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ON WINGS WITH NON-ELLIPTIC LIFT DISTRIBUTIONS

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Abstract

Non-elliptic lift distributions were originally conceived by Prandtl as a solution for wings having optimal aerodynamic and structural efficiency. Since 1933, many researchers have expanded on this theory to confirm that it explicitly facilitates the trade-off between induced drag and weight, and have unveiled other potential benefits such as proverse yaw. This theory is seen as a promising alternative to redefine the underlying principles of conceptual wing design in the search for a step increase in efficiency. Given the recent revival of this approach, in this paper the authors present a concise review of its theoretical basis, the developments carried out in this area and their relevance to the future of aviation.

Keywords: Non-elliptic lift distributions, wing design, aero-structural efficiency, induced drag, proverse yaw

1. Introduction

The study of wing aerodynamics and design have been central to the work of aeronautical engineers from the earliest days of human flight. The theories and methods developed have been consolidated through the engineering excellence reflected in the safety standards of aviation. Today, optimisation processes used in conceptual wing design tools are still grounded in these established approaches, such as Prandtl's lifting line theory[1]. However, the current socio-economic prospect has fuelled the prevailing need for a technological future that is in closer harmony with the environment, yet accommodating to our ever-growing demands. The consequential strive for more efficient flight has led many to challenge a diversity of conventionalities in civil aircraft design. In particular, a community of researchers and designers have addressed it through the reconsideration of the underlying theoretical principles of conceptual wing design, in order to minimise induced drag and advance towards a holistic flight model. This endeavour has led to the re-inspection of an alternative approach first published by Prandtl in 1933[2]. Here, he proposed to remove the fundamental constraint of fixed span, established in the lifting line theory and instead prescribe wing bending moment. By doing so, he synthesised both aerodynamic and structural efficiency into a single theory, of which the solution for minimum induced drag yielded a non-elliptic lift distribution (NELD). Today, a similar 'span-extension' solution can be seen on wings with winglets and elegantly designed wing twist distributions, the latter often obtained through state-of-the-art multidisciplinary design optimisation (MDO) processes. Researchers are now trying to integrate Prandtl's 1933 approach in wing design frameworks so that traditional assumptions, such as the elliptic spanload being the optimal, do not constrain their proposed concept.

Given the recent revival in the development of this approach to conceptual wing design, in this paper the authors present a concise review of the work and developments carried out in the field. This begins by providing the reader with a short summary of lifting line theory, which is the theoretical basis used to define the key concepts of aero-structural efficiency and span extension in the NELD approach. This is followed by a more detailed discussion of the derivation, assumptions and limitations of Prandtl's 1933 work, which also includes: (1) further developments carried out by researchers such as Jones, Klein and Viswanathan and Hunsaker et al. and, (2) the need for revisiting the traditional perspectives on induced drag and span efficiency. The reader is also provided with a summary of flight dynamics implications through the discussion of the experimental approaches of the Horten brothers and Bowers. The paper concludes by a short review of current conceptual design frameworks and highlights the potential routes for integrating the NELD approach in modern tools.

2. Lifting line theory

It is widely acknowledged that Prandtl's work in fluid dynamics has shaped modern wing theory. The lifting line theory[3, 4] arose from the collaboration with Betz in the early 1900's, and is now so fundamental to conceptual aircraft design that it can be found at the core of almost all modern wing design tools. The theory defines a finite wing as a bound vortex filament consisting of an infinite number of horseshoe vortices that are shed downstream. This approximation captured the main aerodynamic features of a lifting surface despite the considerable number of assumptions: inviscid, incompressible and irrotational flow only at small angles of attack. In the interest of conciseness, the authors direct the reader to Reference [5] for the detailed theoretical derivations. Yet, Prandtl's early work exposes the struggle he faced to find an expression of circulation distribution that would yield a "plausible"[3] downwash after solving the singular integral given by the Biot-Savart law, which determines the velocities induced at the wing by the trailing vortices. The desired solution was initially found in the equation of a half-ellipse as follows:

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{y}{s}\right)^2} \tag{1}$$

With this, the circulation distribution $\Gamma(y)$ at each spanwise position y would be determined by the maximum circulation at the root Γ_0 and the wing semi-span s. In addition to the elegance of the constant downwash provided by Glauert's integral, the elliptic lift distribution also yielded the minimum induced drag for a given wing. Hereby, the spanwise loading could be effectively represented as the sum of all the velocities induced on the wing, or presented in terms of angles-of-attack through the following linear integro-differential equation:

$$\frac{-2\Gamma(y)}{C_{l_\alpha}(y)c(y)V_\infty} - \frac{1}{4\pi V_\infty}\int_{-s}^{s} \frac{d\Gamma(y_0)}{dy}\,\frac{dy_0}{y-y_0} + \alpha + \alpha_T(y) - \alpha_{0L}(y) = 0 \tag{2}$$

The terms in the lifting line equation presented in Equation 2 equate to, respectively: the effective angle-of-attack, the induced angle-of-attack, wing incidence angle α, geometric twist α_T and zero-lift angle-of-attack α_{0L}; and is presented as a function of: circulation Γ, lift curve slope C_{l_α}, chord c, free-stream velocity V_∞ and vortex strength $(d\Gamma/dy)dy$.

The first formal mathematical model of a wing's lifting properties was published by Prandtl[1] in 1918, but it could be argued that this could not have been achieved without Lanchester's earlier work, published in 1907[6]. Here, he acknowledged the existence of asymmetric vortices being shed from the wingtips of a finite wing, which would dissipate energy into the wake of a three-dimensional flow. This has lead to historical controversy on the origins of lifting line theory and thin aerofoil theory[7], for which Lanchester has received further credit in recent years. Be that as it may, this theory is built upon the work of Kutta-Joukowski[8, 9], Biot-Savart[10], Helmholtz[11] or Munk[12], and altogether, they lay down the theoretical and mathematical foundations behind the aerodynamics of the aircraft we see today.

The profound relevance of lifting line theory has led many to generalise the approach: it has now been extended for the study of curved, swept or other arbitrary-shaped wings[13, 14, 15, 16, 17]; it can consider unsteady flow conditions[18, 19, 20, 21] and integrate nonlinear data[22, 23]; or include compressibility effects[24, 25, 26]. An extensive discussion covering the numerous adaptations of lifting line theory could easily commence, but that is beyond the scope of this review. The reader interested in such developments is directed to the References[5, 27, 28, 29, 17, 30, 31, 32, 33].

3. Theory of non-elliptic lift distributions

Prandtl continued exploring the mathematics behind the established lifting line theory. The elliptic circulation distribution was just the first order term of a series equation, which had been presented as[3]:

$$\Gamma(\xi) = \sqrt{1-\xi^2}(\Gamma_0 + \Gamma_2\,\xi^2 + \Gamma_4\,\xi^4 + ...) \tag{3}$$

where $\xi = y/s$. This formulation describes a variational problem that could yield many different solutions. However, the design concepts that Prandtl was previously evaluating always considered a fixed wing. With this constraint, the elliptic curve would always be the optimal spanwise loading.

However, his perseverance took him to an alternative solution, presented in 1933[2], which removed the span constraint and instead considered structural limitations[1]. Conscious of the fact that span extension is sometimes impractical due to operational limitations, he still believed it unseemly to consider the elliptic spanload as the only optimum. Therefore, Prandtl reformulated the problem for a given design flight condition: knowing that increasing aspect ratio reduces induced drag, he enquired whether it was possible to extend the wing while minimising the increase in structural weight. To do so, the integrated bending moment due to spanload was prescribed as the parameter driving structural weight, while wing area remained fixed. Prandtl expanded Equation 1 according to Equation 3 to account for second order circulation terms, becoming:

$$\Gamma(y) = \left[\Gamma_0 + \Gamma_2\xi^2\right]\sqrt{1-\xi^2} \tag{4}$$

which would associate with a quadratic downwash distribution necessary to satisfy the problem conditions. This generated a family of curves that would yield the spanload for minimum induced drag and structural weight at each span extension. While the half ellipse still provided the ideal aerodynamic characteristics for the baseline wing, the optimal solution would have a bell-shaped form[2]. The latter provided an 11% induced drag reduction with a 22.5% span extension, therefore minimising induced drag while maintaining the same structural weight as its equivalent wing with elliptic loading.

