# MULTISTAGE MISSILE TRAJECTORY AND STAGE OPTIMIZATION VIA DELTA-V APPROACH

**Seong-Min Hong*, Min-Jea Tahk***
***KAIST**

**Keywords**: *Trajectory optimization, Stage optimization, Multistage missile*

## Abstract

*This paper addresses a design optimization problem of a multistage missile during a preliminary design phase. One of the most important points of the design of long-range surface-to-air missiles is to find the minimum weight available to carry out the mission. In this study, the weight and trajectory are optimized at the same time to solve the problem. Delta-v approach, commonly used in simple analysis of rocket performance, is used to deal with simultaneous trajectory and weight optimization problems. The velocity changes during the flight is assessed considering mission constraint, gravity, and aerodynamic drag forces. Validation and verification are also conducted by comparing the results of two different optimization methods.*

## 1 Introduction

Multistage surface-to-air missiles are designed to protect friendly bases from long-range threats or to intercept targets at a higher altitude. However, design of the missiles is a great challenge because of its highly complex and complicated nature. In the beginning of the design process, there exist various rough design concepts and requirements depending on specific mission objectives. During the preliminary design phase, tedious and iterative steps are repeated to find the optimal missile configuration.

Thus, integrated tools for system level optimization are required to reduce cost and time. Some researchers have solved a missile design problem using multi-objective optimization (MOO) or multi-disciplinary optimization (MDO) [3-5]. Anderson has optimized 29 design parameters at the same time but long calculation time is inevitable due to the use of several software to calculate subsystem specifications [6].

In this paper, to reduce the calculation time, delta-v approach is used. Velocity constraint is imposed considering the velocity losses due to gravity and aerodynamic drag force. Trajectory and stage optimization is conducted including additional mission constraints. Two different optimization methods are used to verify the results. The rest of this paper is organized as follows. The optimization problem is formulated in section 2. Section 3 briefly introduces optimization methods. Then, optimization results are depicted and compared in section 4. Finally, section 5 concludes the paper.

## 2 Problem Formulation

### 2.1 Missile System

The surface-to-air missile system is composed of two propulsion stages and a kill vehicle. The objective of the rocket motors is to deliver the kill vehicle to the predicted intercept point (PIP). At the beginning, the first stage rocket motor is ignited and the missile is launched vertically. After the first boosting phase, the empty rocket motor is jettisoned and the missile glides. The second rocket motor ignites and the missile steers its direction to the target. At the end of the second boosting phase, the second stage is separated and the kill vehicle flies to the PIP.

The thrust is proportional to the weight of each stage at the end of the boost phase. Aerodynamic coefficients are calculated using Missile DATCOM. The missile is 0.4m in cross sectional diameter and 3.6m in length, which is defined as the reference shape to calculate the aerodynamic coefficients. It is assumed that even if the weight of each stage is changed in the subsequent stage optimization problem, the slenderness ratio does not change. Thus, the change of the weight does not significantly affect the change of aerodynamic coefficients.

## 2.2 Delta-v approach

The ideal rocket equations, which is derived from Newton's second law of motion, is expressed as follows [1, 2].

$$\Delta V_{missile} = v_e \cdot \ln \Lambda \tag{1}$$

where $\Delta V_{missile}$ is the maximum change of speed of the missile, $v_e$ is the effective exhaust velocity, and $\Lambda$ is the propellant mass ratio, or burn-out mass ratio. The effective exhaust velocity can be expressed by using specific impulse, and the relationship to each other is defined as the following equation.

$$v_e = g_0 \cdot I_{sp} \tag{2}$$

where $I_{sp}$ is the vacuum specific impulse in seconds, and $g_0$ is the standard gravity. The burn-out mass ratio is defined as:

$$\Lambda = \frac{m_0}{m_f} = \frac{m_0}{m_0 - m_p} = \frac{m_s + m_p + m_{pl}}{m_s + m_{pl}} \tag{3}$$

where $m_0$, $m_f$, $m_s$, $m_p$, and $m_{pl}$ are the initial, final, structural, propellant and payload mass, respectively.

