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KINEMATIC MODEL OF REFINED THEORY AND EXACT ANALYTICAL SOLUTIONS OF STATIC AND BUCKLING PROBLEMS OF STRUCTURALLY-ANISOTROPIC COMPOSITE PANELS OF AIRCRAFT

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Abstract

New design problem – design to cost – can be solved by combining exact models and modern computer technologies. Aircraft composite structure design in the field of production technology is the outlook research trend. New mathematical model relations for the stress-strain state investigation of structurally-anisotropic panels comprising composite materials are presented in this study. The mathematical model of a stiffening rib being torsioned under one-sided contact with the skin is refined. The primary scientific novelty of this research is the further development of the theory of thin-walled elastic ribs. The resolved equation with an eight order differential operator and natural boundary conditions are obtained with the variation procedure of Lagrange. Exact analytical solutions for edge problems are considered according to the general treatment of physical boundary conditions for structure components. The solution in closed form is designed by a single trigonometric series. For arbitrary boundary conditions at the contour the method of uniform solutions is employed. The new method algorithm for solutions as a series with a non-orthogonal system of general proper functions and analytically defined constants is given. The mathematical model relations for the pre-critical stress state investigation of structurally-anisotropic panels made of composite materials are presented. Furthermore, the mathematical model relations for the buckling problem investigation of structurally-anisotropic panels made of composite materials are presented in view of the pre-critical stressed state. The critical force definition of the general bending form of the thin-walled system buckling and the critical force definition of the many-waved torsion buckling are of the most interest in accordance with traditional design practices. In both cases, bending is integral with the plane stress state. Thus, the buckling problem results in the boundary value problem when solving for the eighth order partial derivative equation in the rectangular field. A computer program package is developed using the MATLAB operating environment. The computer program package has been utilized for multi-criteria optimization of the design of structurally-anisotropic aircraft composite panels. The influence of the structure parameters on the level of stresses and displacements, of critical buckling forces for bending and for torsion modes has been analyzed. The results of testing series are presented.
Keywords: composite panels, stress-strain state, buckling

1. Introduction

This paper discusses the stress-strain state and buckling problems of flat rectangular multilayer panels made of composite materials, the casing of which is eccentrically supported by the longitudinal-lateral stiffening set. The panel is subjected to transverse loading arbitrarily distributed in the stationary temperature field and to distribute loading applied to the edges in casing plane. The boundary conditions at the contour are assumed to be general.

One considers the residual thermal stresses arising during cooling after hardening and the pre-stressed tension of the reinforcing fibers with respect to panel production technology.

The theory of thin-walled elastic ribs is used without the hypothesis of the zero shear deformation for the reinforcing elements which are in the complex resistance with respect to two-plane bending and limited torsion. The primary scientific novelty of this study is the further development of the theory of thin-walled elastic ribs related to the contact problem for the skin and rib with an improved rib model.

The schematization of the panel as structurally anisotropic has been proposed as a design model when the stress-strain state and the critical forces of total bending form of buckling were determined. For many-waved torsion buckling study one should use the generalized functions set.

The problem of determining stress-strain state of structurally anisotropic panels is reduced to the solution of the boundary value problem for an equation of the eighth order in the partial derivatives in a rectangular field. The eight-order differential equations are resolved for the pre-critical stress-strain state and for the buckling problem. The closed-form solution is designed by a unitary trigonometric series for the particular case of conformable boundary conditions on two opposite sides and by the method of uniform solutions for arbitrary non-conformable boundary conditions at the contour. One examines all possible variants of the boundary edge restrictions in relation to the connecting plane and bending problems.

A computer program package in MATLAB was performed for the multi-criteria optimization of the design of structurally-anisotropic composite panels of FA.

2. Main relations of mathematical model

Several studies (e.g. Adumitroaie and Barbero 2015; Auricchio et al. 2014; Boutin and Viverge 2016; Cerracchio et al. 2015; Celkovic 2015; Fernandes et al. 2010; Goodsell et al. 2013; Kant and Shiyekar 2013; Le et al. 2015; Li et al. 2015; Luan et al. 2011; Mantan and Guedes 2014; Nath and Afsar 2009; Petrolo and Lamberti 2016; Tran et al. 2015) [1, 2, 3, 6, 7, 9, 13, 15, 18, 19, 21, 22, 24, 25, 27] describe the static problems of structurally-anisotropic composite panels, but the following mathematical model is novel (e.g. Introduction). The classification and survey of the main directions of the stress-strain state theory development of the structurally-anisotropic composite panels demonstrates the need and scope for the refined theory paying due attention to the features of the composite panel deformation behavior.

The geometrical and physical relations are assumed at each *k*-ply on the macro-level for the skin according to the Kirchhoff hypothesis.

The stress-strain state components of each *k*-ply of composite stringers are calculated within the contact problem for the skin and rib with an improved rib model considering the plate and rib displacement and rotation angle equalities. The cross-section warping is considered free here. Shear strain occurring because of the thin-walled bar torsion is connected with its rotation relative to the chosen pole and also relative to the bending center in view of «pure» torsion theory additives (e.g. Boytsov and Gavva 2017) [4]

$$\left.\begin{aligned} u_1^{(k)}(x,s) &= u_0(x) - \frac{\partial w}{\partial x}(x) z^{(k)} - \frac{\partial v_0}{\partial x}(x) y^{(k)} + \left[u_4(x)\right]_1 \omega_1^{(k)}(s) \\ v_1^{(k)}(x,s) &= \qquad\qquad w(x) z'^{(k)} + v_0(x) y'^{(k)} - \frac{\partial w}{\partial y}(x) \rho_1^{(k)}(s) \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned}
\sigma_{x1}^{(k)} &= \bar{Q}_{11}^{(k)}\left[\frac{\partial u_0}{\partial x} - \frac{\partial^2 w}{\partial x^2}z^{(k)} - \frac{\partial^2 v_0}{\partial x^2}y^{(k)} + \frac{\partial\left(u_4\right)_1}{\partial x}\omega_1^{(k)} - \bar{\alpha}_1^{(k)}\Delta T - \bar{\varepsilon}_{H_1}^{(k)}\right] \\
\tau_{xy_1}^{(k)} &= \bar{Q}_{66}^{(k)}\left[-\frac{\partial^2 w}{\partial x \partial y}\left(\rho_1 + \rho_1^0\right)^{(k)} + \left(u_4\right)_1 \rho_{\omega_1}^{(k)} - \bar{\alpha}_6^{(k)}\Delta T - \bar{\varepsilon}_{H6}^{(k)}\right]
\end{aligned}\right\} \qquad (2)$$

