A GENERAL APPROACH TO SUPERSONIC AERO–ELASTIC VIBRATIONS PROBLEMS.

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ABSTRACT

Flutter prevention can a significant factor in aircraft design with implications on structural design and thus on weight and performance.

It is well known, however, that in experiments the plate amplitude is limited by the non-linear behaviour of the structure.

A most important non-linear factor which limites the amplitudes in the case of flutter of plates is the non-linearity of a geometric kind associated mainly with the occurrence of tensile stresses in the middle surface.

These stresses strongly depend on the in-plane boundary conditions that can be of whatever kind.

This makes it difficult to select functions which satisfy the necessary conditions and then to apply, as is usual done, Galerkin's method.

The purpose of this paper is to present a new general mathematical approach which has been developed and adapted specifically to three-dimensional panel flutter taking into account coupling of out-of-plane bending and in-plane stretching.

The method here presented can be considered as a generalization of the Galerkin's method and, with the recent advances in computer technology, computations for boundary-value problems either linear or non-linear are of practical interest.

1. – *Introduction*

The response of three – dimensional plates in a high supersonic flow to a disturbance is studied.

Linear theory [1] [2] indicates there is a critical dynamic pressure above which the plate motion grows exponentially with time.

A large deflections, however, non-linear effects come into play and generally restain the motion to a bounded limit cycle. In Refs [2]-[8], the non-linear panel flutter has been investigated.
Also in the present paper the non linear membrane forces induced by the plate motion are included in the analysis and the linear aerodynamic theory is employed.

The present article will concentrate on effects of the in-plane boundary conditions on the amplitude and frequency of limit cycle and on the stress distribution.

2. – A General Approach to a Modified Galerkin's Method

Let us consider a uniform road, fixed at its ends (0,1) under conditions of axial vibration :

It is well known that its modes are expressed by

(1) $$u_k = \sin k \pi x$$

whereas natural frequencies, made conveniently non-dimensional are $\omega_k = k\pi$.

Now, if end-conditions are modified, and an elastic constraint of rigidity β_i ($i = 1$ for the left end; $i = 2$ for the right end) are added to the structure, we have the new end – conditions:

(2) $$\text{for } x = 0 \; ; \quad \frac{d\,u}{d\,x} - \beta_1 u = 0$$
$$\text{for } x = 1 \; ; \quad \frac{d\,u}{d\,x} + \beta_2 u = 0$$

which are not satisfied by (1) which would imply zero end displacement. However, by denoting by u_o, u_1 the actual end values of $u(x)$ we may write:

(3) $$u = u_o (1-x) + u_1 x + \sum_{1k}^{\infty} a_k \sin k\pi x$$

since it is well known that the functions $\sin k\pi x$ constitute a complete set to represent functions vanishing at the ends, such as is the case of $u(x) - u_o(1-x) - u_1 x$. However, by introducing (3) into the equation of dynamic equilibrium of the rod, we obtain a relationship of the kind:

(4) $$F\,(x; u_o, u_1, a_x) = 0$$

Similarly, the end–conditions (2) yield

(5) $$f_i\,(u_o, u_1, a_k) = 0$$

Since (4) is to be valid everywhere in (0,1), the coefficients of expansions of F in terms of $\sin k\pi x$ must be zero.

Thus, if N waves are taken in the series (3) to approximate $u(x)$, the above said conditions, together with (5), constitute a set of N + 2 equations from which the constants u_o, u_1, a_k (and the frequencies as well) can be determined. As N approaches infinity, the solution will converge to the exact one. (Appendix I)

In a similar way, it is known that the $\sin k\pi x$ are the modes of the uniform simply supported beam; for a non simply supported beam, we may obtain the modes as $\sum_{k=1}^{\infty} a_k \sin k\pi x + \sum_{r=o}^{3} w_r x^r$.

