NEW APPROACH FOR INVESTIGATION OF MAXIMUM-RANGE AIRCRAFT TRAJECTORIES

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Abstract

The problem of maximization of the aircraft range is considered. The solution is based on the Pontryagin maximum principle. New analytical solutions taking into account constraints on the path angle rate are obtained within a framework of the approximate pseudo-conservative model of motion. The solution for the comprehensive model of motion is obtained by the homotopy method and the approximate analytical solution as the initial base solution.

1 Introduction

The maximum range problem is a classical problem in aircraft design considered in many publications. The interest in this problem is caused, on the one hand, by its importance for practice, and on the other hand, by the known objective difficulties of its solving in a rigorous statement, caused by the degeneracy of the optimization problem.

The aim of this paper is to present, following for [1], some new qualitative results of investigation of the problem on the base of the Pontryagin maximum principle [2], especially alternative solutions to the traditional steady flight.

Let us point some results of other authors, which are mainly close to the subject considered in this paper.

The main statements are as follows:

1. The range maximization with fixed fuel mass,
2. The range maximization with fixed time,
3. The range maximization with fixed fuel mass and time.

In [3] the models of motion that differs from each other by simplifications are investigated. Authors suggest the point mass model where the lift equals the weight and the flight path angle is small. The flight path angle and the thrust throttle are the control variables.

A qualitative analysis of the statements 1 and 2 is performed in [4]. The problem is considered by the energy approach, where the altitude and the thrust are the control variables.

In [5] the statement 1 is considered. It is proved that the steady level flight fails the Jacobi optimality conditions. It is also shown that the oscillatory trajectory can increase the range when compared with the level flight.

In [6]-[8] on specific examples it is shown that the periodic solutions have an advantage over the level flight. In [6] the thrust throttle and the lift coefficient are the control variables. In [7] the thrust throttle and the flight path angle are the control variables, and the flight path angle and its rate are small. In [8] solution obtained by selection of parameters of a sine law for altitude changes.

In [9] and [10] some examples of the periodic optimal solutions are demonstrated in the cruise-dash problem. The thrust and lift coefficient are the control variables here. In [10] the existence of the singular optimal control is investigated.

In [1] statements 2 and 3 are considered. The analytical solution with oscillatory optimal trajectories are obtained in the statement 2 suggesting the pseudo-conservative model of the motion with the path angle as the control variable.

This paper considers the extension of the statement [1] of the range maximization with fixed time problem. The procedure of transition from the pseudo-conservative model, where the flight path angle is the control variable, to a comprehensive model is described. This procedure is held in two stages. On the first stage the extended pseudo-conservative model is used but the lift coefficient is taken as the control variable. The analytical solutions are obtained for this statement. It is proven that the optimal control has a singular arc. Then, the transition from the pseudo-conservative to the comprehensive model, where the lift coefficient and the thrust are the control variables, is performed by the homotopy method. The numerical results are presented.

2 Problem Statement

The motion is governed by the equations [3]:

$$\begin{aligned} \dot{V} &= \frac{\mu T_{\max} - D}{m} - g\sin\gamma, \\ \dot{\gamma} &= \frac{L}{mV} - \frac{g\cos\gamma}{V}, \\ \dot{h} &= V\sin\gamma, \\ \dot{x} &= V\cos\gamma, \end{aligned} \tag{2.1}$$

where V is the velocity, h is the altitude, γ is the flight path angle, x is the range, m is the mass assumed constant, $T_{max}(V,h)$ is the maximum thrust, μ is the throttle setting, $D = C_D qS$ is the aerodynamic drag, $L = C_L qS$ is the aerodynamic lift, g is the acceleration of the gravity, $C_D(M, C_L)$ is the drag coefficient, M is the Mach number, C_L is the lift coefficient, q is the dynamic pressure, S is the reference area.