Let us expand on the derivations to show the mathematical implications of the above problem statement. Prandtl assumed the structural weight W_s of the wing to be proportional to the integrated bending moment $M_I(y)$ at each location as:

$$W_s(y) = \frac{1}{K_s} M_I(y) = \frac{1}{K_s}\rho_\infty V_\infty \int_y^s \Gamma(y_0)\,(y_0 - y)\,dy_0 \tag{5}$$

with ρ_∞ representing the free-stream density. Here, the spar was assumed to have a constant section, which implies a rectangular planform and a constant proportionality factor K_s along the span. The prescribed structural weight is computed through the integration of the bending moment $M_I(y)$, which can be expressed in terms of the aerodynamic load as the moment of inertia of the lift:

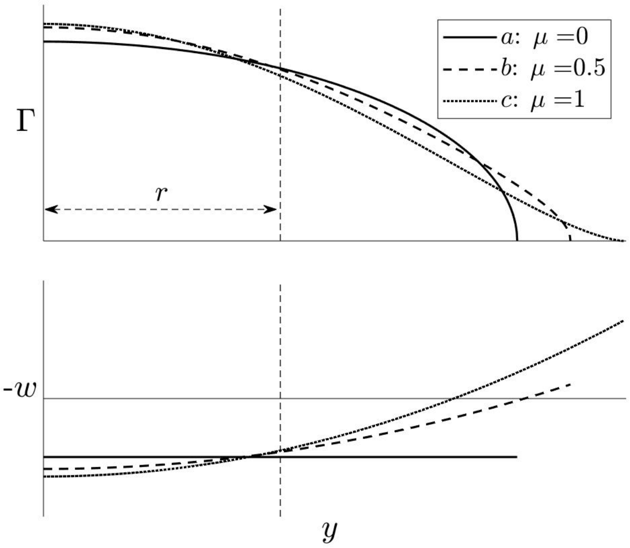
$$Lr^2 = \rho_\infty V_\infty \int_{-s}^{s} \Gamma(y)\, y^2 dy \tag{6}$$

where r is radius of gyration of the lift distribution and L is the total lift. Betz derived the respective auxiliary constraints by applying a variation $\delta\Gamma$ to an auxiliary wing placed far downstream. Here, the streamwise induced velocities are negligible, and thus the wing is only subjected to a downwash given by $2w$. If the required circulation must satisfy $\Gamma(y)$, the addition of $\delta\Gamma$ will cause no variation on the induced drag D_i, complying with Helmotz's theorem. With this, the expression for induced drag became:

$$D_i = \frac{L^2}{8\pi\rho_\infty V_\infty^2 r^2}\frac{(1-\mu/2)\left(1-\mu/2+\mu^2/4\right)}{(1-\mu/4)^3} \tag{7}$$

where $\mu = -\Gamma_2/\Gamma_0$. With this, the span b becomes a function of the spanload in the form of:

$$b = 4r\sqrt{\frac{1-\mu/4}{1-\mu/2}} \tag{8}$$



μ	$b/4r$	Γ_0	D_i/D_{i_E}
0.00	1.0000	1.0000	1.0000
0.25	1.0351	1.0305	0.9458
0.50	1.0801	1.0581	0.9096
0.75	1.1402	1.0795	0.8921
1.00	1.2247	1.0887	0.8889

Figure 1 – Reproduction of Prandtl's 1933 solutions[2]. Curves for circulation and downwash distributions (left). Tabulated span, circulation and induced drag factors for different values of μ (right).

The results of this approach are shown in Figure 1: curves a show the well-known elliptic circulation and constant downwash distributions, which agree with the earlier solution derived from the lifting line theory, as there is no span extension; curves b show an intermediate case given by an arbitrary span extension; curves c show the optimal bell-shaped distribution, having the same lift and integrated bending moment as the elliptic case but providing the aforementioned 11% reduction in induced drag with a 22.5% longer span. Prandtl examines this "optimal" point as a function of μ. This parameter expresses the ratio of the second linearly superposed circulation, driven by the coefficient Γ_2, over the constant term in the expansion, which determines the elliptic spanload. Later it will be shown how the second term is directly linked to the structural weight[35]. Thus, when $\mu = 0$ the solution is elliptic and no additional circulation is superimposed. As μ increases, the second term re-shapes the original curve by superposition, while induced drag decreases with increasing span at a fixed radius of inertia r. However, for $\mu > 1$ the resulting circulation distributions start yielding negative values from the wingtip, which is not physically consistent nor aerodynamically efficient. The optimal point is therefore given by the curve that has a null gradient at the wingtips, corresponding to $\mu = 1$. With this, Prandtl elegantly found an alternative wing design approach that would suit both operational and structural design requirements at a conceptual design stage. The solution could be derived analytically and would provide minimum induced drag for a given design point.

3.1 Other classical approaches to aero-structural efficiency

Many have tried to derive similar theoretical models for minimum induced drag and optimal structural weight. In 1950, Jones[36, 37] supported a design philosophy for practical wing design driven by structural requirements, and addressed it independently by prescribing the wing root bending moment M_R, given by:

$$M_R = \rho_\infty V_\infty \int_0^s \Gamma(y) y dy \tag{9}$$

To reach Jones solution, Munk's mutual drag theorem[12] was assumed to find an additional distribution that varies the loading without increasing induced drag. As previously, far-field wake assumptions was applied to determine the downwash through potential theory. Herein, the downwash is defined by a first order polynomial to satisfy the boundary conditions of this approach, as opposed to the quadratic variation demanded by Prandtl's 1933 proposal.

[1]Oddly enough, it is only recently that an official English translation of Prandtl's 1933 paper was published[34].

[2]Hereafter, the authors refer to the bell-shaped spanload as the optimal non-elliptic curve for minimum induced drag corresponding to each of the theories.

Prandtl[2] had defined the radius of inertia of the lift distribution and held this parameter fixed and equal to that of the elliptic loading. Unlike Prandtl, Jones determines the circulation as a function of the non-dimensional spanwise position of the spanload centroid over the semi-span $\overline{y_C}$. This parameter needs to vary with span extension for the circulation to to redistribute the loading. However, what remains fixed and equal to the baseline elliptic spanload is the dimensional centroid $y_C = \overline{y_C}s$. The loading distribution for a given lift and root bending moment can be expressed as a function of the non-dimensional centroid as:

$$\Gamma(y) = \frac{L}{2\rho_\infty V_\infty s}\left(\frac{12}{\pi} - 6\overline{y_C}\right)\frac{\sqrt{s^2-y^2}}{s} + \left(18\overline{y_C} - \frac{24}{\pi}\right)\frac{y^2}{s^2}\cos^{-1}\frac{s}{|y|} \tag{10}$$

and the expression for minimum induced drag yields:

$$D_i = \frac{L^2}{\pi\rho_\infty V_\infty 4s^2}\left(\frac{9}{2}\pi^2\overline{y_C}^2 - 12\pi\overline{y_C} + 9\right) \tag{11}$$

With this, Jones had obtained another set of non-elliptic spanloads that provided minimum induced drag and structural weight for any span extension. The optimal solution was now determined pragmatically, given that the rate of induced drag reduction with span increase is insignificant for $1.15 < r_s < 1.33$, and further extensions will again lead to negative circulation at the wingtips. Here, r_s designates the ratio of span extension given by the span of the extended wing over the span of the baseline elliptic planform as $r_s = b/b_E$. The upper limit of this span extension range can be found by computing the first and second derivatives of Equation 11. The root of the first derivative is found at $r_s = 1.33$, and, like Prandtl's solution, it will not be a global minima, but an inflection point. Thus, the optimal point can be set at the lower limit, where Equation 10 yields a non-elliptic spanload that reduces induced drag by 15% with a 15% increase in span. If the span remains fixed, the resulting circulation curve agrees with the solution of an elliptic spanload, and any other solution will hold the same root bending moment as the elliptic case.