Nor should be concluded that only the $\sin k\pi x$'s are suitable functions: any set of proper modes will work provided a sufficient number of additional functions be introduced, capable of re-establishing the missing quantities in the pertinent boundary conditions. Thus, in the example previously stated, the form

$\Sigma_k\, a_k \sin k\pi x + u' x^2 + u'' x^3$ would not be adequate, since it always vanishes as $x = 0$, although it depends on two additional functions, as (3).

At this points we may re-state Galerkin's well known method, by expressing the unknown function $u(x)$ under the form

$$\sum_{1}^{\infty}{}_k\, a_k\, \psi_k(x) + \sum_{1}^{2}{}_r\, w_r\, \chi_r(x) \tag{6}$$

where the $\psi_k(x)$ are the modes corresponding to arbitrary homogeneous end-conditions and the functions $\chi_r(x)$ must be such that the term corresponding to them do not vanish identically when boundary conditions corresponding to the φ_k^s are applied Inorder words, one must have:

$$\begin{bmatrix} \left(\frac{d\chi_1}{dx} - a_1 \chi_1\right)_{x=o} & \left(\frac{d\chi_2}{dx} - a_1 \chi_2\right)_{x=o} \\ \left(\frac{d\chi_1}{dx} + a_2 \chi_1\right)_{x=1} & \left(\frac{d\chi_1}{dx} + a_2 \chi_2\right)_{x=1} \end{bmatrix} \neq 0$$

Introduction of (6) into the equation of dynamic equilibrium and the expansion of the result in terms of a convenient complete set of functions (preferably, but not necessarily the $\psi_k(x)$) provides, together with the boundary conditions, the necessary and sufficient number of equations for the solution of the problem.

3. – The two–dimensional case

Let us now consider a rectangular panel, where at every point, we have a normal displacement $w(x, y)$ and membrane displacements $u(x, y)$, $v(x, y)$.

Normal displacements are expressed as a series of flexural modes of the panel:

$$w = L \sum_{1}^{\infty}{}_r\, w_r\, W_r(x, y) \tag{7}$$

As for as membrane displacements are concerned, let us consider all point lying on a strip $y = \text{const.}$ We may repeat for such points what has been said under Art. 2 and write consequently:

$$\begin{aligned} \frac{L}{h^2}\; u(x, y) = u_o(y)\left(1 - \frac{x}{a}\right) + u_1(y)\,\frac{x}{a} \\ + \sum_{1}^{\infty}{}_k\, a_k \sin k\,\frac{\pi x}{a} \end{aligned} \tag{8}$$

A similar formula will hold for $v(x, y)$.

But since we can analogously formulate $u_o(y)$, $u_1(y)$, $a_K(y)$ we are finally led to the expression:

$$u(x)= u_{oo}\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)+u_{o1}\left(1-\frac{x}{a}\right)\frac{y}{b}+u_{11}\frac{x}{a}\frac{y}{b}$$

$$+u_{10}\left(1-\frac{y}{b}\right)\frac{x}{a}+\left(1-\frac{x}{a}\right)\sum_{1}^{\infty}{}_{s}\,U_{os}\sin\frac{s\pi y}{b}$$

(9) $$+\frac{x}{a}\sum_{1}^{\infty}{}_{s}\,U_{1s}\sin\frac{s\pi y}{b}$$

$$+\left(1-\frac{y}{b}\right)\sum_{1}^{\infty}{}_{r}\,U_{ro}\sin\frac{r\pi x}{a}+\frac{y}{b}\sum_{1}^{\infty}{}_{r}\,U_{r1}\sin\frac{r\pi x}{a}$$

$$+\sum_{1}^{\infty}{}_{r}\sum_{1}^{\infty}{}_{s}\,a_{rs}\sin\frac{r\pi x}{a}\sin\frac{s\pi y}{b}$$

An analogous expression is valid for $v(x, y)$. By introducing u, v into the membrane equation of equilibrium:

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\,\partial y}=$$

(10) $$=-\frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 w}{\partial y^2}\right)+\frac{\partial w}{\partial y}\frac{1+\nu}{2}\frac{\partial^2 w}{\partial x\,\partial y}$$

expanding into a series of $\sin\frac{k\pi x}{a}$ we have a relationship containing the unknown constants appearing in the previous expression plus the coordinates (x, y)

We now express the conditions that all the harmonics of the previous relationship must vanish, and we obtain the final result:

$$C_{11}A+C_{12}B+D_{01}U_{01}+D_{11}U_{11}+D_{02}U_{02}+D_{12}U_{12}+$$

$$+E_{01}V_{01}+E_{11}V_{11}+E_{02}V_{02}+E_{12}V_{12}$$

(11) $$+G_{00}v_{00}+G_{01}v_{01}+G_{10}v_{10}+G_{11}v_{11}=$$

$$=\sum_{m}\sum_{n}w_m\,w_n\,Q_{mn}$$

The rather lengthy expressions of the coefficients appearing in (11) are given in Appendix II. Here the vectors A denote the vector obtained by the a_{mn} (the coefficients of v_{mn}), etc.

We now turn our attention to boundary conditions.

We shall take them in the very general form.

(12)
$$\lambda_1 N_x + \mu_{11} u + \mu_{22} v = 0$$
$$\lambda_2 N_{xy} + \mu_{21} u + u_{22} v = 0$$

on the side $x = 0$; analogous expressions are valid for the other three sides. If we calculate the N_x , U_y from the expressions of u , v again we well have two relationship depending on the unknown constant and upon x .

We must formally state that they are valid for $x = 0$ and $x = 1$, and the coefficients of their expansion in term of $\sin \frac{k \pi x}{a}$ is equal to zero. By repeating the same operations on the other three sides, we have with (11) a sufficient set to eliminate the constants as bilinear functions of the W_r's .

In order to prove that the conditions so obtained are necessary and sufficients we must of course refer to the realistic case when a finite numbers of modes $\sin \frac{k\pi x}{a} \sin \frac{s\pi y}{b}$ is taken ; let M be the number of modes in x , N the number of modes in y . Then Eqs (11) (and the equivalents in y) are $2MN$; the equations derived by (12) are $2M + 4$ and analogously on the other three sides, we will have $2M + 4$, $2N + 4$, $2N + 4$ conditions respectively.

So, all together, we will have $2MN + 4M + 4N + 16$ conditions. But it can be proven that independent corner conditions are halved (suffice here to think that for every corner we have only one u and one v), so that we have a total of $2MN + 4M + 4N + 8$ conditions which is exactly the double of the constant appearing in (9) (an equal number refers to v).

When all the quantities are eliminated, as bi-linear functions of the coefficients of w, introduction into the equations of dynamic equilibrium of the vibrating aeroelastic panel and subsequent development into a series of $W_r (x , y)$ will yield the final well known non linear equations [2] .

It is pointed out that the difference between the procedure here described and other approach will results in the coefficient of third order nonlinearities.

4. — Use of Airy's function

For some specific purpose, use of Airy's function may be more suitable. In this case, we must write the expression to be used will be, in adimentional form

(13)
$$\psi = \sum_{2}^{4}{}_i \sum_{o}^{i}{}_j c_{ij} \xi^{i-j} \eta^j + \sum_{o}^{3}{}_i \sum_{1}^{\infty}{}_n D_{in} \xi^i \sin n \pi \eta + \sum_{o}^{3}{}_j \sum_{1}^{\infty}{}_m E_{mj} \eta^j \sin m \pi \xi$$
$$+ \sum_{1}^{\infty}{}_m \sum_{1}^{\infty}{}_n \psi_{mn} \sin m\pi \xi \sin n \pi \eta$$

to be introduced into the equation of in plane equilibrium [see glossary for simbols]

$$(14) \qquad a_1^4 \frac{\partial^4 \psi}{\partial \xi^4} + 2 a_1^2 b_1^2 \frac{\partial^4 \psi}{\partial \xi^2 \partial \eta^2} + b_1^4 \frac{\partial^4 \psi}{\partial \eta^4} = \gamma^2 b_1^2 \left(\frac{\partial^2 w}{\partial \xi \partial \eta} \right)^2 - \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2}$$