Let us use the dimensionless variables:

$$\begin{aligned} &V = \left(\frac{V}{V_i}\right)_{\dim}, h = \left(\frac{hg}{V_i^2}\right)_{\dim}, x = \left(\frac{xg}{V_i^2}\right)_{\dim}, \\ &T_{\max} = \left(\frac{T_{\max}}{mg}\right)_{\dim}, D = \left(\frac{D}{mg}\right)_{\dim}, \\ &L = \left(\frac{L}{mg}\right)_{\dim}, t = \left(\frac{tg}{V_i}\right)_{\dim}, \end{aligned} \tag{2.2}$$

where the index «dim» designates dimensional variables, V_i is the initial velocity.

Then, equations (2.1) take the following dimensionless form:

$$\begin{aligned} \dot{V} &= \mu T_{\max} - D - \sin\gamma, \\ \dot{\gamma} &= \frac{L}{V} - \frac{\cos\gamma}{V}, \\ \dot{h} &= V\sin\gamma, \\ \dot{x} &= V\cos\gamma. \end{aligned} \tag{2.3}$$

We take C_L and μ as the control variables that are subjected to the inequality constraints:

$$\begin{aligned} &C_{L\min} \le C_L \le C_{L\max}, \\ &\mu_{\min} \le \mu \le 1. \end{aligned} \tag{2.4}$$

The following boundary conditions are used. At the initial time t_i=0:

$$V(0) = V_i,\ h(0) = h_i,\ x(0) = 0. \tag{2.5}$$

At the final given time t_f:

$$V(t_f) = V_f,\ h(t_f) = h_f. \tag{2.6}$$

The functional is the range:

$$\Phi \equiv x(t_f) \Rightarrow \max. \tag{2.7}$$

3 Optimality Conditions

According to the maximum principle [2] the optimal control $u^T = \{C_L, \mu\}$ is determined by the condition:

$$u_{opt}(t) = \underset{u}{\operatorname{argmax}}\, H, \tag{3.1}$$

where $H = \boldsymbol{\lambda}^T \mathbf{f}$ is the Hamiltonian, $\mathbf{f}$ is the vector of the right sides of equations (2.3), $\boldsymbol{\lambda}^T = \{\lambda_V, \lambda_\gamma, \lambda_h, \lambda_x\}$ is the vector of conjugate variables, which corresponds to the variables of the phase vector $\mathbf{x}^T=\{V, \gamma, h, x\}$ and satisfies the vector differential equation:

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial H}{\partial \mathbf{x}}\right)^T. \tag{3.2}$$

The transversality conditions are reduced to:

$$\begin{aligned} &\lambda_x(t_f)=1,\\ &\lambda_\gamma(0)=0,\\ &\lambda_\gamma(t_f)=0. \end{aligned} \tag{3.3}$$

When $\lambda_V > 0$ the optimal control variables are:

$$C_L = \begin{cases} \tilde{C}_L, C_{L\min} \le \tilde{C}_L \le C_{L\max}, \\ C_{L\min}, \eta < 0, \\ C_{L\max}, \eta > 0, \end{cases} \tag{3.4}$$
$$\mu = 1,$$

where $\eta = H(C_{L\max}) - H(C_{L\min})$ and $\tilde{C}_L$ is obtained from:

$$\lambda_\gamma - \lambda_V V \frac{\partial C_D(\tilde{C}_L)}{\partial C_L} = 0. \tag{3.5}$$

When $\lambda_V < 0$:

$$C_L = \begin{cases} C_{L\max}, \eta > 0, \\ C_{L\min}, \eta < 0, \end{cases} \tag{3.6}$$
$$\mu = \mu_{\min}.$$

When $\lambda_V = 0$:

$$C_L = \begin{cases} C_{L\max}, \eta > 0, \\ C_{L\min}, \eta < 0, \end{cases} \tag{3.7}$$
$$\mu = \mu^*,$$

where μ^* is the singular control that satisfies (2.4). The existence of the singular arc and the chattering control ($\eta = 0$) is not considered here.

Thus, the problem consists of finding $\gamma(0)$, $\lambda_V(0)$ and $\lambda_h(0)$ to solve the two-point boundary-value problem for the differential equations (2.3) and (3.2), and the boundary conditions (2.5), (2.6) and (3.3).

