrishnan analyzed the relation between the proposed solution and the optimal solution, showing that when neglecting certain terms, the HJB equation reduces to the state-dependent differential Riccati equation (18). They gave an approximate solution, according to Eq. (7), to solve Eq. (18), namely, freezing the state $\boldsymbol{x}$ in the coefficient and weight matrixes at the current time point. Then

$$\boldsymbol{P}(\boldsymbol{x},t) \approx \left[e^{\left(\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})\boldsymbol{R}(\boldsymbol{x})^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_{ss}\right)(t-t_f)} \left((\boldsymbol{F} - \boldsymbol{P}_{ss})^{-1} - \boldsymbol{D}\right) e^{\left(\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_{ss}\right)^{\mathrm{T}}(t-t_f)} + \boldsymbol{D} \right]^{-1} + \boldsymbol{P}_{ss} \tag{21}$$

where $\boldsymbol{P}_{ss}$ is the solution of the following algebraic Riccati equation

$$\boldsymbol{P}_{ss}\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_{ss} - \boldsymbol{P}_{ss}\boldsymbol{B}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_{ss} + \boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{0} \tag{22}$$

and $\boldsymbol{D}$ satisfies the following Lyapunov equation

$$\left(\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_{ss}\right)\boldsymbol{D} + \boldsymbol{D}\left(\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_{ss}\right)^{\mathrm{T}} - \boldsymbol{B}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x}) = \boldsymbol{0} \tag{23}$$

Since the analytic form solution involves the matrix inverse computation, $\boldsymbol{P}_{ss}$ should be negative semi-definite in order to avoid the numerical difficulty. In particular, the differential Riccati equation (5) is inverse-time stable [20]. Therefore, an alternative way of directly integrating the differential equation with the coefficient matrixes fixed, upon the efficient numerical computation capacity, may also produce an accurate solution efficiently. To get $\boldsymbol{P}$, introduce $\tau = t_f - t$; we then have the following Initial-value Problem (IVP)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P} + \boldsymbol{Q}(\boldsymbol{x}),\ \boldsymbol{P}\big|_{\tau=0} = \boldsymbol{F} \tag{24}$$

Note that the coefficient and weight matrixes are fixed at the current time point in Eq. (24), and the IVP will produce the same results as Eq. (21).

3. SDRE Controller for Vehicle Landing

Landing is a crucial flight phase for vehicles. The goal during this process is for the vehicle to land at a desired runway in a limited downrange with a near-zero vertical velocity [24]. The dynamics of the vehicle landing are non-linear and traditional guidance methods rely on the predetermined reference trajectories to be followed, while with the SDRE method, the desired reference trajectory is not required because the control law may achieve the closed-loop landing control with tolerable touchdown vertical velocity. Assuming zero cross range to the runway, the dynamic model for a RLV in the vertical plane is [19]

$$\dot{V} = -\frac{D}{m} - g\sin\gamma \tag{25}$$

$$\dot{\gamma} = \frac{L}{mV} - \frac{g}{V}\cos\gamma \tag{26}$$

$$\dot{h} = V\sin\gamma \tag{27}$$

$$\dot{s} = V\cos\gamma \tag{28}$$

where V is the velocity magnitude, γ is the flight-path angle, h is the height, and s is the downrange. m is the mass of the aircraft. g is the gravity acceleration. α is the angle-of-attack. $L = QS_aC_L$ is the aerodynamic lift and $D = QS_aC_D$ is the aerodynamic drag. The lift and drag coefficients are modeled as $C_L = C_{L_0}\alpha$ and $C_D = C_{D_0} + K_I C_L{}^2$, respectively. $Q = \frac{1}{2}\rho V^2$ is the dynamic pressure and S_a is the reference area. The air density is calculated by $\rho = \rho_0 \exp(-h/H)$, where ρ_0 is the sea-level air density and H is the scale height. Note that the system is not affine in the control α. To obtain an affine form, the time derivative of α is introduced as the new control input, i.e.,