At this point, by the same technique already described, we way obtain the expressions of u, v and introduce them into the boundary conditions, and state equivalence to zero the resulting sine-harmonics. Same applies to Eq (14) and again we can express all the quantities of interest as bilinear functions of the coefficients of w. Subsequent developments remain unchanged.

5. – The free plate

For the free plate case the boundary conditions simply state that ψ and its normal derivative vanish along the boundary. So we may take for ψ the expression

$$(15) \qquad \psi = \sum_{m}^{\infty}{}_{1} \sum_{n}^{\infty}{}_{1} \psi_{mn} \chi_m (\xi) \chi_n (\eta)$$

where now the χ_r' s are the modes of the clamped-edges beam, which has formally the same boundary conditions than the case here considered, and need not any bune be considered.

We limit ourselves to present some typical results for plate of Fig. 1 free in-plane and simply supported out of the plane x–y.

In Fig. 2 is shown the shape of the limit cycle panel deflection for $\sigma = 10$ and $\rho = 1$. The deflection shape is very similar (as function of ξ, η) to that predicted by linear theory. In Fig. 3 limit cycle amplitude is drown vs. dynamic parameter pressure σ for panel of different lenght to width ratios ($\rho = 1$; 1.5 ; 2.0).

For comparison results of Bibl. [4] [6] are given. In making this comparison we must take into account the different in-plane boundary conditions. Here the free-edges conditions have been imposed; the zero in plane stretching, in average sense in [4] and locally in [6] are assumed.

In Fig. 4 the limit cycle frequency vs dynamic parameter is given for comparison with the results of [4] [6].

Adimentional stresses distribution N_ξ, N_η, $N_{\xi\eta}$ along panel for $\rho = 1$ and $\sigma = 10$ are presented in Figs. 5, 6, 7.

6. – The fixed edges plate

For a fixed edges plate, in plane x y, it is preferable to use the u, v approach:

$$(16) \qquad u = \sum_{1}^{\infty}{}_{m} \sum_{1}^{\infty}{}_{n} u_{mn} \sin m \pi \xi \sin n \pi \eta$$

(17) $$v = \sum_{1}^{\infty}{}_{r} \sum_{1}^{\infty}{}_{s} v_{rs} \sin r\pi\xi \sin s\pi\eta$$

Also in this case we limit ourselves to present some typical results for plate simply supported in z – direction.

In Fig. 8 the shape of the limit cycle panel deflection for $\sigma = 10$ and $\rho = 1$ is given.

In Figs. 9, 10 respectively the limit cycle amplitude and frequency vs. dynamic parameter are given.

APPENDIX I

Consider a uniform rod with the following end–conditions

$$(I.1) \qquad \text{for} \quad x = 0 \qquad u = 0$$
$$\text{for} \quad x = 1 \qquad \frac{d\,u}{d\,x} = 0$$

by introduction of (3) becomes:

$$(I.2) \qquad \sum_{k=1}^{\infty} a_k \,[\omega^2 - (k\,\pi)^2] \sin k\,\pi\,x + \omega^2\,[u_o(1-x) + u_1\,x] = 0$$