The cross-section warping - $u_4(x)$, $\omega_1^{(k)}(s)$ is designed by the sector square epure for the open contour. $\rho_1^{(k)}(s)$, $\rho_{\omega_1}^{(k)}(s)$ - the perpendicular length from the contact point for the stringer and skin and from the bending center to the contour point, $\rho_1^0(s)$ is for «pure» torsion theory.

$\bar{Q}_{i,j}^{(k)}$, $i, j = 1, 2, 6$ – the k-ply stiffness, $\bar{\alpha}_j^{(k)}$, $j = 1, 2, 6$ – the temperature extension coefficients of each k-ply, $\bar{\varepsilon}_{Hj}^{(k)}$, $j = 1, 2, 6$ – the tension deformation of each k-ply, ΔT – the difference between the common temperature and hardening temperature for the residual thermal stress calculation or the external temperature field intensity.

The stress-strain state components of each k-ply of composite stiffness ribs along y axis are calculated with the analogue expressions.

The problem consists of finding the base surface displacements $u_0(x,y)$, $v_0(x,y)$, $w(x,y)$.

The ribs are considered operating in tension-compression, out-of-plane plate bending, in-plane plate bending and torsion.

The equilibrium equations relative to the five unknown functions, namely, the longitudinal and tangential displacements, the deflection, the axis displacements due to cross section warping, and corresponding natural boundary conditions are obtained with the variation procedure of Lagrange.

An equilibrium differential equation system is written in operator form with the linear differential operators. By using the symbol integration method, when four from five expressions are identically satisfied, the system can be reduced to a single eighteenth order linear non-uniform resolving partial differential equation for the potential function $\Phi(x,y)$, with the help of which all accounting values are expressed

The cross partial derivatives are of an eighteenth order, the derivatives by each coordinate are of twelfth order with respect to the linear differential operator of the resolved equation comprising the edge effects

The natural boundary conditions help determine internal force factor relations. Forces and moments both depend on the longitudinal and tangential skin plane displacement functions $u_0(x,y)$, $v_0(x,y)$, and on the deflection function $w(x,y)$. Thus**,** this shows that the problem is integral; in other words, it cannot be divided into a plane part and a bending part.

If the structurally-anisotropic composite panel structure derived as a result of energy averaging becomes orthotropic, the linear differential operator of the resolving equation and linear differential operators of the relations between symmetrical stress-strain state components and potential function $\Phi(x, \boldsymbol{y})$ will contain only even numbered derivatives with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$. The **o**blique symmetrical components are determined with odd numbered derivatives as follows:

$$L^{(18)}\Phi = q$$

$$
\begin{aligned}
L^{(18)} = & \frac{K_{108}}{a^{10}b^{8}}\frac{\partial^{18}}{\partial x^{10}\partial y^{8}} + \frac{K_{810}}{a^{8}b^{10}}\frac{\partial^{18}}{\partial x^{8}\partial y^{10}} + \frac{K_{124}}{a^{12}b^{4}}\frac{\partial^{16}}{\partial x^{12}\partial y^{4}} + \frac{K_{106}}{a^{10}b^{6}}\frac{\partial^{16}}{\partial x^{10}\partial y^{6}} + \frac{K_{88}}{a^{8}b^{8}}\frac{\partial^{16}}{\partial x^{8}\partial y^{8}} + \\
& + \frac{K_{610}}{a^{6}b^{10}}\frac{\partial^{16}}{\partial x^{6}\partial y^{10}} + \frac{K_{412}}{a^{4}b^{12}}\frac{\partial^{16}}{\partial x^{4}\partial y^{12}} + \frac{K_{122}}{a^{12}b^{2}}\frac{\partial^{14}}{\partial x^{12}\partial y^{2}} + \frac{K_{104}}{a^{10}b^{4}}\frac{\partial^{14}}{\partial x^{10}\partial y^{4}} + \frac{K_{86}}{a^{8}b^{6}}\frac{\partial^{14}}{\partial x^{8}\partial y^{6}} + \\
& + \frac{K_{68}}{a^{6}b^{8}}\frac{\partial^{14}}{\partial x^{6}\partial y^{8}} + \frac{K_{410}}{a^{4}b^{10}}\frac{\partial^{14}}{\partial x^{4}\partial y^{10}} + \frac{K_{212}}{a^{2}b^{12}}\frac{\partial^{14}}{\partial x^{2}\partial y^{12}} + \frac{K_{12}}{a^{12}}\frac{\partial^{12}}{\partial x^{12}} + \frac{K_{102}}{a^{10}b^{2}}\frac{\partial^{12}}{\partial x^{10}\partial y^{2}} + \\
& + \frac{K_{84}}{a^{8}b^{4}}\frac{\partial^{12}}{\partial x^{8}\partial y^{4}} + \frac{K_{66}}{a^{6}b^{6}}\frac{\partial^{12}}{\partial x^{6}\partial y^{6}} + \frac{K_{48}}{a^{4}b^{8}}\frac{\partial^{12}}{\partial x^{4}\partial y^{8}} + \frac{K_{210}}{a^{2}b^{10}}\frac{\partial^{12}}{\partial x^{2}\partial y^{10}} + \frac{K_{012}}{b^{12}}\frac{\partial^{12}}{\partial y^{12}} + \\
& + \frac{K_{100}}{a^{10}}\frac{\partial^{10}}{\partial x^{10}} + \frac{K_{82}}{a^{8}b^{2}}\frac{\partial^{10}}{\partial x^{8}\partial y^{2}} + \frac{K_{64}}{a^{6}b^{4}}\frac{\partial^{10}}{\partial x^{6}\partial y^{4}} + \frac{K_{46}}{a^{4}b^{6}}\frac{\partial^{10}}{\partial x^{4}\partial y^{6}} + \frac{K_{28}}{a^{2}b^{8}}\frac{\partial^{10}}{\partial x^{2}\partial y^{8}} + \\
& + \frac{K_{010}}{b^{10}}\frac{\partial^{10}}{\partial y^{10}} + \frac{K_{80}}{a^{8}}\frac{\partial^{8}}{\partial x^{8}} + \frac{K_{62}}{a^{6}b^{2}}\frac{\partial^{8}}{\partial x^{6}\partial y^{2}} + \frac{K_{44}}{a^{4}b^{4}}\frac{\partial^{8}}{\partial x^{4}\partial y^{4}} + \frac{K_{26}}{a^{2}b^{6}}\frac{\partial^{8}}{\partial x^{2}\partial y^{6}} + \frac{K_{08}}{b^{8}}\frac{\partial^{8}}{\partial y^{8}}
\end{aligned} \quad (3)
$$