$$\dot{\alpha} = u \tag{29}$$

Since a predetermined and fixed downrange is more interesting in landing [19], the dynamic equations (25)-(29) are changed with the downrange s as the independent variable. Denoting the derivative of a variable with respect to s by the prime notation ‘'’, then there are

$$V' \equiv \frac{\mathrm{d}V}{\mathrm{d}s} = \frac{1}{V\cos\gamma}\left(-\frac{D}{m} - g\sin\gamma\right) \tag{30}$$

$$\gamma' \equiv \frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{1}{V\cos\gamma}\left(\frac{L}{mV} - \frac{g}{V}\cos\gamma\right) \tag{31}$$

$$h' \equiv \frac{\mathrm{d}h}{\mathrm{d}s} = \tan\gamma \tag{32}$$

$$t' \equiv \frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{V\cos\gamma} \tag{33}$$

$$\alpha' \equiv \frac{\mathrm{d}\alpha}{\mathrm{d}s} = \frac{u}{V\cos\gamma} \tag{34}$$

Using the SDRE method, the suboptimal controller for a quadratic performance index may be designed. The performance index is

$$J = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(s_f)\boldsymbol{F}\boldsymbol{x}(s_f) + \frac{1}{2}\int_{s_0}^{s_f}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + Ru^2)\mathrm{d}s \tag{35}$$

where the state vector is $\boldsymbol{x} = \begin{bmatrix} V & \gamma & h & t & \alpha \end{bmatrix}^{\mathrm{T}}$, s_0 and s_f denote the initial downrange and the prescribed terminal downrange, respectively. $\boldsymbol{F}$, $\boldsymbol{Q}$ and R are appropriate weight matrixes. Referring to the transformation in Ref. [19], we have the linear-like equation (17) for the vehicle landing dynamic model where

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix}, \boldsymbol{B}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{V\cos\gamma} \end{bmatrix} \tag{36}$$

The specific entries for $\boldsymbol{A}(\boldsymbol{x})$ are

$$A_{11}=\frac{-\rho_0\exp(-x_3/H)S_a(C_{D_0}+K_I C_{L_0}^2 x_5^2)}{4m\cos x_2}-\frac{d_1\rho_0 S_a C_{D_0}}{4m\cos x_2}$$

$$A_{12}=\frac{-g\sin x_2}{x_1 x_2\cos x_2}$$

$$A_{13}=\frac{-\rho_0 S_a C_{D_0} x_1(\exp(-x_3/H)-d_1)}{4m x_3\cos x_2}$$

$$A_{14}=0$$

$$A_{15}=\frac{-\rho_0\exp(-x_3/H)S_a K_I C_{L_0}^2 x_1 x_5}{4m\cos x_2}$$

$$A_{21}=-\frac{g}{x_1^3},A_{22}=0,A_{23}=\frac{\rho_0(\exp(-x_3/H)-d_2)SC_{L_0}x_5}{4m x_3\cos x_2}$$

$$A_{24}=0,A_{25}=\frac{\rho_0\exp(-x_3/H)SC_{L_0}}{4m\cos x_2}+\frac{d_2\rho_0 SC_{L_0}}{4m\cos x_2}$$

$$A_{31}=0,A_{32}=\frac{\sin(x_2)}{\cos(x_2)x_2},A_{33}=0,A_{34}=0,A_{35}=0$$

$$A_{41}=\frac{1}{x_1^2\cos(x_2)},A_{42}=0,A_{43}=0,A_{44}=0,A_{45}=0$$

where d_1 and d_2 are the hyper-parameters that regulate the linear-like form. Then we get the control law

$$\boldsymbol{u}=-R^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}(s)\boldsymbol{x} \tag{37}$$

upon the following equation

$$-\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{P}=\boldsymbol{P}\boldsymbol{A}(\boldsymbol{x})+\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}(\boldsymbol{x})R^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}+\boldsymbol{Q},\ \boldsymbol{P}(s_f)=\boldsymbol{F} \tag{38}$$

Fixing the coefficient and weight matrixes at current time point, $\boldsymbol{P}(s)$ may be solved following the approximate analytic method, i.e., using Eqs. (21)-(23), or alternatively, it may be obtained numerically by solving the IVP transformed from Eq. (38).