Klein and Viswanathan also explored the added value that the non-elliptic spanloads can provide in terms of aero-structural efficiency. Their work insists on the importance of incorporating structural considerations in aerodynamic design as a way of coupling both design limitations into a single theory. Initially, they derived a solution providing minimum induced drag for a prescribed wing-root bending moment[38], as defined in Equation 9. Interestingly, this proposal was built upon Prandtl's 1933 paper alone and with no knowledge of Jones' work, and could be applied to higher aspect ratio wings with relatively small sweep. As Jones had shown, the downwash was required to vary linearly along the span. The circulation distribution they obtained was as follows:

$$\Gamma(y) = r_s\left[\sqrt{1-\left(\frac{y}{s}\right)^2}\left(2C_1 + 2\frac{C_2}{\pi}\right) - \frac{C_2}{\pi}\left(\frac{y}{s}\right)^2\log\frac{1-\sqrt{1-\left(\frac{y}{s}\right)^2}}{1+\sqrt{1-\left(\frac{y}{s}\right)^2}}\right] \tag{12}$$

where the constants C_1 and C_2 are a function of the span ratio r_s.

Although Klein and Viswanathan employ a different approach to obtain the circulation distribution in Equation 12, this can be rewritten in terms of the inverse hyperbolic cosine to yield Jones' solution in Equation 10, and therefore yield the same load and induced drag behaviour with span extension, given that the boundary conditions of the problem are the same. In this case, Klein and Viswanathan defined the optimal point at the upper bound of r_s, where a reduction of 16% in induced drag is achieved with a 33% span extension. This result shows the negligible reduction in induced drag compared to span increase that Jones had observed previously.

Later, Klein and Viswanathan expanded the above theory to include more precise approximation of the wing structural weight[39, 40], in which the structural boundary conditions were now determined by the integrated bending moment M_I and shear force F_s distributions, expressed as:

$$\begin{aligned}\int_0^s F_s(y)dy &= \rho_\infty V_\infty \int_0^s \int_y^s \Gamma(y_0)dy_0 dy \\ \int_0^s M_I(y)dy &= \rho_\infty V_\infty \int_0^s \int_y^s \Gamma(y_0)(y_0 - y)dy_0 dy\end{aligned} \tag{13}$$

which now required a quadratic downwash distribution. Altogether, the circulation distribution derived from these assumptions follows:

$$\Gamma(y) = \frac{s}{s_E}\left[\sqrt{1-\left(\frac{y}{s}\right)^2}\left(2C_1+2\frac{C_2}{\pi}+C_3\right)-\frac{C_2}{\pi}\left(\frac{y}{s}\right)^2\log\frac{1-\sqrt{1-\left(\frac{y}{s}\right)^2}}{1+\sqrt{1-\left(\frac{y}{s}\right)^2}}-\frac{2}{3}C_3\sqrt{\left(1-\left(\frac{y}{s}\right)^2\right)^3}\right] \tag{14}$$

where $C_1 = C_2 = C_3 = f(r_s)$, and the ratio of induced drag over the elliptic case is expressed as:

$$\frac{D_i}{D_{i_E}} = 4\frac{9r_s^4-40r_s^3+\frac{145}{2}r_s^2-60r_s+\frac{75}{4}}{r_s^6} \tag{15}$$

According to their publication, minimising Equation 14 would yield a NELD with a 7% reduction in induced drag with a 16% span extension. Herein, they also discussed further design criteria that should be considered in the design process, such as the reduction of wing weight during flight due to fuel consumption. However, it is not straight forward to analytically insert such a design parameter into the variational problem.

3.2 Extensions and limitations

The early theories for aero-structural wing design are insightful and elegant approximations that can be employed as a prelude to the conceptual design phase. They add emphasis on wing shape optimisation as a variational problem, the solution of which is not unique but dependant on the assumptions and design criteria adopted. Indeed, one of the main limitations of these early approaches arises from the determination of wing loads. In all cases, the structural constraint is set at the same flight condition at which drag is minimised: cruise flight. Nevertheless, structural strength requirements must be met at all regions of the flight envelope to ensure structural integrity. Not only it is a critical design driver, but its omission can also bring dramatic induced drag increase due to the sensitivities of the spanload to off-design conditions. The large washout needed to yield the traditional non-elliptic spanloads penalises performance during high load manoeuvres, since twist has a direct impact on lift coefficient C_L. Similarly, to select a critical load condition as the design point will also increase drag considerably during cruise.

These considerations have been addressed by many via extensions of the early theories presented above. A more practical approach would allow to minimise induced drag at a different C_L at which the bending moment must be prescribed[41] in order to reduce off-design penalties while alleviating wing structural loads during high load manoeuvres[35, 42]. Moreover, the choice of aerodynamic and structural models alters the trade-off between span extension and wing structural weight, and impacts the theoretical proportionality coefficient K_s between bending moment and weight[43]. However, more accurate models for wing weight can be embedded to an analytical approach. For a simple beam, the integrated bending moment divided by the section thickness[44] can capture the variation of material properties carried along the span due to changes in chord or thickness, which would affect the wing box strength and mass[45]. The inclusion of aerofoil thickness as a variable provides further insight on various off-design considerations. It is a determining factor in the relation between weight and bending moment K_s, and becomes a crucial parameter when total drag is considered[46, 47, 48], on which the characteristic twist distribution of wings with NELD will also have a significant impact[49]. Constraints on the local maximum stress as a function of the beam cross-section and material would allow further consideration of arbitrary planforms through a numerical approach. Wing area is another variable that will compromise the results due to its direct relation with profile drag and area-dependant weight[48, 50]. Moreover, the resulting high aspect ratio together with the required non-linear twist distribution increases the risk of encountering adverse aeroelastic phenomena[51], which has been assessed through its impact on wing deformation[52] and with consideration of aeroelastic feedback on the bending moment constraints[43, 46, 47, 53].

The use of NELDs might face further limitations in transonic flight regimes. Here, compressibility effects are unavoidable and aircraft designers invest great effort to address such phenomena. Existing methods focus on cross section and planform design to increase the speed at which drag divergence occurs while simultaneously weakening the shock strength. These include supercritical aerofoils and wings, laminar flow wings, bodies shaped under the Whitcomb area rule, and, of course, the established wing sweep. Wing warp is another design approach that is gaining interest due to the growing MDO capabilities[54, 55]. Here, twist and camber distributions have proved to be significantly more critical than spanwise thickness; parameters which are key features of wings with NELDs. These designs require a non-linear downwash that is commonly achieved through a bell-shaped twist distribution, yielding a washout that can be notably higher than that found on conventional wings. Alternatively, the same effect could be achieved through careful design of camber distribution. However, these requirements could lead to an increased the likelihood of shock waves and drag onset at both design and off-design conditions.