The phase system (2.3) is autonomous, so the first integral takes place at the optimal control:

$$H\big|_{u_{opt}} = const. \tag{3.8}$$

Note that the same integral holds at a constant control:

$$H\big|_{u=const} = const. \tag{3.9}$$

The properties (3.8) and (3.9), as was shown in [11], are useful for verification of a mathematical model of an optimization software package.

4 The Analytical Optimal Solution for the Pseudo-Conservative Model of Motion

The analytical solution for the pseudo-conservative model of the motion has been obtained in [1].The following assumptions has been made:

- the thrust is equal to the aerodynamic drag,
- the path angle, as a fast variable, is taken as the control.

The assumptions are quite appropriate for studying the motion of aircraft with high lift to drag ratio. The first assumption is consistent with the quasistationary conditions. The second assumption broadens significantly the class of acceptable trajectories in comparison with quasistationary ones due to no restrictions on the path angle.

After these assumptions, the next analytical solution has been obtained:

$$\begin{aligned} &V = \frac{V_i}{\cos\gamma_i}\cos\left(\frac{\cos\gamma_i}{V_i}t + \gamma_i\right),\\ &\cos\gamma = \cos\left(\frac{\cos\gamma_i}{V_i}t + \gamma_i\right),\\ &L = \frac{V_i}{2\cos\gamma_i}t - \frac{V_i^2}{4\cos^2\gamma_i}\sin 2\gamma_i + \\ &\frac{V_i^2}{4\cos^2\gamma_i}\sin\left(\frac{2\cos\gamma_i}{V_i}t + 2\gamma_i\right). \end{aligned} \tag{4.1}$$

According to (4.1) the optimal trajectories are oscillatory. The local extremals (satisfying necessary optimality conditions) are not unique and their quantity depends on task parameters.

If *V* and *h* at the left and right ends of the trajectory are equal then the level flight is among admissible trajectories. However, it has been proven [1] that the level flight does not satisfy the necessary optimality conditions.

The following result of the analytical investigation [1] is of interest for the next. The aerodynamic load is always limited on the optimal trajectory independently on the task parameters:

$$\left(\frac{L}{mg}\right)_{u_{opt}(t)} \leq 2.$$

The simulation of maneuverable aircraft motion with the ideal flight director system that follows up a local extremal oscillatory control program has shown a high coincidence with the analytical solution in the range (inaccuracy is less than 1%).

5 The Analytical Optimal Solution for Extended Pseudo-Conservative Model of Motion

Let us make the same assumption as in previous section:

- the thrust is equal to the aerodynamic drag,

but assume γ as the state variable and C_L as the control variable.

Then the equations (2.3) take the form:

$$\begin{aligned}\dot{V} &= -\sin\gamma,\\ \dot{\gamma} &= (C_L qS - \cos\gamma)/V,\\ \dot{h} &= V\sin\gamma,\\ \dot{x} &= V\cos\gamma.\end{aligned} \tag{5.1}$$

We can omit the third equation since the altitude depends on the velocity due to the energy integral in frame of the pseudo-conservative model:

$$E = 0.5V^2 + h = const. \tag{5.2}$$

So we get the following motion equations:

$$\begin{aligned}\dot{V} &= -\sin\gamma,\\ \dot{\gamma} &= (C_L qS - \cos\gamma)/V,\\ \dot{x} &= V\cos\gamma.\end{aligned} \tag{5.3}$$

The Hamiltonian for (5.3) is:

$$H = -\lambda_V \sin\gamma + \lambda_\gamma (C_L qS - \cos\gamma) + \lambda_x V\cos\gamma. \tag{5.4}$$

According to (3.2) and (5.4) the conjugate system is:

$$\begin{aligned}\dot{\lambda}_V &= -\lambda_\gamma C_L S\frac{dq}{dV} - \lambda_x\cos\gamma,\\ \dot{\lambda}_\gamma &= -\lambda_V\cos\gamma - (\lambda_\gamma + V)\sin\gamma,\\ \lambda_x &= 1.\end{aligned} \tag{5.5}$$