4. Simulation Results

An RLV landing example with a downrange horizon of 20000ft from [19] is considered. We use flight simulation with the closed-loop suboptimal SDRE control to generate the landing trajectory that has the least vertical velocity and flight-path angle. The following values are used for the RLV parameters: C_{L_0}=2.3, C_{D_0}=0.0975, K_I= 0.1819, S/m=0.912 ft^2 / slug. The sea-level air density is ρ_0=0.0027 slugs / ft^3 and the scale height is H=2.7887×10^{4}ft. The hyper-parameters are d_1=1 and d_2=1. In the flight simulations, the ordinary differential equation integrator “ode45”, with a relative error tolerance of 1×10^{-6} and an absolute error tolerance of 1×10^{-9}, was employed to solve the IVP regarding the Riccati matrix.

Two cases with different weight matrixes in the performance index are investigated. In Case 1, the weight matrixes were selected as R=10, $\boldsymbol{Q}$=diag(0, 0.0365, 1×10^{-9}, 2.5×10^{-13}, 1.6414), and $\boldsymbol{F}$ =diag(0, 7.2951×10^7, 1, 0, 0). The simulations with the approximate analytic method and the numerical integration method were carried out. As shown in Fig. 1, they both achieve the desired target and their solutions are the same. We test the computation time for the control command generation. This analytic form solution is efficient and the maximum consumption is only 0.0018 s. For the numerical integration method, the maximum consumption is 0.0636 s, small as well, and one may expect the time further reduced on a specialized onboard computer. In addition, we checked the approximate analytic solution with the positive semi-definite matrix $\boldsymbol{P}_{ss}^{+}$. It is shown that now the approximate analytic method underperforms in that the control commands are coarse, which arises from the inverse singularity. This may be avoided with the negative semi-definite matrix $\boldsymbol{P}_{ss}^{-}$ in employing the method. See Fig. 2.

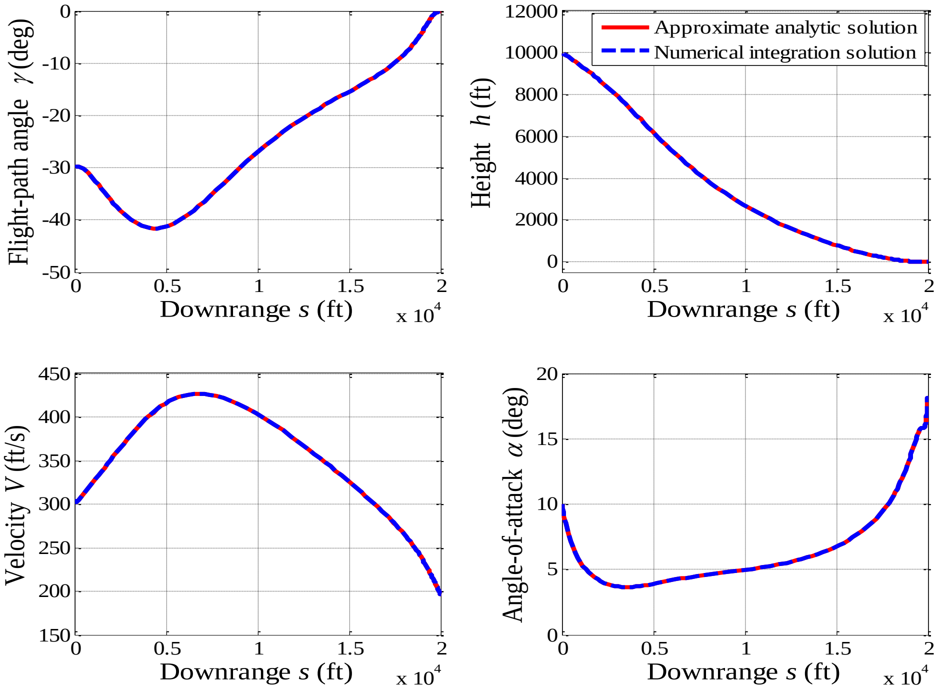

Figure 1 – State solutions of RLV landing with the approximate analytic method and the numerical integration method for Case 1.