Emphasis must be kept on span extension to achieve aero-structural optimisation under the assumptions of the above methodologies. Past work[56, 57] that did not account a span extension did not see significant improvements in performance. Weight savings can be achieved on a fixed wing by exchanging an elliptic spanload for a NELD as the load is shifted inboard. Without a span extension, these savings will be rapidly overcome by drag penalties as the spanload deviates from the elliptic. In practise, however, it is frequently unfeasible to extend the span on an existing aircraft. When operational aspects are considered, weight reduction at expense of induced drag becomes a debatable trade-off. The optimal solution for such a problem would be given by the spanload that provides the desired structural weight reduction with the smallest increase in drag. But this would not necessarily improve the aircraft's performance. However, this approach to aero-structural efficiency does not essentially represent Prandtl 1933 proposal. Alternatively, such an approach could be chosen if the design trade-off is to be made within the traditional vision of wing equivalence. The standards[58, 59] define that two wings will be equivalent if they have an equal exposed wing area, tip chord and fixed gross span. But this definition arises from the fact that majority of available methods to predict wing aerodynamic properties are predominantly based on tapered planforms. Thus, this guideline fulfils the need to make allowances for shape variations, such as cranks, and its implementation is done through the geometric representation of such wings in their equivalent tapered form. However, it could be argued that this traditional definition, which is only a function of geometric parameters, is not suitable in the context of NELD. Instead, it appears more sensible to define wing equivalence in relation to the parameters that are to be compared. NELD wing designs are aerodynamically and structurally equivalent because they yield the same prescribed aerodynamic and structural properties. Equivalence might therefore become a malleable criterion that is shaped to provide the appropriate comparison required in each analysis.

A major benefit of the equivalent extended wing lies in the fact that induced drag is reduced while structural penalties are minimised. But wing extension comes in other forms. Wingtip devices provide the means to achieve similar aerodynamic improvements to those gained by increasing the wing aspect ratio, but also entail many of the trade-offs discussed above[60, 61, 62, 63, 64, 65]. The value of wingtip technologies excels in their use as retrofitting devices, retaining the design aerodynamic and handling characteristics while avoiding the redesign of the wing. The type of wingtip device that best suits a given aircraft will depend on various factors, such as the spanload carried by the baseline wing, airfield gate and manoeuvring limitations, on the convenience for wing structure modifications, as well as the type of design mission of the aircraft. Anyhow, careful design of twist and camber is necessary to extract significant improvements from the integration of this device[66, 67, 68, 69, 70, 71]. Generally, planar wingtip extensions can provide higher reductions in induced drag than winglets of similar size. However, planar extensions induce higher loads on the wing structure, which will necessitate reinforcement unless combined with a wing lift distribution that alleviates such loads. Aerodynamic and structural features become again trade-offs between added weight and loads, which in turn will determine the concessions made on performance and handling characteristics.

4. Induced drag

Drag prediction is a challenging task that has amassed many methods to estimate its magnitude and just as many others to minimise its presence. Generally, drag forces manifest themselves by altering the balance of momentum and pressure, while conservation of energy is achieved by an increase in heat and kinetic energy. The latter is particularly specific to induced drag, which is associated with the motion of large air masses surrounding the wing and characterised by a downwards motion perpendicular to the direction of flight. This phenomena can be treated in both the near-field and far-field planes. The near-field requires surface pressure integration and the far-field approach necessitates wake contour integration in the Trefftz plane[72]. Each of the methods carry their corresponding assumptions and limitations. The problem is exacerbated by the fact that any attempt of validation requires a separation of total drag into its subcomponents (impossible to do with current flight test techniques).

Theoretical derivations differ in their physical conception but work accordingly within theoretical fluid mechanics, and have been used in the analysis of planar and non-planar lifting surfaces[73, 74, 75]. Analytical methods, however, not always address the accuracy of their predictions[76, 77, 78], although various methodologies have achieved improved agreement[79, 80], while also considering higher order terms neglected in the classical theories[81]. Earlier methods for approximate solutions still show favourable predictions when compared to later procedures developed for accurate computation of minimum induced drag[61, 66, 82]. These discrepancies suggest that previous theories assumed excessively simplified boundary conditions. Lifting surface theories such as the vortex-lattice method include chordwise and spanwise effects, and panel methods also consider wing thickness. However, induced drag predictions from these latter methodologies are usually sensitive to discretisation[83, 84, 85, 86].

Although immensely brief, this overview on induced drag intends to recall the complex phenomena that we are trying to address. The methods and tools available have been successful in its estimation, but there exists a persistent strive to find the right mathematical approach to translate such physical events into an expression that allows further understanding and manipulation. For more detailed discussions on induced drag, the reader is the directed to the references [87, 88, 89, 90, 91, 92, 93] .

4.1 Induced drag in NELDs

Countless efforts have been made to improve aircraft performance through induced drag reduction. Nowadays, most solutions deviate the spanload from the elliptic ideal, which suggests that our baseline theory is not the most efficient in real flight. Recent trade-off concepts have led to incremental but successful gains in performance, including planform optimisation, highly non-planar surfaces, and diverse tip devices. Still, this undeniably leads to compromise, either penalising on weight, increasing systems complexity, or reducing internal airframe space, and it is to the judgement of the designer to determine which sacrifice will be less detrimental to satisfy the mission specifications. Nevertheless, non-elliptic spanloads seem to promise a more favourable deal in the trade between induced drag and weight.

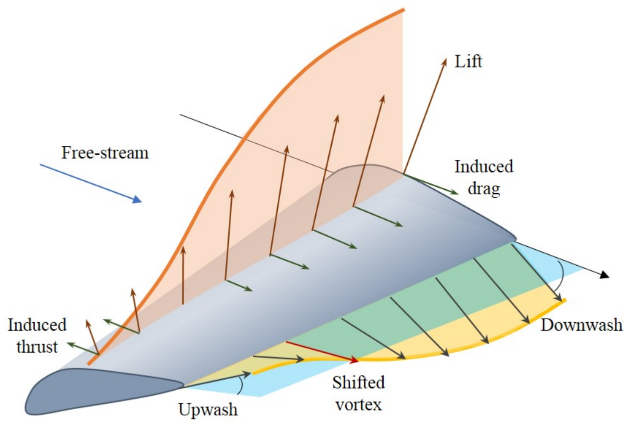


Figure 2 – Representation of the aerodynamic forces on a wing with non-elliptic lift distribution. Reproduced from Reference [94].

The optimal bell-shaped curve is associated with a span extension, which reduces induced drag. To relieve the increased structural loads derived from the extended span, the centre of lift is shifted towards the root. Effectively, the lift is redistributed to be locally high at the inboard wing and have lightly loaded tips. Here, upwash is developed, generating local negative induced drag, or induced thrust. But induced thrust, as many experts say, is one of those things that one would rather sweep under the carpet. The division between the concepts of drag and thrust may vary with the reference frame or context used, and such a divergence is typically subjected to theoretical simplifications. The projection of aerodynamic forces into the flight direction is classified as drag, and excludes thrust forces produced by the engine. Within this framework, negative induced drag could be understood as a propulsive self-induced aerodynamic force, and could be represented by the sailboat analogy used in the context of winglets. This remarkably popular definition of induced thrust depicts the winglet as a sail that takes advantage of the distorted inflow at its tip to produce lift, in the same manner as a sailboat navigates through the sea. The net resultant force will sometimes have a forward component in the reference frame of the moving body. This is occasionally identified as thrust. Although it can be a matter of nomenclature convention used to simplify the explanation of this phenomena, it is just lift and drag forces generated locally on a portion of a lifting surface. But it is misleading to state that winglets use the energy from the wingtip vortex to reduce induced drag by generating thrust. A winglet will not simply reduce the strength of the wingtip vortex, its presence will alter the three-dimensional flow-field at the wingtip as projected on the Trefftz plane and will redistribute the whole spanload so that its expended kinetic energy is reduced. Drag can be regarded as the contribution from the dissipative effects of the flow, and induced thrust could be assigned as a contribution to propulsive forces. However, fuel burn will be reduced not due to the apparent thrust generated by the winglets, but because the modified spanload will expend less energy. Winglets will combine an appropriate toe angle, aerofoil camber and height for the wing to develop optimal loads. Likewise, wings with NELDs redistribute the three-dimensional flow-field by means of a careful wing twist design. A similar outcome is therefore obtained, where apparent thrust can be found at the outboard wing. Horten[95] explained it as the induced propulsion at the wingtips shown by bell-shaped lift distributions, which is favourable with aileron deflection. Bowers[94] describes it as the resultant force locally tilted upstream as a consequence of the induced upwash, represented in Figure 2.