For multistage missiles (or rockets), the ideal velocity increment is the sum of the velocity increment of each propulsion stage.

$$\Delta V_{missile} = \sum_{i=1}^{N} v_{e,i} \cdot \ln \Lambda_i \tag{4}$$

The missile must provide the required velocity to intercept the target with sufficient velocity. In addition, the missile loses its velocity due to various factors: gravity, aerodynamic drag, steering, ambient pressure change, and performance margin. Among them, gravity and aerodynamic drag have the greatest influence on velocity loss. From the missile kinematics in Fig. 1, the gravity loss can be calculated by integrating gravity acting on the velocity direction.

$$\Delta V_g = \int_{t_0}^{t_f} g \sin \gamma(t) \cdot dt \tag{5}$$

where $\gamma(t)$ is the missile flight path angle, which is a function of time. The gravity also varies depending on the altitude. In general, as the altitude increases, the magnitude of gravity decreases, however, this study assumes that the gravity is fixed as 9.8m/s$^2$ relative to sea level and does not decrease with altitude. When integrated, a greater value is calculated than the actual loss of velocity due to gravity, which is used as a design margin for missile performance.

The aerodynamic drag force is defined as :

$$F_D = \frac{1}{2} \rho V^2 S_{ref} C_D \tag{6}$$

Then, the velocity loss by aerodynamic force can be calculated by integrating eq. (6) divided by the missile weight.

$$\Delta V_d = \int_{t_0}^{t_f} a_D \cdot dt = \int_{t_0}^{t_f} \frac{F_D}{m(t)} \cdot dt \tag{7}$$

Finally, $\Delta V_{missile}$ must satisfy following inequality.

$$\Delta V_{missile} \geq \Delta V_{mission} + \Delta V_g + \Delta V_d \tag{8}$$

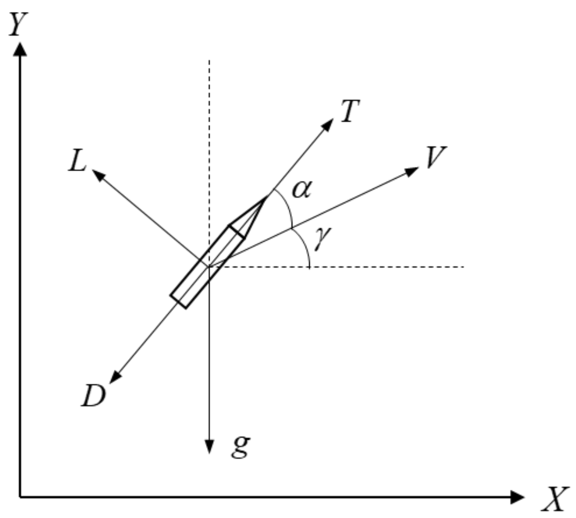


**Fig. 1 Missile Kinematics**

## 2.3 Stage Optimization

We can define the objective function for the stage optimization problem to find minimum initial weight of the missile (or gross lift-off weight, GLOW).

$$m_0 = \sum_{i=1}^{N} m_i + m_{pl} \tag{9}$$

Dividing by $m_{pl}$ and taking the natural logarithm, Eq. (9) can be expressed as follows.

$$\min_{u,\Lambda_i} J = \ln\left(\frac{m_0}{m_{pl}}\right) = \sum_{i=1}^{N} \ln \frac{(1-\varepsilon_i)\Lambda_i}{1-\varepsilon_i \Lambda_i} \tag{10}$$

where $\varepsilon_i$ is the structural ratio and it depends on material technology. Constraint for the mission is given as follows.

$$g = \sum_{i=1}^{N} v_{e,i} \ln \Lambda_i - \Delta V_g - \Delta V_d \geq \Delta V_{mission} \tag{11}$$

## 2.4 Trajectory Optimization

Assume that the missile moves along the pitch plane, then, the equations of motion for the trajectory optimization problem can be expressed as Eq. (12). Here, the missile weight is also considered as a state variable. The initial value of the weight and burning rate $c$ is predetermined in the previous stage optimization step.

$$\begin{aligned}
\dot{x} &= V_x \\
\dot{y} &= V_y \\
\dot{V}_x &= (T\cos\theta - D\cos\gamma - L\sin\gamma)/m \\
\dot{V}_y &= (T\sin\theta - D\sin\gamma + L\cos\gamma)/m - g \\
\dot{m} &= -c
\end{aligned} \tag{12}$$

where $x$, $y$, $V_x$, and $V_y$ are the downrange, altitude, $x$-directional velocity, and $y$-directional velocity, respectively. $T$, $D$, and $L$ are the thrust, drag force, and lift force, respectively. $\theta$ is the pitch attitude angle and $\gamma$ is the flight path angle. Two angles are related to each other by:

$$\theta = \gamma + \alpha \tag{13}$$

where $\alpha$ is the angle of attack, which is used as the control input in this study.