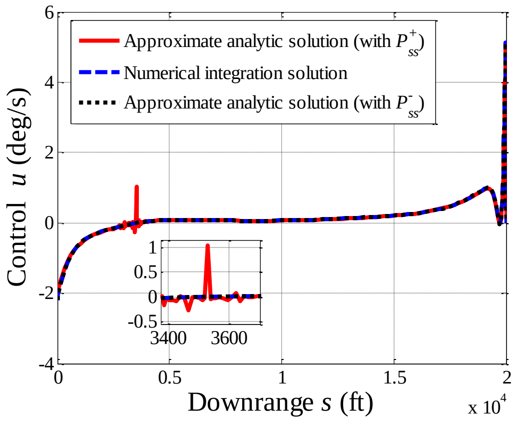

Figure 2 – Control solutions of RLV landing with the approximate analytic method and the numerical integration method for Case 1.

In Case 2, the weight matrix $\boldsymbol{Q}$ is slightly adjusted as $\boldsymbol{Q}$ =diag(0, 0.0365, 1×10^{-9}, 2.5×10^{-13}, 3.2828) in comparison with Case 1. It is found that with the positive semi-definite matrix $\boldsymbol{P}_{ss}^{+}$, the approximate analytic method fails to give the solution, while the right solution may be obtained with the negative semi-definite matrix $\boldsymbol{P}_{ss}^{-}$ or the numerical integration method.

Furthermore, we compared the suboptimal landing trajectory with the optimal trajectory planned off-line with the PS method for Case 1. As shown in Fig. 3, it is found that the suboptimal control also well realizes the control objective and the performance, measured by Eq. (35), is only larger by 8.9% than that of the optimal solution, even if the curves are different.

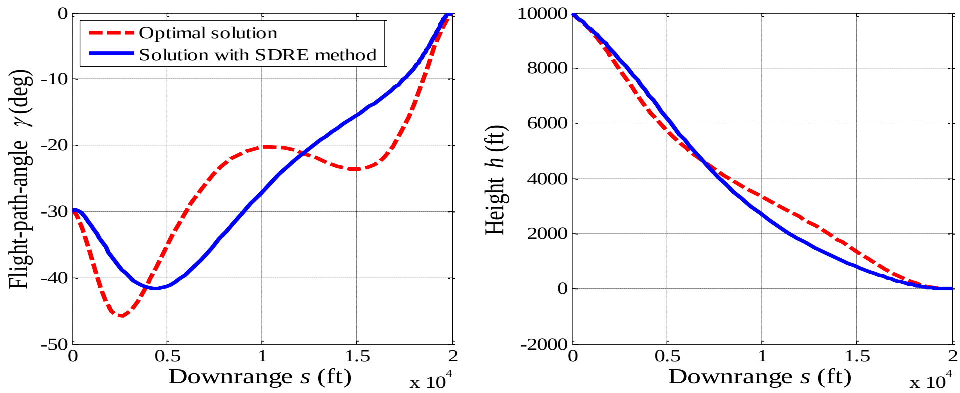

Figure 3 – Comparison between the optimal solution and the suboptimal solution of Case 1.

5. Further Investigation

In order to improve the performance of the control solution, three approaches are further investigated.

5.1 Reformulating the Linear-Like Form

Obviously, different linear-like forms will produce different results. In Section 3, we introduce two hyper-parameters, i.e., d_1 and d_2, to change the concrete form of $\boldsymbol{A}(\boldsymbol{x})$. To find whether there is better choices of d_1 and d_2 than those in last section. We investigate a series of values for d_1 and d_2, each of which varies from 0 to 2 with an interval of 0.2. Therefore, 121 simulations, under the setting of Case 1, were carried out. It is found that 56 sets of parameters suffer numerical difficulty and fail to give a solution due to the violation of Assumption 2, but we did find some better choice of the parameters, and they may produce solution closer to the optimal (with a perforce index of 523.4793). Table 1 gives the performance of the representative solutions among these.

Table 1 – Representative solutions under different hyper-parameter settings.