The treatment of total forces intuitively simplifies the isolation of local contributions, where negative induced drag will be subtracted from the total drag and never added to the engine thrust. However, even this distinction is penalised by, for instance, the presence of spurious drag in high-fidelity methods, and the classification of drag-producing mechanisms becomes even more complex[96, 97, 98]. Furthermore, this problematic dichotomy can reach the fundamental definition of induced drag, as given by Equation 16. Thus, arisen from dissatisfaction with the underlying theory, some have tried to demonstrate or disprove mathematically the existence of negative induced drag. Spalart[99] raised essential questions on the analysis and evaluation of induced drag as defined in the Trefftz plane due to the paradoxical negative squared term of the perturbation velocity u^2 along the wake in the following equation:

$$D_i = \frac{1}{2}\rho_\infty \int_{S_{far}} \left(v^2 + w^2 - u^2\right) dS \tag{16}$$

whereas intuitively, the integral for induced drag should contain the expression for kinetic energy given by the perturbation velocities (u, v, w). Such a definition appears to allow the prediction of negative induced drag, since the integrand is not positive-definite, and thus the perturbation along the wake could overcome the velocities perpendicular to it. Moreover, this expression adds complexity to the frequently used formulation that only considers the perturbation of kinetic energy within the plane perpendicular to the free-stream:

$$D_i = \frac{1}{2}\rho_\infty \int_{S_{far}} \left(v^2 + w^2\right) dS \tag{17}$$

This simplification is derived from the assumption that the trailing vortices far downstream cannot induce velocities along their own axis, which now follows the free-stream direction, and thus any parallel velocity perturbation must fade. Spalart provides a rigorous assertion that removes the ambiguity in the negative sign of u^2, while deriving a mathematical proof for consistent positivity of induced drag within a set of idealised but generally valid assumptions. Yet, he believes that a concise mathematical and physical description of induced drag is still unconcluded, and is thus imperative. The complex formulations can be derived into simple and desirable results, for which physics can be justified through reasoned language, but which do not fully comply with the mathematical grounds in which they are established. This was the case of a particular approach to Prandtl's small-perturbation wing theory from a force-impulse balance perspective[100, 101, 102, 103]. Here, a particularly provocative outcome resulted in the static pressure at the Trefftz plane being greater than the atmospheric pressure, which would produce small forward increments of velocity in the wake. Thereof, concepts such as forward flow of energy or wake buckling were derived. Although it is counter-intuitive for such a phenomena to occur given the association of induced drag with kinetic energy, it is possible to find regions of positive perturbation pressure behind a lifting wing. However, a combination of an unsuitable assumptions had led to this surprising interpretation of conditions downstream[99, 104, 105]. It is clear that any appearance of negative induced drag is usually received with curiosity and scepticism, as it ignites the need to give a reason for and resolve this oddity, regardless of how desirable its existence may be.

4.2 Wing wake, downwash and upwash

Sections 2 and 3 convey the need for accurate prediction of downwash and upwash distributions to estimate the aerodynamic characteristics of a finite wing. Furthermore, the problem of induced drag minimisation could be posed as the manipulation of the flow-field around the wing, since induced drag can be viewed as a by-product of the downwash associated with the vortical wake.

The downwash distribution induced by NELDs is determined by the degree of the functions that compose the problem conditions, which is quadratic for an integrated bending moment constraint and linear if the root bending moment is prescribed. The gradient of the bell-shaped circulation distributions approaches zero at the wingtip, which features a weak circulation strength, and sheds stronger vortices inboard. With this, it can be interpreted that the downwash at the vicinity of the wing root can overcome the downwash at the wing tip, transforming it into upwash[94, 106]. Conversely, the gradient of an elliptic spanload approaches infinity at the tip to satisfy a null circulation, leading to an immediate formation of strong wingtip vortices. Figure 3 illustrates the contrast between both downwash distributions. The wake characteristics of wings with NELDs could be observed recently at NASA after a set of flight tests carried out under the direction of Bowers[94, 107]. The inboard vortex roll-up and the outboard upwash were captured during flight by streamers attach on the trailing edge of the flying wing. The fundamental design approach followed Prandtl's 1933 theory and was inspired by the Horten aircraft and aimed to gather experimental data on the performance and handling qualities of such configurations.

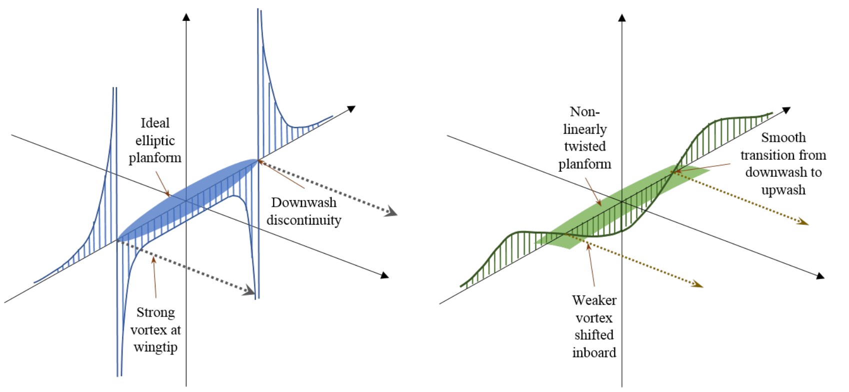


Figure 3 – Diagram comparing the downwash induced by an elliptic circulation distribution (left) and a bell-shaped circulation distribution (right). Reproduced from Reference [94].

Induced downwash is expressed as the vortex strength induced by the trailing vortex sheet. Thus, variations of the self-induced velocity profile behind the wing will dictate the roll-up behaviour of the wake vortex sheet. A recent comparative study[108] was performed on the wake characteristics of elliptic and NELD configurations, with the use a RANS approach at the near-field and free-wake simulations at the far-field. The results show an unconventional wake development for the aero-structurally optimised wings, which agree with past studies on wakes shed by similar spanloads[109, 110, 111]. Firstly, Prandtl and Jones' configurations featured a substantial delay in wake roll-up (approximately 6 and 12 times farther respectively, compared to an equivalent elliptic spanload). Moreover, the roll-up rate behaved correspondingly to the quadratic and linear downwash distribution characteristic of each configuration. Secondly, the trailing vortices of the non-elliptic configurations did not develop at the wingtips but were formed further inboard. Thirdly, the vortex span was smaller in the wake of the non-elliptic spanloads. However, it was identified that the separation of the vortex cores corresponded to the distance between the maxima and minima of the shed vortex strength, multiplied by the ratio of the vortex span to the wingspan of the elliptic configuration. The vortex mitigation achieved by means of non-elliptic spanloads can challenge existing methods for passive alleviation of the vortex roll-up process. These results could be beneficial in the context of formation flying and address other operational interactions.