The constants $K_{12-i,i}, i = 0, 2, 4, 6, ..., 12$– in (3) depend on geometrical and stiffness structure characteristics; *x = x/a, y = y/b* are– non-dimensional coordinates with reference to the panel half-length *a* and to its width *b*.

The resolved equation (3) comprises the main stress-strain state and edge effects.

The general boundary value problem in the rectangular field is specified by elastic fixity conditions

$$
\begin{aligned}
x = const \quad & (\gamma_1 u_0 + \delta_1 N_x) = (\gamma_2 v_0 + \delta_2 N_{xy}) = (\gamma_3 w + \delta_3 \bar{Q}_x) = (\gamma_4 w'_x + \delta_4 M_x) = \\
& (\gamma_5 w''_{xy} + \delta_5 C_x) = [\gamma_6 (u_4)_1 + \delta_6 B_{xy}] = 0 \\
y = const \quad & (\alpha_1 u_0 + \beta_1 N_{yx}) = (\alpha_2 v_0 + \beta_2 N_y) = (\alpha_3 w + \beta_3 \bar{Q}_y) = (\alpha_4 w'_y + \beta_4 M_y) = \\
& (\alpha_5 w''_{xy} + \beta_5 C_y) = [\alpha_6 (u_4)_2 + \beta_6 B_{yx}] = 0
\end{aligned} \quad (4)
$$

where $\gamma_i, \delta_i, \alpha_i, \beta_i,\ i = 1, 2, ..., 6$ are changed from 0 to 1 and

$N_x, N_y, N_{xy}, N_{yx}, M_x, M_y, \bar{Q}_x, \bar{Q}_y, C_x, C_y, B_{xy}, B_{yx}$ -- inner force factors.

Equations (4) can be used to examine all possible boundary restraint combinations for connected plane and bending problems: free edge, simply supported edge, sliding fixation and constraining. In other words edge conditions may be cinematic, static or mixed.

One further discusses the mathematical model with the proposal of the infinitesimal normal stresses caused by bar bending in the panel plane and bar cross-section warping. One considers the ribs operating in tension-compression, out-of-plane plate bending and torsion.

The edge effects are neglected owing to the asymptotic aspects and the problem of main stress-strain state determination of structurally anisotropic panels is reduced to the solution of the boundary value

problem for the equation of the eighth order in the partial derivatives in a rectangular field. The edge effects are considered if the stress-strain state of the boundary field is determined
Composite panel equilibrium equations and natural boundary conditions are formed by the variation method using the Lagrange concept as a result of the minimization of the entire potential energy of the system. This case the cross section warping is proportioned to the distributed torsion angle.
An equilibrium differential equation system is written in operator form with linear differential operators. By using the symbol integration method, when the first and second expressions are identically satisfied, the system can be reduced to a single eighth order linear non-uniform resolving partial differential equation for the potential function $\Phi(x,\boldsymbol{y})$, with the help of which all accounting values are expressed.
If the structurally-anisotropic composite panel structure derived as a result of energy averaging becomes orthotropic, the linear differential operator of the resolving equation and linear differential operators of the relations between symmetrical stress-strain state components and potential function $\Phi(x, \boldsymbol{y})$ [5] will contain only even numbered derivatives with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$. The **o**blique symmetrical components are determined with odd numbered derivatives as follows:

$$L^{(8)}\Phi = q, \tag{5}$$

$$L^{(8)} = \frac{K_{80}}{a^8}\frac{\partial^8}{\partial x^8} + \frac{K_{62}}{a^6 b^2}\frac{\partial^8}{\partial x^6 \partial y^2} + \frac{K_{44}}{a^4 b^4}\frac{\partial^8}{\partial x^4 \partial y^4} + \frac{K_{26}}{a^2 b^6}\frac{\partial^8}{\partial x^2 \partial y^6} + \frac{K_{08}}{b^8}\frac{\partial^8}{\partial y^8}$$