Value of hyper-parameters	Performance index of solution
d_1=0.8, d_2=0.8	547.3863
d_1=1, d_2=1	569.9775
d_1=0, d_2=0	952.9076

5.2 Reduced-Horizon Control Strategy

The SDRE method assumes a constant matrixes of $\boldsymbol{A}$ and $\boldsymbol{B}$ for the whole time-to-go, and one way that may promote the method is to reduce the length of the horizon. Rearrange the performance index (35) as

$$J = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(s_f)\boldsymbol{F}\boldsymbol{x}(s_f) + \frac{1}{2}\int_{s_0+\Delta s}^{s_f}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + Ru^2)\mathrm{d}s + \frac{1}{2}\int_{s_0}^{s_0+\Delta s}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + Ru^2)\mathrm{d}s \tag{39}$$

where Δs is the horizon parameter. According the Bellman principle, denote the optimal performance index as $J^*(\boldsymbol{x}(s),s)$; then without changing the optimal solution, Eq. (39) may be reformulated as

$$J = J^*(\boldsymbol{x}(s_0+\Delta s), s_0+\Delta s) + \frac{1}{2}\int_{s_0}^{s_0+\Delta s}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + Ru^2)\mathrm{d}s \tag{40}$$

However, we cannot obtain the analytic expression of $J^*(\boldsymbol{x}(s),s)$ and we have to find a substitute way. Assuming that the optimal performance may be expressed as

$$J^*\left(\boldsymbol{x}(s),s\right) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{P}\left(\boldsymbol{x}(s),s\right)\boldsymbol{x}(s) \tag{41}$$

It was shown in Ref. [19] that neglecting certain terms, the matrix $\boldsymbol{P}$ satisfies the differential Riccati equation (38). Heuristically, we may plan a path with the approximate analytic method to obtain $\boldsymbol{x}(s)$, and then compute $\boldsymbol{P}(s)$ along $[s_0, s_f]$ upon Eq. (38). Consider the reduced horizon $[s, s+\Delta s]$, the control law may be calculated as

$$\boldsymbol{u} = -R^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_r(\boldsymbol{x},s)\boldsymbol{x} \tag{42}$$

where $\boldsymbol{P}_r(\boldsymbol{x},s)$ is computed by

$$\boldsymbol{P}_r(\boldsymbol{x},s) = \left[e^{-\left(\boldsymbol{A}(\boldsymbol{x})-\boldsymbol{B}(\boldsymbol{x})R^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_{ss}\right)\Delta s} \left((\boldsymbol{P}(s+\Delta s) - \boldsymbol{P}_{ss})^{-1} - \boldsymbol{D} \right) e^{-\left(\boldsymbol{A}(\boldsymbol{x})-\boldsymbol{B}(\boldsymbol{x})R^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}_{ss}\right)^{\mathrm{T}}\Delta s} + \boldsymbol{D} \right]^{-1} + \boldsymbol{P}_{ss} \tag{43}$$

$\boldsymbol{P}_{ss}$ is the solution of the algebraic Riccati equation (22) and $\boldsymbol{D}$ is the solution of the Lyapunov equation (23).

To test the strategy, a simulation case, called Case 3, with same initial conditions as Case 1 is carried out. The weight matrixes were R=10, $\boldsymbol{Q}$ =diag(0, 0.1824, 1×10^{-10}, 2.5×10^{-11}, 0.3283), and $\boldsymbol{F}$ =diag(0, 7.2951×10^{7}, 0.2, 0, 0). The horizon span Δs was set as $\Delta s = 10$. Figures 4 and 5 plot the solution under the reduced-horizon control strategy, whose performance index is 483.52. In comparison, the approximate analytic solution (with a performance index of 483.60) and the optimal solution (with a performance index of 474.26) are also presented. With the reduced-horizon control strategy, the performance of the solution is improved but the improvement is small. We also test other settings of Δs, with the values of 1, 50, and 100 respectively. It is found that the solutions are nearly the same. However, when the reduced-horizon control strategy is applied to Case 1, this approach fails to give a solution. About the reason, it is found that the velocity of the vehicle approaches zero near the terminal downrange and hence the numerical singularity occurs.