# 3 Optimization Methods

To solve optimal control problems numerically, a wide range of approaches have been developed. These methods are divided into two categories, indirect methods and direct methods [7]. In direct methods, the state and/or control of the original optimal control problems are approximated using various interpolation functions. The direct methods are re-classified according to how the discretization is conducted. When only the control inputs are approximated, the method is called a control parameterization method. When both state and control are approximated, the method is called a collocation method.

Collocation methods are further subdivided into two methods: (1) Local collocation method and (2) Global collocation method. In this paper, the given optimization problem is solved through these two methods.

## 3.1 Local Collocation Method

In local collocation, time interval is divided into a series of subintervals. The optimal solution $(x(t),u(t))$ is approximated by suitable functions for each interval, and the dynamic constraints are imposed to derive necessary algebraic equations. Among the local collocation methods, Hermite-Simpson method transcribes the states into a cubic polynomial in the interval $(t_k, t_{k+1})$.

$$\begin{aligned}
\tau &= (t-t_k)/(t_{k+1}-t_k) = (t-t_k)/h \\
\tilde{x}(\tau) &= c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3
\end{aligned} \tag{14}$$

where $\tilde{x}(\tau)$ with respect to $t$ is calculated as:

$$\frac{d}{dt}\tilde{x}(\tau) = \frac{1}{h}\frac{d}{d\tau}\tilde{x} = (c_1 + 2c_2\tau + 3c_3\tau^2)/h \tag{15}$$

For each interval, the coefficients should be calculated to satisfy the boundary conditions. Furthermore, in order to satisfy the dynamics constraints, the slope of the interpolation function at $\tau = 1/2$ should be equal to the derivative calculated from the state equation.

$$\left.\frac{d}{dt}\tilde{x}(\tau)\right|_{\tau=\frac{1}{2}} = h \cdot f\left(\tilde{x}(t_c), u(t_c), p, t_c\right) \quad (16)$$

where $c = k + 1/2$. In summary, we obtain

$$x_{k+1} - x_k = \frac{h}{6}\left(f_k + 4f_c + f_{k+1}\right) \quad (17)$$

which is well-known as Simpson's rule.

### 3.2 Global Collocation Method

Among the global collocation methods, pseudospectral methods are widely used. In this method, the states are approximated by combination of interpolating polynomials and collocation is performed at the chosen points. The exact number of polynomials depends on the collocation points, but in general,

$$x(\tau) \approx \mathbf{X}(\tau) = \sum_{i=1}^{N} \mathbf{x}(\tau_i) L_i(\tau) \quad (18)$$

where $(\tau_1, \cdots, \tau_N)$ are the $N$ collocation points. The cost is approximated as

$$J \approx \frac{t_f - t_0}{2} \sum_{i=1}^{N} \omega_i L\left[\mathbf{x}(\tau_i), \mathbf{u}(\tau_i), \tau_i\right] \quad (19)$$

where $\omega_i$ $(i = 1, \cdots, N)$ are the weights. The resulting collocation equation forms also depend on the collocation points, in general,

$$\mathbf{D}_k \times \mathbf{X}_k = \frac{t_f - t_0}{2} \mathbf{F}(\mathbf{X}, \mathbf{U}) \quad (20)$$

After a continuous infinite dimensional optimal control problem is transcribed and converted into a nonlinear programming problem (NLP), then a NLP solver is required to obtain optimization results. Most common methods are sequential quadratic programming (SQP) and interior point (IP) methods. In this paper, we combine Hermite-Simpson and SQP method and pseudospectral method and IP method to solve the problem in two different ways.

## 4 Optimization Results

### 4.1 Optimization Scenario

In this study, the mission is given as follows. The PIP is set as (60, 100) km. The objective of the mission is to deliver the kill vehicle of 150 kg to the PIP with sufficient velocity. Eq. (8) can be modified as following equation.

$$\Delta V_{missile} - \Delta V_g - \Delta V_d \geq \Delta V_{misison} = 2{,}000\, m/s \quad (21)$$

where Eq. (21) is defined as an inequality constraint.