The constants $K_{8-i,i}$, $_{i=0,2,4,6,8}$ – in (5) depend on geometrical and stiffness structure characteristics; $x = x/a$, $y = y/b$ – non-dimensional coordinates with reference to the panel half-length a and to its width b.
The general boundary value problem in the rectangular field is specified by elastic fixity conditions

$$\begin{aligned} x = const \quad & \left(\gamma_1 u_0 + \delta_1 N_x\right) = \left(\gamma_2 v_0 + \delta_2 N_{xy}\right) = \left(\gamma_3 w + \delta_3 \overline{Q}_x\right) = \left(\gamma_4 w'_x + \delta_4 M_x\right) = 0, \\ y = const \quad & \left(\alpha_1 u_0 + \beta_1 N_{yx}\right) = \left(\alpha_2 v_0 + \beta_2 N_y\right) = \left(\alpha_3 w + \beta_3 \overline{Q}_y\right) = \left(\alpha_4 w'_y + \beta_4 M_y\right) = 0, \end{aligned} \tag{6}$$

where γ_i, δ_i, α_i, β_i , $_{i = 1, 2, 3, 4}$ are from 0 to 1.
The equations (6) can be used to examine all possible boundary restraint combinations for connected plane and bending problems: free edge, simply supported edge, sliding fixation and constraining. In other words edge conditions may be cinematic, static or mixed.
The closed form boundary value problems solution for (5) is constructed by a single Fourier series

$$\Phi(x, y) = \sum_{n=1,3,5}^{\infty} \Phi_n(x)\sin(n\pi y) \tag{7}$$

for particular boundary conditions along two opposite sides. These restrictions are called conformable and they satisfy the hinging condition of bending. However, the sliding fixation condition is in the tangential direction for the plane problem when the panel is loaded by shear force flows along its longitudinal edges. Face conditions are arbitrary and may be in a state of elastic fixity for symmetric or asymmetric boundary value problems.
For an example, we accounted the static displacements and stresses in the flat angle aluminum panel eccentrically stiffened in the longitudinally-lateral directions and comprising composite bundles with the E_{bund} module, kg/mm^2, as the stringer panel. The deflections and normal stresses are presented in

Table 1. Obviously, it is possible to neglect the edge effects, as the difference in calculating of the stress-strain state components is less than 3%.

Table 1 Deflections, inner force factors and normal stresses for the panel with different composite bundle stiffness

<table>
<tr><td>x</td><td>$(-)$</td><td>0</td><td>1,0</td><td>1,0</td><td>1,0</td><td>0</td><td colspan="5" rowspan="3">Δ</td></tr>
<tr><td>y</td><td>$(-)$</td><td>0,5</td><td>0,5</td><td>0,5</td><td>0,5</td><td>0,5</td></tr>
<tr><td>z</td><td>mm</td><td>___</td><td>___</td><td>___</td><td>-0,5</td><td>-0,5</td></tr>
<tr><td>E_{bund}</td><td>N</td><td>w/q</td><td>N_x/q</td><td>M_x/q</td><td>σ_x/q</td><td>σ_y/q</td><td>Δ w</td><td>ΔN_x</td><td>ΔM_x</td><td>$\Delta \sigma_x$</td><td>$\Delta \sigma_y$</td></tr>
<tr><td>kg/mm^2</td><td>$(-)$</td><td>mm^3/kg</td><td>Mm</td><td>mm^2</td><td>$(-)$</td><td>$(-)$</td><td colspan="5">%</td></tr>
<tr><td rowspan="2">0</td><td>8</td><td>33,761</td><td>-136,82</td><td>-12612</td><td>367,30</td><td>-265,15</td><td rowspan="2">0,18</td><td rowspan="2">1,36</td><td rowspan="2">0,02</td><td rowspan="2">0,35</td><td rowspan="2">0,23</td></tr>
<tr><td>12</td><td>33,701</td><td>-134,98</td><td>-12609</td><td>368,59</td><td>-264,53</td></tr>
<tr><td rowspan="2">18000</td><td>8</td><td>19,232</td><td>-116,57</td><td>-12241</td><td>334,38</td><td>-219,69</td><td rowspan="2">0,34</td><td rowspan="2">1,26</td><td rowspan="2">0,16</td><td rowspan="2">0,13</td><td rowspan="2">0,43</td></tr>
<tr><td>12</td><td>19,167</td><td>-115,12</td><td>-12221</td><td>334,81</td><td>-218,76</td></tr>
<tr><td rowspan="2">30000</td><td>8</td><td>15,789</td><td>-108,68</td><td>-11997</td><td>324,10</td><td>-203,06</td><td rowspan="2">0,42</td><td rowspan="2">1,06</td><td rowspan="2">0,01</td><td rowspan="2">0,06</td><td rowspan="2">0,51</td></tr>
<tr><td>12</td><td>15,723</td><td>-107,54</td><td>-11996</td><td>323,91</td><td>-202,03</td></tr>
</table>

The boundary value problems solution for (5) is constructed in a closed form by the method of uniform solutions for arbitrary non-conformable boundary conditions at the contour. A new method algorithm for solution as a series with a non-orthogonal system of general proper functions and analytically defined constants is proposed.

The regular part of general integral is designed as a series by uniform proper functions $\Phi_k(y)$

$$\Phi(x,y)=\Phi_{01}(x,y)+\Phi_{02}(x,y)+\sum_{k=1}^{\infty}B_k\Phi_k(y)ch(\lambda_k x) \tag{8}$$

The uniform boundary value problem for proper functions by one of the coordinates is formulated with partial solutions - $\Phi_{01}(x,y)$, $\Phi_{02}(x,y)$.

The condition of general orthogonality for uniform functions of the primary and adjoint problems under a vector space contains a weight matrix R

$$\int_{-1}^{1}\left(\overline{\varphi}_k^{*}R\psi_i\right)dy=\begin{cases}0, & \lambda_k\neq\overline{\lambda}_i\\ H_i, & \lambda_k=\overline{\lambda}_i\end{cases}, \tag{9}$$

H_i is the generalized proper function norm.