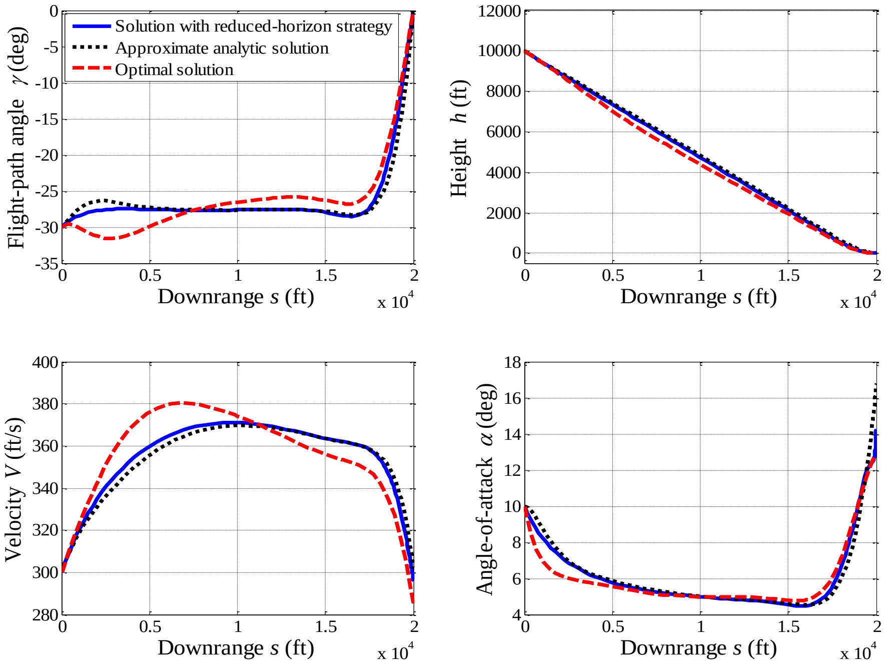

Figure 4 – State solutions of RLV landing with the reduced-horizon control strategy for Case 3.

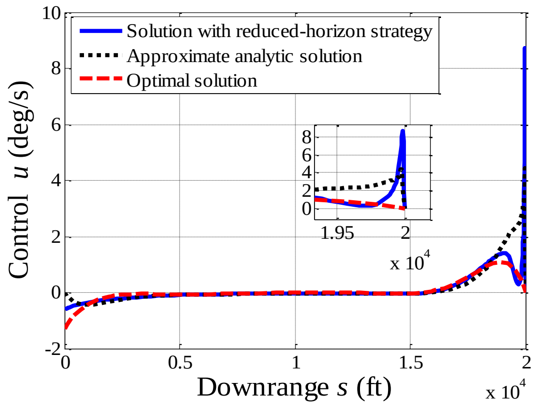

Figure 5 – Control solutions of RLV landing with the reduced-horizon control strategy for Case 3.

5.3 Iterative Control Strategy

By investigating the differential equation in Eq. (38), one may conjecture that if the prior information about $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ along the horizon $[s_0, s_f]$ is available, then this IVP may produce a better solution of $\boldsymbol{P}$. However, such information is impossible to obtain before the solution is determined. Fortunately, if an iterative strategy is employed, it may provide a prediction of $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$. In particular, if the iteration may improve the solution monotonically, it may be boldly supposed that the resulting solutions will converge to the optimal with more iterations. Follow this idea; we first use the approximate analytic method to generate an initial solution, and then use the numerical integration method to compute the control command based on the downrange-varying differential Ricccati equation. A simulation for Case 3 was carried out with such strategy. Figures 6 and 7 plot the solution under the iterative control strategy, the approximate analytic solution and the optimal solution. It is shown that the solution under the iterative strategy is closer to the optimal solution. It has a performance index of 475.44 and the index does improve apparently. However, we also find that this strategy is not generally applicable. For Case 1, it fails to give a solution, because the velocity at the terminal stage becomes too low and the numerical singularity again occurs during this course.