According to the principal maximum the optimal control is:

$$C_L = \begin{cases} C_{L\max}, \lambda_\gamma > 0,\\ C_{L\min}, \lambda_\gamma < 0,\\ C_L^*, \lambda_\gamma \equiv 0.\end{cases} \tag{5.6}$$

Let us investigate the singular optimal arc with the control C_L^*. From (5.5) we obtain:

$$\dot{\lambda}_\gamma = -\lambda_V\cos\gamma - V\sin\gamma = 0 \tag{5.7}$$

or

$$\lambda_V = -V\tan\gamma. \tag{5.8}$$

Taking the time derivative from (5.7) and using (5.3), (5.5) and (5.8) we get the singular optimal control:

$$C_L^* = \frac{2\cos\gamma}{qS}. \tag{5.9}$$

Substituting (5.9) in (5.3) we obtain the analytical solution of (5.1) on the singular arc:

$$\begin{aligned}V^* &= \frac{V_1}{\cos\gamma_1}\cos\left(\frac{\cos\gamma_1}{V_1}t + \gamma_1\right),\\ \gamma^* &= \frac{\cos\gamma_1}{V_1}t + \gamma_1,\\ L^* &= \frac{V_1}{2\cos\gamma_1}t - \frac{V_1^2}{4\cos^2\gamma_1}\sin 2\gamma_1 + \\ &\frac{V_1^2}{4\cos^2\gamma_1}\sin\left(\frac{2\cos\gamma_1}{V_1}t + 2\gamma_1\right),\end{aligned} \tag{5.10}$$

where V_1 and γ_1 are the velocity and the path angle at the initial point of the singular arc correspondingly.

The solution (5.10) coincides with the optimal solution (4.1).

6 The Optimal Solution for the Comprehensive Model of Motion

The analytical solution obtained in previous section we use as the initial approximation for continuation to obtain a solution of the problem (2.3)-(2.7). For this, let us represent the T_{max} in the following form:

$$T_{\max} = C_D^{st} qS + \Delta T, \tag{6.1}$$

where C_D^{st} is the drag coefficient at the level flight with parameters (2.5). The equations of motion (2.3) we can write as:

$$\begin{aligned} \dot{V} &= \alpha_1 \left(C_D^{st} - C_D \right) qS + \alpha_2 \Delta T - \sin\gamma, \\ \dot{\gamma} &= \left(C_L qS - \cos\gamma \right)/V, \\ \dot{h} &= V\sin\gamma, \\ \dot{x} &= V\cos\gamma, \end{aligned} \tag{6.2}$$

where α_1 and α_2 is homotopy parameters.

Equations (6.2) for $\alpha_1 = 0$ and $\alpha_2 = 0$ have the analytical solution (4.1).