4.3 Span efficiency factor and performance

In the search for the spanload for minimum induced drag, it is customary to quantify the aerodynamic efficiency of a given lift distribution in terms of the span efficiency factor e. In lifting line theory, it is derived from the expression for induced drag, given by:

$$D_i = \frac{2L^2}{\pi b^2 \rho_\infty V_\infty^2}\left[1+\sum_{n=2}^{\infty}\frac{nA_n^2}{A_1^2}\right] = \frac{2L^2}{\pi b^2 \rho_\infty V_\infty^2}\left[1+\delta\right] = \frac{2L^2}{\pi e b^2 \rho_\infty V_\infty^2} \tag{18}$$

for a fixed wing area, and is obtained from the general Fourier series formulation for the circulation distribution. Hence, the expression for e becomes:

$$e = \frac{1}{1+\delta} = \frac{1}{1+\sum_{n=2}^{\infty}\frac{nA_n^2}{A_1^2}} \tag{19}$$

Here, the inviscid span efficiency factor measures the departure of a given spanload from the elliptic solution[112], as a result of the contribution of the additional non-zero Fourier coefficients A_n to the circulation distribution. Within the classical lifting line theory framework, the elliptic spanload provides the minimum value of δ, as it is defined by only the first Fourier coefficient A_1 and $A_n = 0$ for $n > 1$. Any other solution will yield $A_n \neq 0$, which deviates the spanload from the elliptic curve and will necessarily yield $\delta > 0$ (or $e < 1$) and will increase induced drag. This criterion initiated the search for wing configurations other than the elliptic planform that could provide the closest value to $e = 1$. Glauert[113] showed that a straight tapered wing with a taper ratio of $\lambda = 0.40$ would be the most advantageous. Nevertheless, as theory and technology were developed along with the endeavour to increase efficiency, many unconventional designs emerged that included highly non-planar configurations. The analysis of such wings required a more generalised expression for the span efficiency factor to account for arbitrary geometries. This could be derived through Munk's optimal normalwash solution, which was obtained as proportional to the cosine of the dihedral angle. A wide range of configurations were found to provide $e > 1$, including wings with an extensive variety of tip forms, box wings, V-wings, etc[61, 114, 115, 116].

Now, it was possible to obtain values of span efficiency higher than the theoretical maximum, despite the contradiction with the original formulation. However, these favourable results would be overcome, among other factors, by the increase in profile drag or weight. Considering that weight is a cardinal determinant for efficient flight, it seems rather inconvenient that it is not integrated in one of the fundamental metrics for efficiency. Spanloads such as the bell-shaped distribution provide reductions in induced drag by outweighing their reduced aerodynamic efficiency with a higher aspect ratio. Yet, this does not reflect that weight has not theoretically increased with span extension, fact that could be considered more efficient from an aircraft performance standpoint. The classical expression for span efficiency factor e is a result of the specific constraints defined for a particular variational problem: a straight wing with a given fixed lift and fixed span; and it conveys how much a spanload will vary from the optimal solution of this particular problem. However, when the lifting line is extended to consider structural weight, the resulting expression for induced drag is now different from Equation 18, and it will vary for each modification of the assumed structural constraints.

Phillips and Hunsaker[35] put this approach into practice to demonstrate the significance of such considerations. The corresponding analytic expressions for induced drag are derived, together with their pertaining efficiency factors as a function of Fourier coefficients. With this, Prandtl's 1933 optimal circulation distribution, shown in Equation 4 and having $\mu = -\Gamma_2/\Gamma_0 = 1$, can be rewritten in trigonometric coordinates as:

$$\Gamma(\theta) = 2bV_\infty A_1 \left[\sin\theta - \frac{1}{3}\sin 3\theta\right] \tag{20}$$

Comparing Equation 20 to the generalised Fourier series expansion for circulation[3], and defining $B_n = A_n/A_1$, the Fourier coefficients for $n > 1$ are now $B_3 = -1/3$ and $B_{n>3} = 0$. Introducing B_3 into Equation 19, which assumes fixed gross lift and span, reveals a lower span efficiency factor than that of the ideal elliptic spanload. This result does not indicate that the drag will be smaller due to the extended span. A more suitable criteria could therefore be to have the highest value of the product b^2e, following Equation 18. Yet, this measure would not consider structural equivalence. To do so, structural parameters must be present in the derivation of the expression for aero-structural efficiency. Nonetheless, it must bear in mind that the choice of constraints remains arbitrary and bound to the assumed critical design points, as to encompass all the structural variables is, again, an unattainable task that demands compromise. For instance, one could fix weight and maximum stress on the structure due to limit loads on a rectangular wing, and assume the integrated bending moment to be proportional to the weight so that the proportionality factor is constant. The integrated bending moment could be defined from its dependency on the aerodynamic loads as well as the weight distribution, which could be a function of the wing structural weight, the total aircraft weight and the specified maximum load factors. From these assumptions it derives that the wing structural weight is a function of B_3 only, and that this coefficient can be traded with span to keep the weight to a minimum while reducing induced drag. The corresponding expression for induced drag would yield:

$$D_i = \frac{K_g W_r}{16\pi\rho_\infty V_\infty^2} \frac{W^2}{K_s W_s} \left(1 + B_3 + 3B_3^2 + 3B_3^3\right) \tag{21}$$

Because all Fourier coefficients contribute to induced drag, this study assumed that $B_n = 0$ for $n \neq 3$. Thus, the expression for induced drag is composed of a constant term, which is a function of various weight parameters, including the aircraft's total weight W, the proportionality factor K_s and dynamic pressure, multiplied by a new efficiency function that depends solely on B_3. Now, if values of B_3 that yield any negative circulation are likewise excluded, the minimum of Equation 21 corresponds to Prandtl's 1933 solution, and can be visualised in Figure 4.

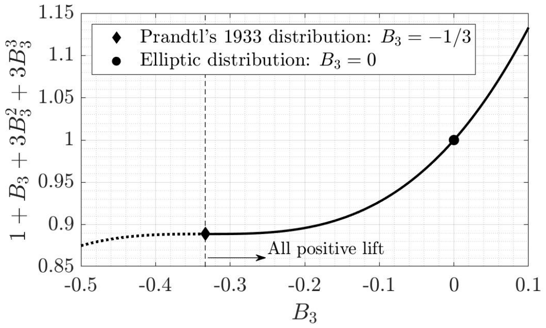


Figure 4 – Induced drag efficiency factor from Equation 21 as a function of B_3 on a rectangular wing with fixed weight, maximum stress, and constant K_s. Reproduced from Reference [35].

As aforementioned, this is just one arbitrary solution subjected to some arbitrary set of constraints. If, instead of the above conditions, the weight, maximum stress, and wing loading were to be fixed for a same structural model and weight distribution, the resulting induced drag would now be a function of:

$$D_i = f\left(\left(1+3B_3^2\right)(1+B_3)^{2/3}\right) \tag{22}$$

The coefficients that minimise the above expression are now $B_3 = -3/8+\sqrt{9/64-1/12}$ and $B_n = 0$ for $n \neq 3$, the trend of which is shown in Figure 5. A final design approach presented by Phillips and Hunsaker evaluates maximum deflection constraints under the previous structural and weight assumptions. The corresponding expression for induced drag as a function of B_3 is shown in Equation 23, and its trend is illustrated in Figure 6.

$$D_i = f\left(\left(1+3B_3^2\right)(1+B_3)^{1/3}\right) \tag{23}$$

In this case, the solution for minimum induced drag is given by $B_3 = -3/7+\sqrt{9/49-1/21}$ and $B_n = 0$ for $n \neq 3$.

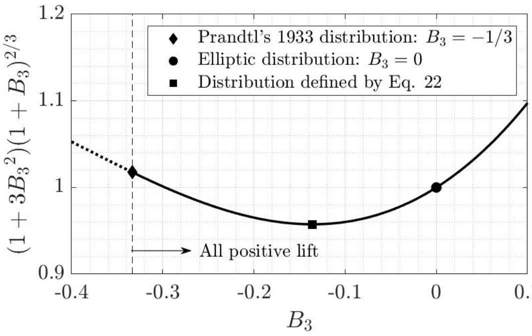


Figure 5 – Induced drag efficiency factor from Equation 22 as a function of B_3. Extracted from Reference [35].