The bounds of state variables, control input and the other optimization parameters are given below.

$$\begin{aligned} &0km \leq x \leq 100km && -10\deg \leq \alpha \leq 10\deg \\ &0km \leq z \leq 120km && 0.5\sec \leq t_{coast} \leq 10\sec \\ &-1km/s \leq V_x \leq 8km/s && 150kg \leq m_1 \leq 1{,}000kg \\ &-1km/s \leq V_z \leq 8km/s && 150kg \leq m_2 \leq 1{,}000kg \end{aligned}$$

The optimization using local collocation method is conducted with MATLAB build-in function and global collocation method is conducted by using GPOPS, which is an optimal control problem solver. In Hermite-Simpson method, optimization results are very sensitive to the number of nodes, scaling factor, initial condition and many other factors. To verify the results from GPOPS, the bounds for the stage weights in the local collocation method are set near the results from the global collocation method, and we solely observe the convergence of the optimization result. Other missile specifications are summarized in the table below.

**Table. 1 Missile Specifications**

| Variables | Stage 1 | Stage 2 |
|---|---|---|
| Ref. Diameter (m) | 0.4 | 0.4 |
| Ref. Length (m) | 1.6 | 1.2 |
| Propellant Ratio | 0.83 | 0.85 |
| $I_{sp}$ (sec) | 250 | 265 |
| $a_{\max}$ (g) | 20 | 10 |

The cost function is to minimize the total GLOW with Eq. (10).

## 4.2 Results

The optimization results of two methods are compared and presented through Fig. 2 to 5.

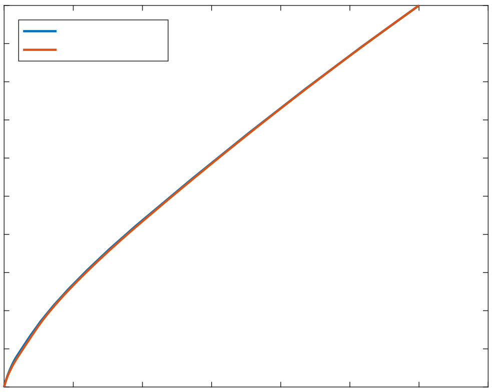

**Fig. 2 Trajectory Comparison**

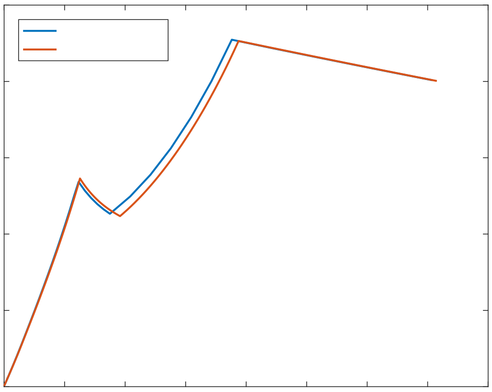

**Fig. 3 Velocity Comparison**

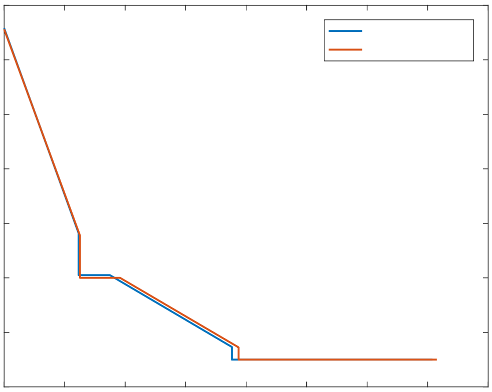

**Fig. 4 Mass Profile Comparison**

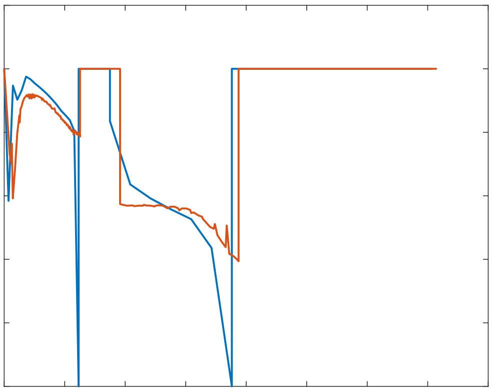

**Fig. 5 Angle of Attack Comparison**

In both methods, almost identical trajectory is achieved in Fig. 2, and at the end of the flight, the kill vehicle reaches the designated PIP. The state variables, including weights, and free flight time between first and second stage rocket are defined as the other optimization parameters. As illustrated in Fig. 3 and 4, there are some difference in calculated free flight times for each method. The optimal free flight time in Hermite-Simpson method is 5.17 seconds and that of the GPOPS is 6.6 seconds, but they are still similar.