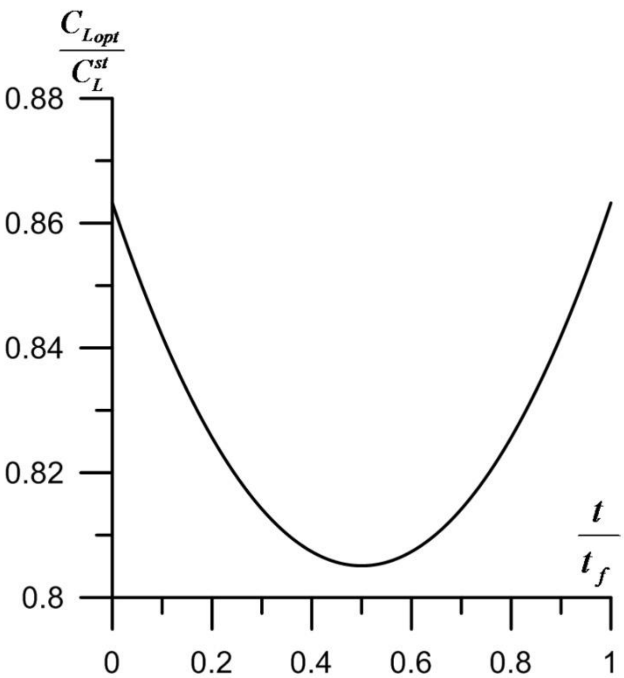

Fig.1. Normalized optimal lift coefficient history

The example of the optimal control program $C_{Lopt}(t)$ of (4.1) is shown in Fig.1. The equality of V=330 *m/s* and h=7 *km* at the left and right ends of the trajectory is assumed here. Changing α_1 from 0 to 1 we obtain the solution to the model that is close to the pseudo-conservative model, where we assumed that the thrust is equal to the drag. But now we use C_L as the control variable.

Then, assuming:

$$\Delta T = \mu T_{\max} - C_D^{st} qS \tag{6.3}$$

and changing α_2 from 0 to 1 we obtain the solution of the initial problem (2.3)-(2.7).

7 Numerical Results

Let us consider the application of the obtained results for the example of the range optimization for the aircraft F-4 [6], which characteristics in Appendix are given.

The atmosphere parameters are taken from [12].

The optimal dependences of the altitude, velocity and path angle on the range are shown in Figures 2-4. It is seen, the range maximization is achieved by acceleration due to descent.

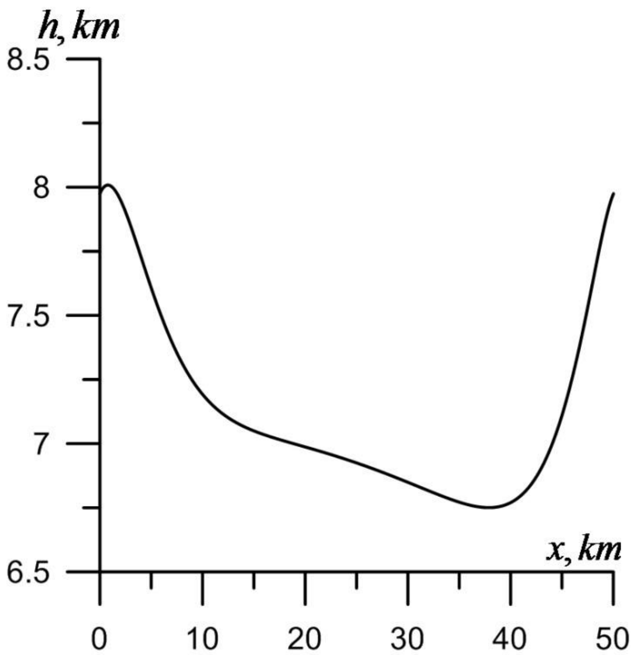

Fig. 2.The optimal dependence of the altitude on the range.

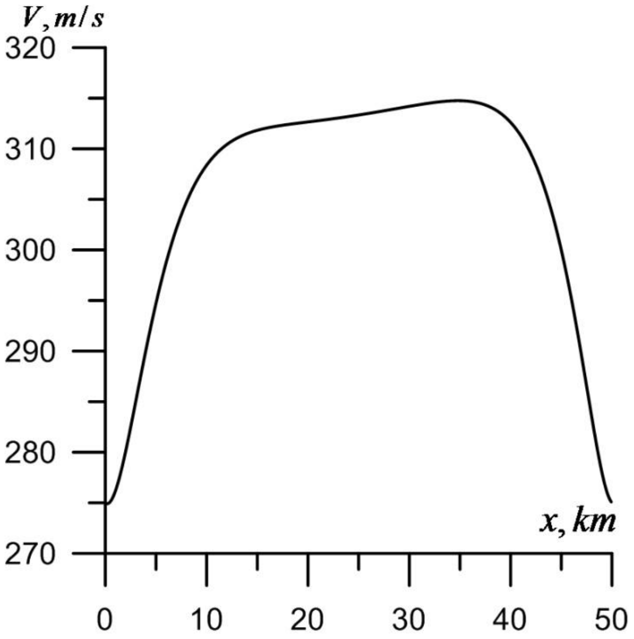

Fig. 3.The dependence of the optimal velocity on the range.