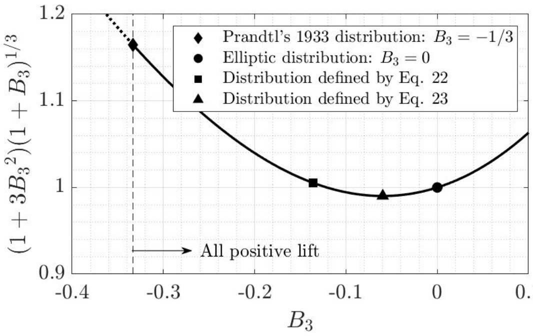


Figure 6 – Induced drag efficiency factor from Equation 23 as a function of B_3. Extracted from Reference [35].

Figures 4 to 6 show the variation of the induced drag efficiency factor as the spanload changes with the third Fourier coefficient. These results manifest how each particular problem yields its own optimal solution in accordance with the chosen set of constraints. Moreover, the above expressions for induced drag are independent of the span, which has become a function of the prescribed parameters and the spanload, specified by B_3. Hence, when structural considerations are incorporated in the lifting line theory, the trade-off between span and weight is balanced through B_3. For a more extensive discussion of this approach the reader is directed to Reference [35].

The derivation of alternative efficiency factors can provide significant insight into the aero-structural sensitivities of a particular design problem at a conceptual design level. Such an analysis allows the quantification of the trade-offs between aerodynamic and structural coupling independently of the span. In other cases, it could further include considerations such as coupling between stability and performance[117]. Efficiency metrics serve as a means to provide absolute comparisons in relation to a defined set of parameters. However, there are many other factors that are neglected, and the comparison is often subjected to the optimal solution of a specific variational problem. Therefore, it would be appropriate to consider the use of further comparative indicators for wings designed under the same underlying considerations. However, an overall comparison of efficiency might require alternative approaches. In such cases, the lift-to-drag ratio L/D is widely chosen in the juxtaposition of overall aerodynamic performance of any aircraft configuration. Its extensive use is due to the simple concept that it carries, in which the invariable goal of achieving the most lift with the least drag can be quantified at any flight condition. Furthermore, it is directly linked with the Breguet range equation, which interconnects the main parameters that define a mission. However, this metric does not explicitly incorporate the trade-off between induced drag and weight. Hence, there is still the need to define a metric that captures aero-structural efficiency as a distinct parameter.

5. Flying with non-elliptic lift distributions

5.1 Early efforts of the Hortens

Not much experimental research has been carried out on wings with NELDs. The pioneers in this field are the Horten brothers, whose strong advocacy of flying wings prevails in their legacy of enigmatic aircraft designs. Turbulent times had ignited the struggle for aeronautical innovation, and the attractive possibilities that flying wings had unveiled was enough justification for them to dive into the intricacies of such unconventional designs[118, 119]. The Hortens devised their aircraft through intuition rather than on a theoretical basis. The challenge was addressed under the terms of handling qualities by exploring planforms with careful non-linear wing twist distributions. To do so, they used the Horten H.II[120] to assesses the aerodynamic characteristics of various circulation distributions, which included various trigonometric curves, as well as the Normal and the elliptic distribution for a given lift, fixed span and fixed wing area. The non-elliptic configurations would show lightly-loaded wingtips, particularly the bell-shaped distribution, which was especially beneficial for tailless aircraft to prevent wingtip stall. Nevertheless, other trigonometric curves provided either higher induced drag or poor handling qualities that demanded great effort from experienced pilots.
Thus, the anticipated spanload, coined by them as 'bell-shaped', provided a downwash distribution that would smoothly become upwash at the outboard wing while the lift remained positive. With this, it was possible to produce induced thrust at the wingtips. Furthermore, it could be used to manipulate the aircraft's yawing moment to make a coordinated turn without the need of any vertical surface. In effect, a combination of the upwash induced at the outboard wing together with an appropriate aileron sizing could yield a drag differential that would cause a favourable yawing moment (proverse yaw). Following this design philosophy, the Hortens conducted experimental research and observed the behaviour of wool tufts along the wing, which pointed in the desired flight direction during turns. With this, the previously required rudder correction would be reduced or even become unnecessary, and so the structure of vertical stabiliser could be significantly reduced. However, the achievement of proverse yaw was critically dependant on the position of the centre of gravity, and these tests were also used to determine its correct position. Flying wings are strongly affected by the location of their centre of gravity, which must closely correspond to the centre of lift in the longitudinal direction to prevent large moments being induced in a flow separation scenario. Herein, the bell-shaped spanload became especially beneficial, since it would readily satisfy this requirement. Moreover, the consequent inboard shift of the centre of lift would improve lateral-directional characteristics[114]. During the Horten H.III project, variable washout was implemented by operating several control surfaces at different deflection angles. In this way, the necessary lift distribution could be attained more accurately. This design vision allowed the Hortens to claim that the aileron and rudder could be completely independent and that it was possible to achieve proverse yaw.

5.2 Proverse yaw implications

For the Hortens the elimination of adverse yaw became one of the primary justifications of adopting the bell-shaped lift distribution. Adverse yaw is usually solved with the aid of flight control systems that couple aileron and rudder deflections. By making use of additional drag, the manoeuvre becomes an aerodynamically inefficient exercise. Still, this energy expense has not been considered significant enough for engineers to invest time and effort. However, various amending devices have been conceived to reduce the magnitude of adverse yawing moments, often at the expense of higher induced drag and only effective within a limited range of the flight envelope. Here, the trade-offs arise from the attached mechanical complexity, cost increase and weight penalty[121]. Along with flying wings, adverse yaw can be particularly augmented in aircraft with high aspect ratio wings. Even though the longer moment arm reduces the magnitude of the loads required to roll, a full rudder deflection is sometimes still insufficient to compensate for adverse yaw.

Theoretical research on spanloads that minimise drag during turn are scarce. However, the potential benefits of NELDs on aircraft handling qualities has drawn inquisitive minds towards the investigation theoretical approaches for synthesised roll control and adverse yaw elimination. Nickel used lifting line theory to investigate the existence of load distributions that could yield null yawing moment and minimum induced drag for any prescribed lift and rolling moment[122]. This theory was tested experimentally with the construction of the Falter 1, an ultra-light glider with no vertical surfaces or drag

rudders. Nickel wanted to proof that the motion around the vertical axis could be controlled by an adequate distribution of induced drag caused solely by the deflection of horizontal control surfaces. The design had two independent ailerons on each semi-span that would be covered with a single fabric to produce a morphing effect. Such design enabled the pilot to hold control around the three axes, where turn initiation would yield no adverse yaw and steady sideslip would be achieved without any opposing moments.
Hunsaker et al. have recently carried out theoretical research in this field, building upon Prandtl's lifting-line theory and following a similar approach as that of Nickel. Specifically, the control of lateral-directional flying characteristics was addressed through wing twist design, in which elliptic and non-elliptic loadings were contrasted in the context of pure rolling motion. With this, the theory explored contraints on turn initiation, steady rolling rate and transition[123]. Analytic solutions confirm that a symmetric non-elliptic spanload must be superposed to the baseline lift distribution to achieve proverse yaw, and can be implemented either in the clean configuration or through trailing edge deflection. Moreover, this work concluded that no antisymmetric spanwise loads that can be added to an elliptic spanload to produce proverse yaw[124], as derived from this analytical approach. Secondly, for each asymmetric contribution with minimum drag there exists a symmetric load that will also yield minimum induced drag[34]. Third, the induced drag present during roll will always be higher than the induced drag of the clean configuration. As discussed in previous sections, any non-elliptic lift contribution to a fixed wingspan will increase induced drag. Hence, the more elliptic is the symmetric contribution to the roll, the more aerodynamically efficient the manoeuvre will be for a fixed wing. On the other hand, the more non-elliptic, the more roll power and proverse yaw it will have[125].
These solutions are effectively applicable on wings equipped with morphing technologies that enable a smooth and continuous twist distribution. On the other hand, the use of discrete ailerons can yield a set of solutions that satisfy the elimination of adverse yaw for a given spanload. Hence, an aileron design methodology was developed to provide insight into the impact of aileron sizing on the induced drag distribution to provide proverse yaw[126]. This preliminary study assessed the lateral-directional static stability characteristics of a conventional sailplane. The baseline configuration carried an elliptic lift distribution and was compared to modified models, in which equivalent wings with NELDs were retrofitted. In each case, the ailerons had been sized under the constraints of roll authority and proverse yaw. Therefore, the stability and control derivative that quantifies yawing moment due to aileron deflection was regarded as the metric for proverse yaw. The methodology provided a design space of aileron geometries for the NELD configurations that satisfied these conditions, whereas no solutions were obtained for the conventional elliptic wing. It is generally believed that the cause of proverse yaw is brought by induced negative drag increasing with downwards aileron deflection[94]. This study showed that this is true for small deflections. However, larger aileron deflections increase drag in both up-going and down-going motions, contrary to the induced drag differential observed in any conventional wing. Instead, the additional drag contribution in each semi-span is almost identical, being marginally bigger on the wing inside of the turn and consequently producing a favourable yawing moment. Moreover, the increase in induced drag per area was found to be lower in relation to that given by the elliptic case.
Proverse yaw was recently observed at NASA during a large series of flight tests performed with the PRANDTL-D flying wing[94]. Here, Bowers and his team observed improved flying characteristics and coordinated flight over a broad range of flight conditions. Onboard instrumentation data showed that both rolling and yawing motions where headed in the same direction during turn initiation. With this, they defend that these designs can attain induced thrust and proverse yaw[127], improving efficiency over a wider region of the low-subsonic flight envelope.