The formula for unknown constants is constructed in closed form

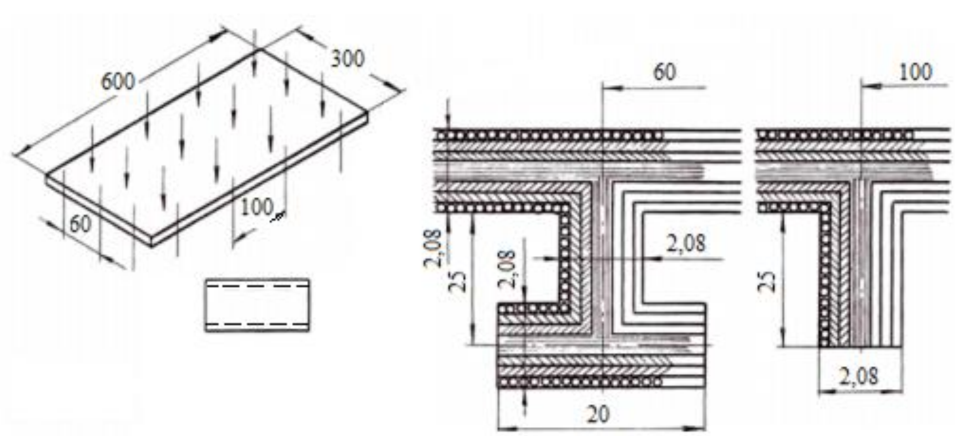
$$B_k = \frac{\hat{H}}{h^{ч}\left(\lambda_k\right)H_k} \tag{10}$$

The exact analytical solutions for edge problems recording the general treatment of the structure elements' physical boundary conditions may be considered.

One can estimate the influence of the production technology factors on the bearing strength of structurally-anisotropic composite panels if the boundary value problem is considered [5, 11] and boundary conditions are non-conformable as (6), the solution is designed by a single trigonometric series or uniform solution method**.**

3. Static problem results and discussion

As an example, we calculated the displacements on the laminate macro-level in the flat angle panel from carbon-plastic eccentrically stiffened in the longitudinally-lateral directions (Figure 1). The edge conditions are asymmetric about the *x* coordinate. The normal loading *q(x, y) = const*.



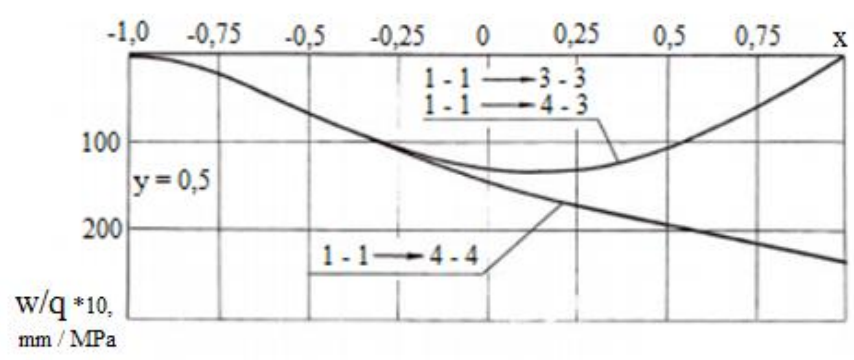


Figure 1 - Panel, stiffened in longitudinally-lateral direction
Asymmetric boundary value problem
Deflections with different edge restrictions
Boundary conditions at lateral edges: plane problem – bending
1 – constraining, 2 – sliding fixation, 3 – simply supported edge, 4 – free edge

The deflections depend on the plane boundary conditions while the lateral edges are simply supported for bending. When these edges are constrained for bending the deflection curves coincide practically while the restriction conditions for the plane problem are varied.

As the next example, one calculated the stresses on the laminate macro- level in the flat angle panel from carbon-plastic eccentrically stiffened in the longitudinal direction (Figure 2). The edge conditions are symmetric about the *x* coordinate, and the normal loading *q(x, y) = const*.

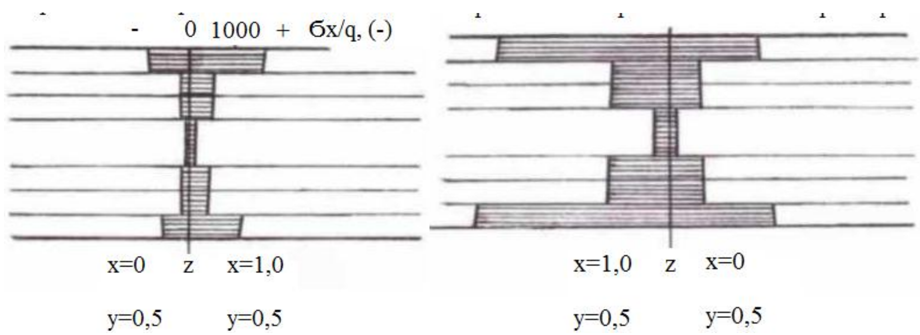


Figure 2 - Panel, stiffened in longitudinal direction
The normal stresses in the skin layers.
The normal stresses in the stringer flange layers

The conclusion for the deflection is similar to the previous one: the deflections substantially depend on the plane boundary conditions while lateral edges are simply supported for bending. When these edges are constrained for bending the deflection curves coincide practically while the restriction conditions of the plane problem are varied.

As the next example, one has calculated the residual thermal stresses induced by cooling after hardening to the flat angle panel made of carbon-epoxy having asymmetric structure along its thickness (Figure 3). The panel contour is simply supported with respect to bending, and the longitudinal side restrictions correspond to the sliding fixation in the tangential direction. However, the lateral edges are free from forces and moments relative to the plane and bending problems