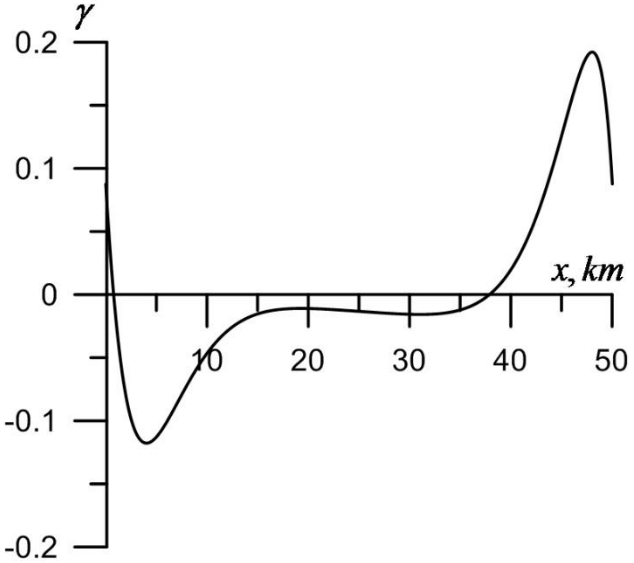

Fig. 4. Dependence of the optimal flight path angle on the range.

The dependence of optimal C_L is shown in Fig. 5 and the optimal throttle μ_{opt} is equal to unity on the whole trajectory.

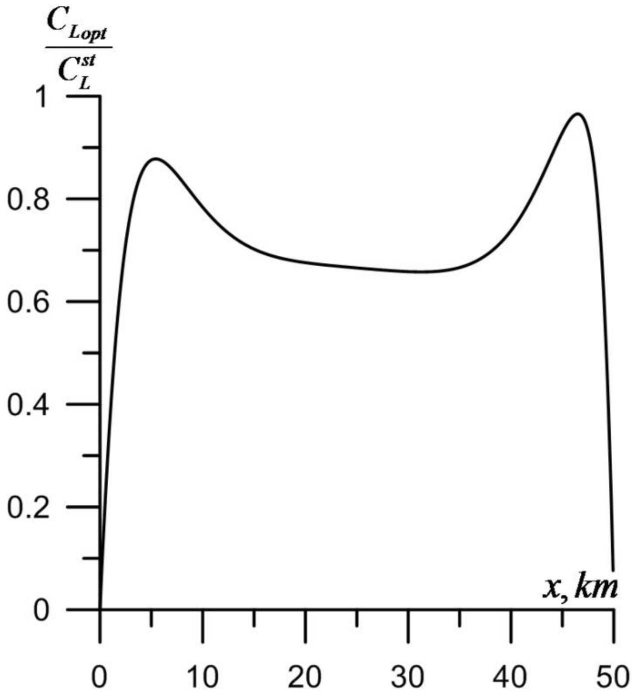

Fig. 5. Dependence of the optimal lift coefficient on the range.

The examples of changing the quantity N_{Jacobi} of iterations with calculations of the Jacobi matrix in dependence on the homotopy parameters α_1 and α_2 are presented in Fig.6 and Fig. 7. The good convergence of iterative process is shown.