The above theoretical and experimental results suggest that the design of wings with NELD can enable dynamic control with very small or absent vertical control surfaces[128, 129, 130, 131, 132]. Vertical tail design is directly linked to the aircraft's directional stability and control, which are always coupled with the lateral dynamics. Certainly, the elimination of the vertical surfaces will lead to unconventional approaches. It could be advantageous then to investigate the optimal compromise between proverse yaw, induced drag and weight derived from the trade-off between spanload and rudder sizing. The penalties associated with this surface can range from 5 to 10% in drag, and can add around

25 to $50kg$ per m^2 of projected side area[50, 133]. Nevertheless, vertical tails compensate these losses through their reliability in providing directional control and stability and yaw damping. Moreover, they ensure safety during a crosswind landing or, more critically, a one-engine-out scenario, which usually drives its design and sizing[134] on any conventional aircraft.

6. Current trends in conceptual design

Today the aerospace sector is predominantly focused on increasing efficiency to reduce fuel consumption and emissions. Developments are being achieved through incremental improvements in the various engineering disciplines, whilst the task of conceptual design is getting ever more complex in attempts to de-risk potential solutions[135, 136, 137]. Within the MDO framework, the conceptual design stage traditionally employs a range of empirical and low-fidelity models to reduce computational cost[138, 139, 140, 141, 142, 143] and puts forward high-level design parameters that will be fixed and passed onto the preliminary design stage. Unplanned factors and decisions may also be introduced at later stages, with significant development implications that lead to trade-offs between aerodynamics, structures and systems. It is in this context that the characteristics of NELD presented above can become beneficial, revolving around the provision of further insight on wing aero-structural trade-offs while enlightening the relevant complexities of the design process. In a sense, the implementation of this approach is a proposal of return to the essence of wing design in order to reach the revolutionary vision for the future of aviation and civil aircraft design.

Aero-structural wing design is one of the most complex and costly processes in MDO. Thus, a better synergy between these disciplines at early stages has proven to be beneficial[144, 145, 146, 147, 148], where the related design space is usually highly constrained. Ideally, such considerations should occur in parallel at each design stage and within each of the components to enable a fast and reliable process. Implementation of the theory of non-elliptic lift distributions into conceptual MDO could readily reveal many of the aero-structural compromises as well as ensure aero-structural equivalence if later modifications are to be confronted.

The established desire to develop unprecedented civil aircraft is becoming more conceivable, and is present in the vision for the near future generations of aircraft[149, 150, 151]. Aircraft configuration is regarded as a critical factor that will redefine the way in which the current technological challenges are addressed. So far, airliners have been restricted to the conventional cantilever-wing configuration due to several limitations on structural and operational aspects, which have hindered revolutionary changes in aviation. Based on the technological state-of-the-art, many trade space analyses and system architecture assessments are identifying very high aspect wings as a near-term achievable solution[152, 153, 154, 155, 156]. These proposals have now been extensively investigated and the potential benefits they can bring on fuel consumption have been established. Nevertheless, such configurations are still penalised by the complexity of the structural requirements, which become a necessity to include reinforcement that increases weight and drag. Here, the integration of the theory of NELD into the design process would allow span-extended configurations under optimal weight considerations. The increasing interest on alternative configurations, such as high aspect ratio wings or blended wing body aircraft[157, 158], aligns with the constant strive to reach a more organic flight model (in which efficient bird-like flight and manoeuvring is possible without the requirement of additional drag or a vertical surface). Moreover, part of this future vision is targeting sub-transonic cruise speeds in order to accommodate to the impending green propulsive technology. This, together with the imminent leap towards urban air mobility, can open another design niche in which the potential penalties derived from compressibility effects faced by wings designs based on NELDs will minimise. Altogether, these approaches could benefit from more efficient cruise and optimal coordinated turns that NELD seem to promise.

Solutions to the problem of time-varying lift distributions have become more relevant as materials, manufacturing and control systems technologies are developed towards the removal of discrete control surfaces. Morphing wings enable key geometric parameters to be changed to produce optimal spanloads and alleviate loads at a wide range of flight conditions[159, 160, 161]. Many configurations[162, 163, 164] have been experimentally tested and show performance and handling qualities improvements while minimising airflow separation. Such adaptive wings could greatly benefit from the theories discussed in this review. The methods expose the relation between spanload and its structural impact. Accordingly, a wing design under the principles of NELD could morph to produce the optimal lift distribution to satisfy the principal efficiency requirement at every flight condition, whether it is aerodynamic, aero-structural, or linked to the aircraft's flight dynamics.

7. Conclusions

Non-elliptic lift distributions promise the consolidation of wing designs with minimum induced drag, optimal structural weight and improved handling qualities. Conventionally, the requirements to reduce induced drag and structural weight are opposed, and current solutions to well-known adverse dynamics are still dependant on the use of drag to correct unwanted aircraft responses. NELDs propose a unified solution to these problems. Nevertheless, it is not straightforward to accept the preached attributes without further validation. The past shortcomings of the Horten flying wings together with traditional prediction methods, in which fixed span is readily presumed, might have narrowed the flourishing of this theory.

The work presented in this paper derives its relevance from the renewed interest of researchers and designers to challenge the fundamental assumptions underlying today's conceptual design methods in search for step improvements in aircraft efficiency. Ideas of structural equivalence together with unconstrained span, first discussed in Prandtl's 1933 paper, are revisited by the authors to propound their capabilities within the current efforts to introduce high aspect ratio wings and morphing technologies. Still, the above overview highlights that the elegance of the foundational theories remain relevant and are necessary to provide economic methods and insightful correlations between design drivers at the conceptual design stage.

Growing attention is debouching these ideas into the broader pursuit of a holistic and green approach to aircraft design. Researchers, such as Bowers at NASA, have combined these principles with observations of bird flight to propose significant benefits in terms of flight dynamic characteristics, while maintaining minimum induced drag and weight. Hunsaker et al. are integrating the theory into the search for optimal spanloads for morphing wings. Others have converged to wings with NELDs in their search for the planforms with high aero-structural efficiency. Aircraft manufacturers are also showing interest and considering the impact of this approach on in-house design tools. However, the effort to comprehend and then to transfer Prandtl's ideas remains unsettled. It is hoped that this review paper will present an opportunity for further technical debate and encourage new approaches to this alternative design philosophy.

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10. Copyright Statement

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