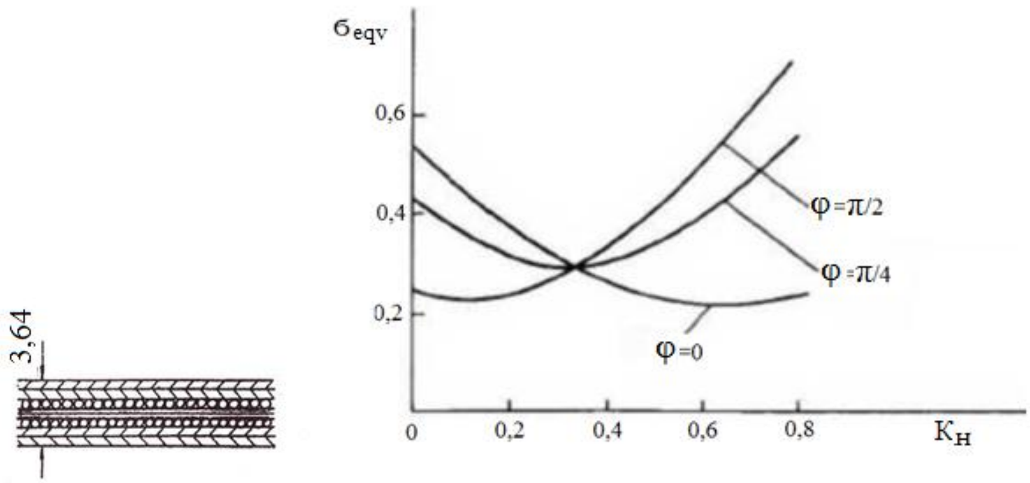


Figure 3 - Panel with asymmetric package structure.
Influence of the reinforcing fibers preliminary tension
on the residual temperature stresses

The estimation of the influence of the reinforcing fibers' preliminary tension on the residual thermal stresses level has been performed. For the composite panel comprising longitudinal, lateral and oblique plies (Figure 3), the equivalent residual thermal stresses decrease while the reinforcing fiber pre-stressed tension increases. The optimum level of allowable *k*-ply fiber deformation is 0, 35 (35%), where the whole package stresses are equal and minimum.

4. Buckling problems

Several studies (e.g. Chen et al. 2011; Huang et al. 2015; Kazemi 2016; Kim et al. 2002; Lindgaard et al. 2010; Mittelstedt and Schroder 2010; Shafei and Kabir 2011; Vescovini and Dozio 2015; Yeter et al. 2014) [8, 14, 16, 17, 20, 23, 26, 28, 29] describe the buckling problems of structurally-anisotropic composite panels, however, the presented mathematical model is new.
The eighth order differential equations are resolved for the pre-critical stress-strained state and for the buckling problems (e.g. Firsanov and Gavva 2017) [10]

$$
\frac{K_{80}}{a^8}\frac{\partial^8\Phi}{\partial x^8}+\frac{K_{62}}{a^6b^2}\frac{\partial^8\Phi}{\partial x^6\partial y^2}+\frac{K_{44}}{a^4b^4}\frac{\partial^8\Phi}{\partial x^4\partial y^4}+\frac{K_{26}}{a^2b^6}\frac{\partial^8\Phi}{\partial x^2\partial y^6}+\frac{K_{08}}{b^8}\frac{\partial^8\Phi}{\partial y^8}=
$$

$$
=\left[\begin{array}{ll}
\dfrac{N_xR_{40}}{a^6}\dfrac{\partial^6\Phi}{\partial x^6} & +\dfrac{\left(N_{xy}+N_{yx}\right)R_{40}}{a^5b}\dfrac{\partial^6\Phi}{\partial x^5\partial y}+ \\
+\dfrac{\left(N_xR_{22}+N_yR_{40}\right)}{a^4b^2}\dfrac{\partial^6\Phi}{\partial x^4\partial y^2} & +\dfrac{\left(N_{xy}+N_{yx}\right)R_{22}}{a^3b^3}\dfrac{\partial^6\Phi}{\partial x^3\partial y^3}+ \\
+\dfrac{\left(N_xR_{04}+N_yR_{22}\right)}{a^2b^4}\dfrac{\partial^6\Phi}{\partial x^2\partial y^4} & +\dfrac{\left(N_{xy}+N_{yx}\right)R_{04}}{ab^5}\dfrac{\partial^6\Phi}{\partial x\partial y^5}+ \\
+\dfrac{N_yR_{04}}{b^6}\dfrac{\partial^6\Phi}{\partial y^6} &
\end{array}\right] \qquad (11)
$$

The deflection function *w(x,y)* is coupled with the potential function Φ*(x,y).* The coefficients R_{ij} , $_{i=4,3,\ldots,0\ ,\ j=0,1,\ldots,4}$ in the relation formulas and the coefficients K_{ij} , $_{i=8,7,\ldots,0\ ,\ j=0,1,\ldots,8}$ in the resolving equation (11) are the constant values, which depend on the elastic characteristics of the material and geometrical structure parameters. All components of the stress-strain state including the inner force factors are related with the potential function Φ*(x, y).*

The critical force definitions of both the general bending form of the thin-walled system buckling and that of the many-waved torsion buckling are of the interest in accordance with traditional design practices. In both the cases bending is integral with plane stress state. Thus, the buckling problem results in the solution of the boundary value problem for the eighth order equation with the partial derivatives in the rectangular field.

The solution is designed in closed form by a double Fourier series for particular conformable boundary conditions along the panel contour.

$$\Phi(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\sin(m\pi x)\sin(n\pi y) \tag{12}$$

The critical force calculated with the main uniform pre-buckling stress state (e.g. Firsanov and Gavva 2017) for the general bending mode of buckling

$$P=\frac{\pi^2}{b^2}\frac{K_{80}\left(\frac{m}{c}\right)^8+K_{62}\left(\frac{m}{c}\right)^6 n^2+K_{44}\left(\frac{m}{c}\right)^4 n^4+K_{26}\left(\frac{m}{c}\right)^2 n^6+K_{08}n^8}{\left[R_{40}\left(\frac{m}{c}\right)^4+R_{22}\left(\frac{m}{c}\right)^2 n^2+R_{04}n^4\right]\left(\frac{m}{c}\right)^2} \tag{13}$$

$c = a/b$

The formula for the critical loading *P* of the multi-wave torsion buckling problem coincides to this formula within the coefficients K_{ij} and R_{ij}. The coefficients $\hat{K}_{ij}$, $_{i,j\,=\,0,\,2,\,4,\,6,\,8}$ and $\hat{R}_{ij}$, $_{i,j\,=\,0,\,2,\,4}$ are determined by generalized stiffness characteristics while stiffness averaging for the elements of longitudinal set up to the skin is replaced with the discrete characteristics

$$\frac{1}{c_1}\to\begin{cases}\frac{2}{b}\sum_{i=1}^{N}\sin^2(n\pi y_i)\\ \frac{2}{b}\sum_{i=1}^{N}\cos^2(n\pi y_i)\end{cases}\to\hat{K}_{ij},\hat{R}_{ij}\quad ,$$

c_1 is the stringer distance; y_i is the coordinate *y* of the discrete stringer.