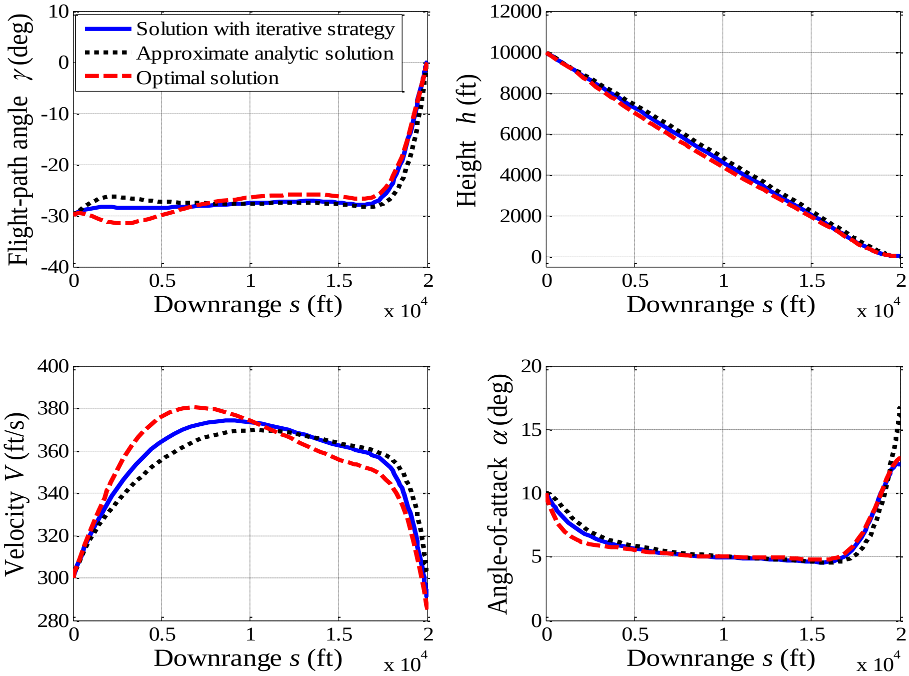

Figure 6 – State solutions of RLV landing with the iterative control strategy for Case 3.

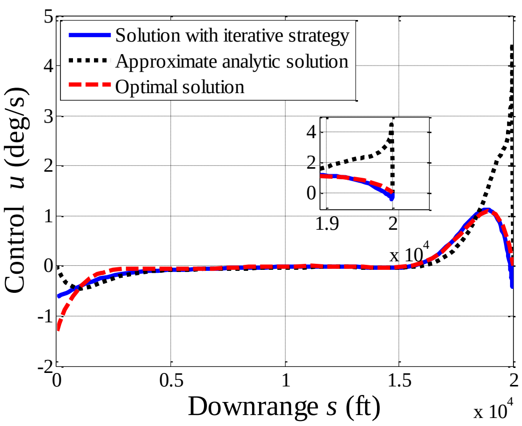

Figure 7 – Control solutions of RLV landing with the iterative control strategy for Case 3.

6. Final Remarks

This paper studies the finite-horizon suboptimal control for the non-linear affine system with the Satedependent Riccati Equation (SDRE) method, in which a differential Riccati equation emerges instead of an algebraic one. Different from the approximate analytic method, which employs the analytic solution for the Linear Quadratic Regulator (LQR) problem, an alternative way of direct numerical integration is proposed for the control command generation, which avoids the inverse computation. Through a reusable launch vehicle landing example, it is found that the integration method may produce the control command rapidly. Furthermore, three approaches to improve the performance of the solution are investigated, including reformulating the linear-like form, the reduced-horizon control strategy, and the iterative control strategy. It is shown that they may generate a better solution.

The title for the paper may be not very appropriate. Actually, after our initial work was accepted by the conference committee in the form of abstract, our understanding towards the SDRE method was gradually deepened during the research. We went on to study some motivating ideas to improve this approach, for example, the reduced-horizon control strategy and the iterative control strategy. However, due to the programme of the conference, we did not alter the title ultimately.

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9. Copyright Statement

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