The end–conditions are now

$$(I.3) \qquad u_o = 0$$

$$(I.4) \qquad u_1 - u_o + \sum_{k}^{\infty}{}_1\, a_k\,(-1)^k\,(k\,\pi) = 0$$

By expressing the condition that each harmonics of the left side of (I.2) mustvanish:

$$(I.5) \qquad [\omega^2 - (k\,\pi)^2\,]\,a_k + \frac{2\,\omega^2}{k\,\pi}\,[u_o - (-1)^k\,u_1] = 0$$

and so, by sostitution into (I. 3) we have:

$$u_o = 0$$

$$(I.6) \qquad u_1[1 + 2\sum_1^{\infty}{}_k \frac{\omega^2}{\omega^2 - (k\,\pi)^2}\,] - u_o\,[1 + 2\sum_1^{\infty}{}_k \frac{(-1)^k\,\omega^2}{\omega^2 - (k\,\pi)^2}\,] = 0$$

By noting that

$$(I.7) \qquad 2\sum_1^{\infty}{}_k \frac{\omega^2}{\omega^2 - (k\,\pi)^2} = \omega \operatorname{ctn} \omega - 1$$

$$(I.8) \qquad 2\sum_1^{\infty}{}_k \frac{(-1)^k\,\omega^2}{\omega^2 - (k\,\pi)^2} = \omega \csc \omega - 1$$

it is to obtain the well know result for the eigen·values

$$(I.9) \qquad \cos \omega = 0$$

for the rod with the end–conditions (I.1).

APPENDIX II

By introducing the expressions (9) into (10), multiplying by $\sin \frac{i \pi x}{a} \sin \frac{j \pi y}{b}$ and integrating to the panel surface one gets the following result

(II.1)

$$a_{ij} \left[(i a_1)^2 + \frac{1-\nu}{2} (j b_1)^2 \right] + (i a_1)^2 \left[c_j'' U_{io} + c_j U_{i1} \right]$$

$$+ \frac{1-\nu}{2} (j b_1)^2 \left[c_i'' V_{oj} + c_j V_{1j} \right] + a_1 b_1 \frac{1+\nu}{2\pi^2} \left[c_j' c_i' + \right.$$

$$+ \sum_1^{\infty} {}_s s \pi k_{js} (V_{1s} - V_{os}) + \sum_1^{\infty} {}_r r \pi k_{ir} (U_{r1} - U_{ro}) +$$

$$\left. + \sum_r \sum_s rs\pi^2 b_{rs} \delta_{ri} \delta_{sj} \right] = a_1 \pi \sum_r \sum_s \sum_p \sum_q w_{pq} w_{rs} L_{pq,rs,ij}$$

where we hqve:

(II.2)

$$c_i = - \frac{\cos i\pi}{i\pi} \quad ; \quad c_i' = \frac{1 - \cos i\pi}{i\pi} \quad ; \quad c_i'' = \frac{1}{i\pi}$$

$$k_{ir} = \frac{4r}{(r^2 - i^2)\pi} \quad \text{if } i \neq r \quad ; \quad k_{ir} = 0 \quad \text{if } i = r$$

(II.3)

$$L_{pqrj,ij} = \left[a_1^2 \, p_1^2 + \frac{1-\nu}{2} b_1^2 q^2 \right] r \, \alpha_{rp}^{(i)} \gamma_{sp}^{(j)} - \frac{1+\nu}{2} b_1^2 \, rsq \, \alpha_{rp}^{(i)} \epsilon_{sq}^{j}$$

$$\alpha_{rp}^{(i)} = \begin{cases} \text{otherwise} & = 0 \\ i = r + p & = 1/4 \\ i = |p - r| & = 1/4 \, \frac{p-r}{i} \end{cases}$$

$$\gamma_{s,q}^{(j)} = \begin{cases} j + s + q \text{ odd} & = \frac{j}{\pi} \left[\frac{1}{(1+q)^2 - j^2} - \frac{1}{(s-q)^2 - j^2} \right] \\ j + s + q \text{ even} & = 0 \end{cases}$$

$$\epsilon^{(j)}_{s,q} = \begin{cases} j+s+q \quad \text{odd} & = \dfrac{1}{\pi}\left[\dfrac{j+n}{(j+n)-q^2} + \dfrac{j-n}{(j-n)^2-q^2}\right] \\ j+s+q \quad \text{even} & = 0 \end{cases}$$

An analogous equation is yield by y–wise equilibrium.