The mathematical model relations for the investigation of pre-critical stress-strain state of structurally-anisotropic panels composed of composite materials are presented. The pre-critical stressed state of plane angle laminated panels made from polymer fiber composite materials with eccentric longitudinally-lateral stiffening set is considered for the further development of refined buckling problems.

The normal forces N_x, corresponding to the pre-critical stress state of the stiffened composite panel compressed along the *x* axis are distributed as [12]

$$N_x=P\sum_{i=1,3,5\ldots}^{\infty}\left\{\left[\sum_{L=1}^{4}(N_x)_{iL}\,ch(\lambda_{iL}x)\right]-\frac{(N_x^T)_i}{P}-\frac{(N_x^H)_i}{P}\right\}\sin(i\pi y) \tag{14}$$

$(N_x)_{iL}$ are the coefficients of single trigonometric series for the normal forces N_x, known after the determination of the constants of single trigonometric series, $\left(N_x^T\right)_i$, $\left(N_x^H\right)_i$ are the coefficients of single trigonometric series for the thermal and tension forces.

Here $\lambda_{iL} = z_L \lambda_{iy} a$, $\lambda_{iy} = \frac{i\pi}{b}$, z_L are the roots of the corresponding characteristic polynomial and are calculated using the MATLAB operating environment.

The variations of the transverse forces N_y and shear forces N_{xy}, N_{yx} are obtained analog to (3).

The critical force formula is designed with the orthogonalization procedure of the general differential equation of the curved surface. The expression

$$P = P^* \left\{ \frac{\pi}{\sum_{i=1,3,5...}^{\infty} \left(-\frac{2}{i} - \frac{1}{2n-i} + \frac{1}{2n+i} \right) \left\{ \left[\sum_{L=1}^{4} \left(N_x\right)_{iL} \frac{\left[(m\pi)^2 - \lambda_{iL}^2\right]}{\left[(m\pi)^2 + \lambda_{iL}^2\right]} \frac{sh\lambda_{iL}}{\lambda_{iL}} \right] - \frac{\left(N_x^T\right)_i}{P} - \frac{\left(N_x^H\right)_i}{P} \right\}} \right\} \quad (15)$$

provides the range of values *P* for additional possible deformation of the base surface at *m* = 1, 2, 3,…, *n* = 1, 2, 3,…, *m* and *n* are the wave parameters.

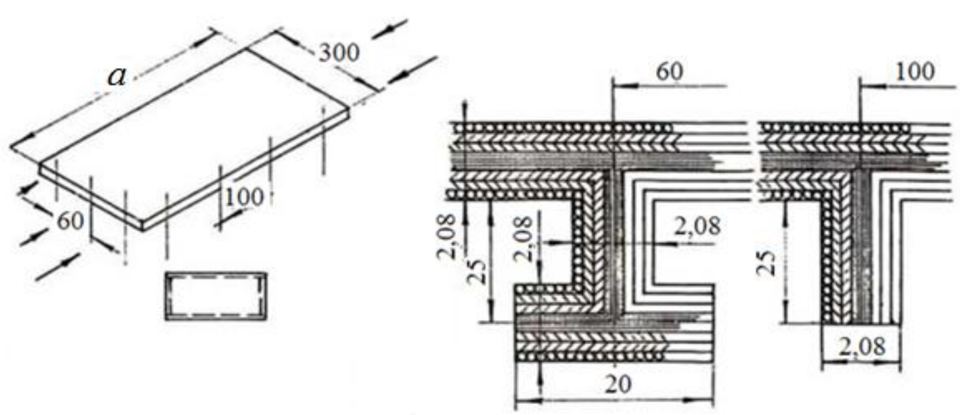
The *P* with «*» is the critical force calculated with the main uniform pre-buckling stress state (13), the panel side ratio *c* = 2*a/b, a, b* are the panel half-length and width, correspondingly.

The step by step method is used to determine the critical forces. The critical force *P* with «*» calculated with the uniform pre-buckling stress state is proposed as an initial first approach.

One can estimate the influence of production technology factors on the bearing strength of structurally-anisotropic composite panels if the pre-critical stressed state is considered, or boundary conditions are non-conformable, and the solution is formed by a unitary trigonometric series or with the help of a uniform solution method.

5. Buckling problem results and discussion

Figure 4 shows the influence of the composite panel size on the force level corresponding to the general bending stability loss form and many-waved torsion buckling. The conditions for the existence of these buckling forms are revealed.