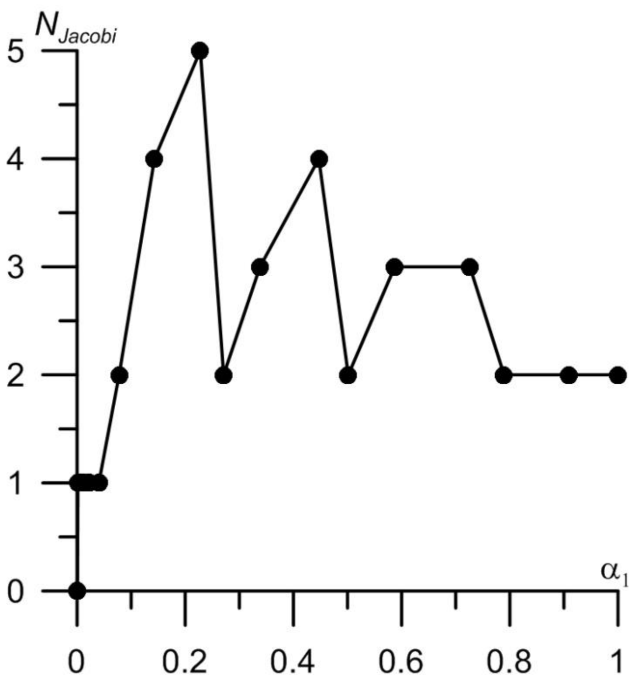

Fig. 6 The number of iteration N_{Jacobi} vs the homotopy parameter α_1.

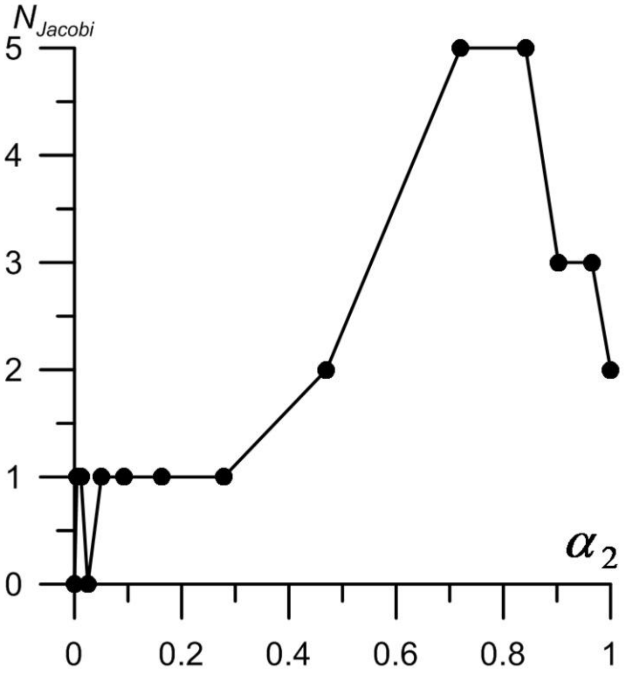

Fig. 7 The number of iteration N_{Jacobi} vs the homotopy parameter α_2 .

8 Conclusions

The new approach of solving the maximization range problem with the fixed flight time is considered. It uses the analytical solution for the pseudo-conservative model as the initial approximation to solve the optimization problem for the comprehensive model by the homotopy method.

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Appendix

The following characteristics of the aircraft F-4 are used in paper. The reference area is 49.25 m^2. The mass is 35000 *kg*.

The drag coefficient is given as function of M and C_L:

$$C_D = a_1 \arctan(a_2(M - a_3)) + a_4 + K(M)C_L^2,$$

where

$$a_1 = 0.007381240246,\ a_2 = 34.165985747,$$
$$a_3 = 0.953467778,\ a_4 = 0.0323999753,$$
$$K = 0.21702097 - 0.35536702M +$$
$$0.36538446M^2 - 0.05068006M^3.$$

The maximum thrust in Newton has the following dependence:

$$T_{\max} = 2g(c_1 + c_2 M + c_3 M^2 + c_4 M^3),$$

where:

$$c_1 = 0.4304853 \times 10^4 - 0.38000872h +$$
$$0.100436 \times 10^{-4} h^2 - 0.6250142 \times 10^{-10} h^3,$$
$$c_2 = 0.1922648 \times 10^4 - 0.7301197h +$$
$$0.5187934 \times 10^{-4} h^2 - 0.10380902 \times 10^{-8} h^3,$$
$$c_3 = -0.9983243 \times 10^3 + 0.1143555 \times 10h -$$
$$0.9524995 \times 10^{-4} h^2 + 0.2094355 \times 10^{-8} h^3,$$
$$c_4 = -0.1314679 \times 10^3 - 0.29088446\, h +$$
$$0.2665875 \times 10^{-4} h^2 - 0.6097151 \times 10^{-9} h^3.$$

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