Fig. 3 shows that the velocity of the kill vehicle is calculated as 2,000m/s at the end of the flight. For the delta-v approach, the velocity changes are only considered during the boost phase. Given the loss of velocity due to gravity and drag, the velocity obtained by the solid rocket motor is specified to be 2,000m/s in Eq. (21). Actually, the velocity at the end of the second boosting phase is calculated as 2,300m/s, which is greater than expected. After the propulsion, the kill vehicle flies towards the PIP, - slowed down by drag and gravity - resulting in 2,000m/s. There are more factors that could cause the loss of velocity in real case, so it is appropriate to use the inequality constraint and to specify the required velocity in this problem.

As mentioned above, the convergence of the local collocation method is determined based on the number of nodes, the initial conditions. Even if the result converges, there may exist large differences in the value of optimization results. Therefore, the results obtained from the global collocation method are used in this study to limit the bounds of the weights in Hermite-Simpson method, and the convergence of the results is observed within a given bound. The weight profile is similarly optimized in Fig. 4. In the Hermite-Simpson method, the weight of each stage is calculated at 453.1kg, 155kg, and for GPOPS, is calculated at 456.1kg, 150kg; the total weight differed by about 3kg.

In Fig. 5, the angle of attack results, defined as optimal control input, are compared. In the Hermite-Simpson method, 17, 5, 6, and 5 nodes are used at each of the four flight phases to approximate the state variables and control inputs. The control inputs are linearly approximated. On the other hand, the pseudospectral method transcribes the control input into interpolation functions. Thus, Hermite-Simpson method has rougher results compared to those of the pseudospectral method, but the overall tendency is similar.

## 5 Conclusion

In this paper, two methods are used to optimize the weight and trajectory of surface-to-air missiles. The objective function is defined as to minimize take-off weight in order to find a minimum weight system capable of achieving a given mission. Since the velocity losses due to gravity, drag, etc. occur significantly in the high altitude task, delta-v approach, which is used for simple analysis of the performance of launch vehicles, is adopted to achieve the sufficient velocity required for the mission. Pseudospectral and Hermite-Simpson method are used to optimize the given problem. In particular, the convergence of Hermite-Simpson method is determined by various factors; or the difference between the converged optimal results is very large. Thus, the weight results obtained from the pseudosepctral method are used to limit the search range and the convergence is observed within the given bound.

The results obtained using the two methods are very similar, and for weight optimization, the total weight difference calculated is within 3kg, although the search bound is limited. For optimal control inputs, the pseudospectral method produces more elaborated results, but the overall trend is similar in both cases.

## Acknowledgement

This work was conducted at High-Speed Vehicle Research Center of KAIST with the support of Defense Acquisition Program Administration and Agency for Defense Development.

## References

[1] David B, Optimum stating of multistage rockets, International Journal of Mathematical Education in Science and Technology, Vol. 5, No. 1, pp. 3-10, 1974

[2] Ezgi C, Multistage launch vehicle design with thrust profile and trajectory optimization, Ph. D. Dissertation, Middle East Technical University, 2014.

[3] Daniel L, Marnix D, and Teresa L, Multidisciplinary design optimization of missile configurations and fin planforms for improved performance, *7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, St. Louis, MO, USA, 1998.

[4] Mustafa O, Multiobjective design optimization of rockets and missiles, M.S. Thesis, Middel East University, 2009.

[5] Qasim Z, Dong Y, Khurram N, Ali K, and Amer R, Multidisciplinary design and optimization of multistage ground-launched boost phase interceptor using hybrid search algorithm, *Chinese Journal of Aeronautics*, Vol. 23, No. 2, pp. 170-178, 2010.

[6] Murray A, John B, and Roy H, Design of a guided missile interceptor using a genetic algorithm, *Journal of Spacecraft and Rockets*, Vol. 38, No. 1, pp. 28-35, 2001.

[7] Anil R, A survey of numerical methods for optimal control, *Advances in the Astronautical Sciences*, Vol. 135, No. 1, pp. 497-528, 2009.

## Contact Author Email Address

mailto:smhong@fdcl.kaist.ac.kr