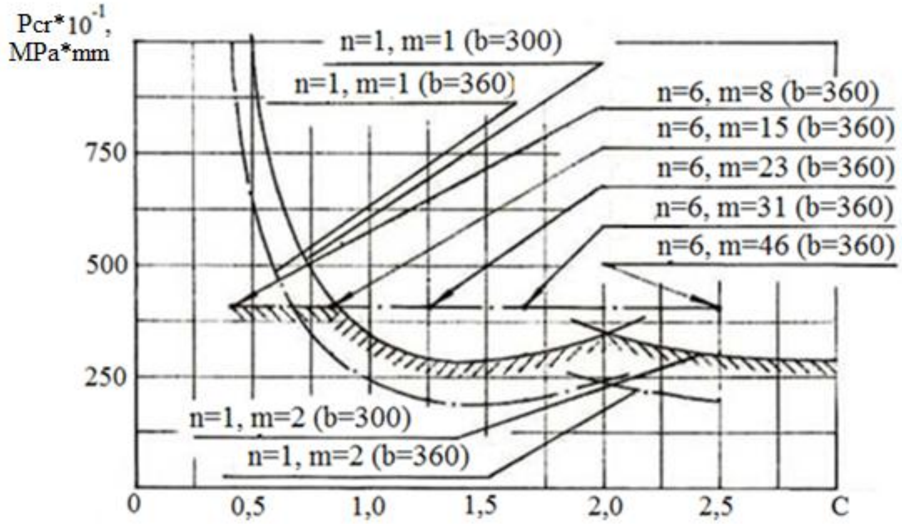


Figure 4 - Panel with longitudinally-lateral stiffening set.
Influence of panel sizes on critical force level. Structurally-anisotropic model. Discrete ribs

For a short panel, if $c < 0{,}75$, the many-waved torsion buckling occurs: $n = 6$, $m = 8$. If $c = 0{,}75$, the panel is equally buckled. The buckling for panels with $0{,}75 < c < 2{,}0$ has the general bending form $n = 1$, $m = 1$. When c > 2,0, the buckling also exhibits the general bending form, but $n = 1$ and $m = 2$.

Figure 5 shows the distribution of inner normal forces $\boldsymbol{N_x}$ for flat angle panes from carbon-plastic eccentrically stiffened in the longitudinal direction. The edge conditions are symmetric about the x coordinate. Compression loading is constant under in-plane bending.

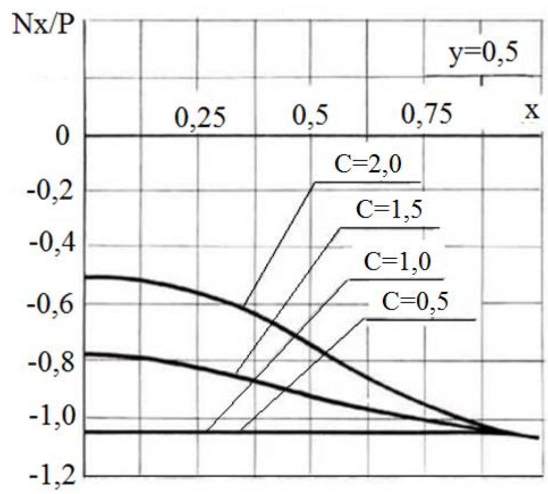


Figure 5 - Panel with longitudinal stiffening set.
Influence of panel sizes on inner normal force distribution under in-plane bending

Only for short panels, if the side ratio $c < 1{,}0$, the inner normal forces are distributed along the panel length almost uniformly

The testing series of uniform compressed stiffened composite panels for carrying the objects to the moment of stability loss (Figure 6) have been made using the special fixture.

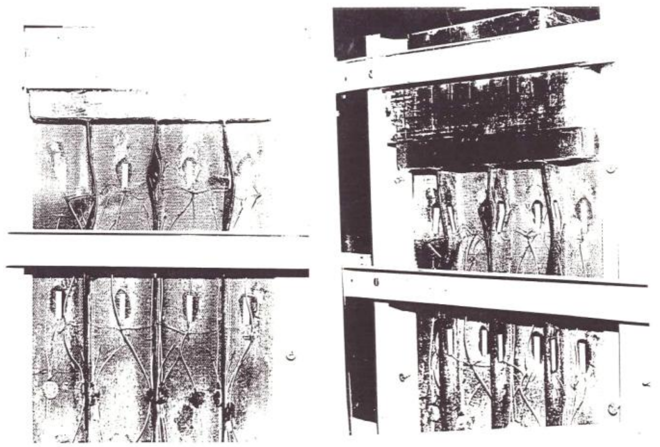


Figure 6 - Experimental specimen

The refined theoretical results and the experimental data are in agreement qualitatively with respect to buckling forms and quantitatively with respect to critical stresses, and they reveal a precision of 12% - 13%. It confirms the authenticity of the mathematical model.
The refined theoretical results and the experimental data are in agreement qualitatively with respect to buckling modes and quantitatively with respect to critical stresses within 12 - 13% if pre-buckling stress state is considered uniform. And they reveal a higher precision of 8 - 10% in view of non-uniform main pre-buckling stress state. Thus, it confirms the authenticity of the presented mathematical model.
The testing series of uniform compressed stiffened panels comprising composite bundles for carrying the objects to the moment of stability loss (Figure 3) have been made using the special fixture for the proposed mathematical model verification.

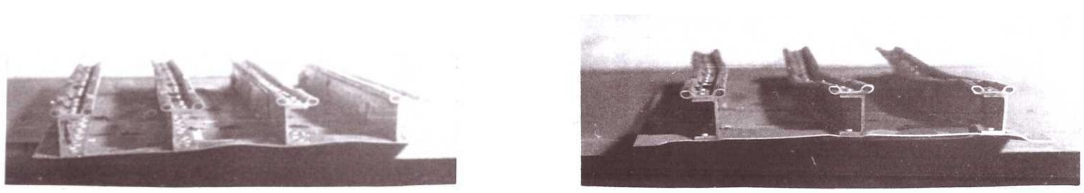


Figure 7 - Stringer panels compressed in the longitudinal direction

The refined theoretical results and the experimental data are in agreement qualitatively with respect to buckling modes and quantitatively with respect to critical forces within 19 - 20% if pre-buckling stress state is considered uniform. The stringer cross section of the panel specimen is non-symmetric, but the numerical results were obtained for the panels with the symmetric cross section of the stringer. Thus, it also confirms the authenticity of the presented mathematical model.

6. Conclusion

Because the solution is obtained by analytical methods, the calculation time is minimal. This is of interest from the perspective of practical design using parametric analysis. The results of the stress analysis calculations, as well as the results of the buckling analysis calculations, offer opportunities for reducing and optimizing the weight characteristics of aircraft elements.

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