Now, in order to reduce the latter to the vector equation (11), we firstly introduce one single subscript for the pair (i,j). If M denotes the number of x–wise waves we may substitute, to the pair (i,j) the index $k = (j-1)\,M + i$. Conversely if the index k is given we have $i = \mathrm{mod}\,(k, M)$, $j = 1+(k-1)/M$. From this, the quantities appearing in (II.1) can been written in matrix form to obtain (11).

GLOSSARY

a	=	plate length
a_1	=	L/a
a_m	=	modal amplitude
b	=	plate width
b_1	=	L/b
h	=	plate thickness
p	=	pressure
q	=	$\rho U^2/2$ dynamic pressure
u , v	=	in–plane displacements (adimentional with respect h^2/L)
u_{ij} , v_{ij}	=	coefficients Eq. (9).
w	=	plate deflection (adimensional with respect to h)
x , y , z	=	plate coordinates
A , B	=	vector Eq (11)
D	=	plate stiffness
C_{ij} , D_{ij}	=	coefficients
E	=	modulus of elasticity
E_{ij} , G_{ij}	=	coefficients
L	=	length unit
M	=	Mach number
N_x , N_y , N_{xy}	=	stress
N_ξ	=	$\dfrac{N_x}{E h b_1^2}$
N_η	=	$\dfrac{N_y}{E h a_1^2}$
$N_{\xi\eta}$	=	$\dfrac{N_{xy}}{E h a_1 b_1}$
Q_{mn}	=	coefficient
T	=	$\left(\dfrac{\mu L^4}{D \pi^4}\right)^{1/2}$

U	=	air velocity
U_{ij} , V_{ij}	=	coefficients
W_i	=	amplitude deflection
β^2	=	$M^2 - 1$
β_i	=	end rigidity
γ	=	$^h/_L$
η	=	$^y/_b$
ϑ	=	$\left(\frac{\beta^2 - 1}{\beta^2}\right)\left(\frac{\bar{\rho}\, L}{\beta\, \mu}\right)$
ξ	=	$^x/_a$
λ_i	=	coefficients Eqs. (12)
μ	=	mass for unit length
μ_{ij}	=	coefficients Eqs. (12)
ν	=	Poisson's ratio
ρ	=	$^a/_b$
$\bar{\rho}$	=	air density
σ	=	$\frac{2\,q}{\beta\, L}\ \frac{T^2}{\mu}$ dynamic pressure
χ	=	eigenvector
ψ	=	$\frac{\varphi}{L^2}$
ϕ	=	Airy's function
ω	=	frequency

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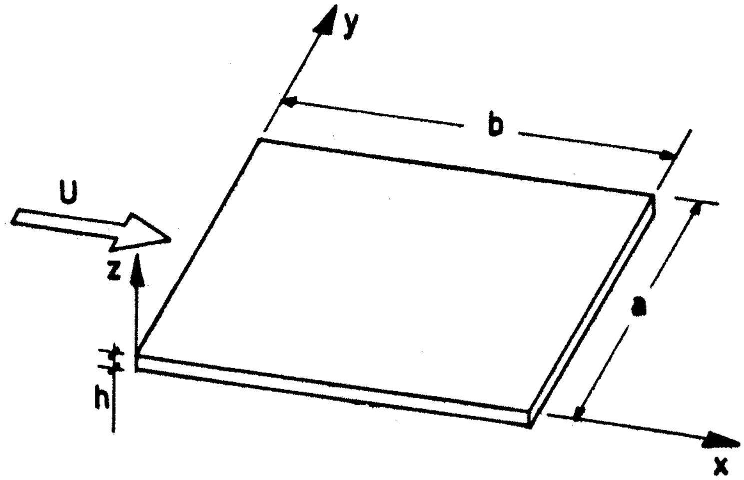

FIG. 1 COORDINATE SYSTEM

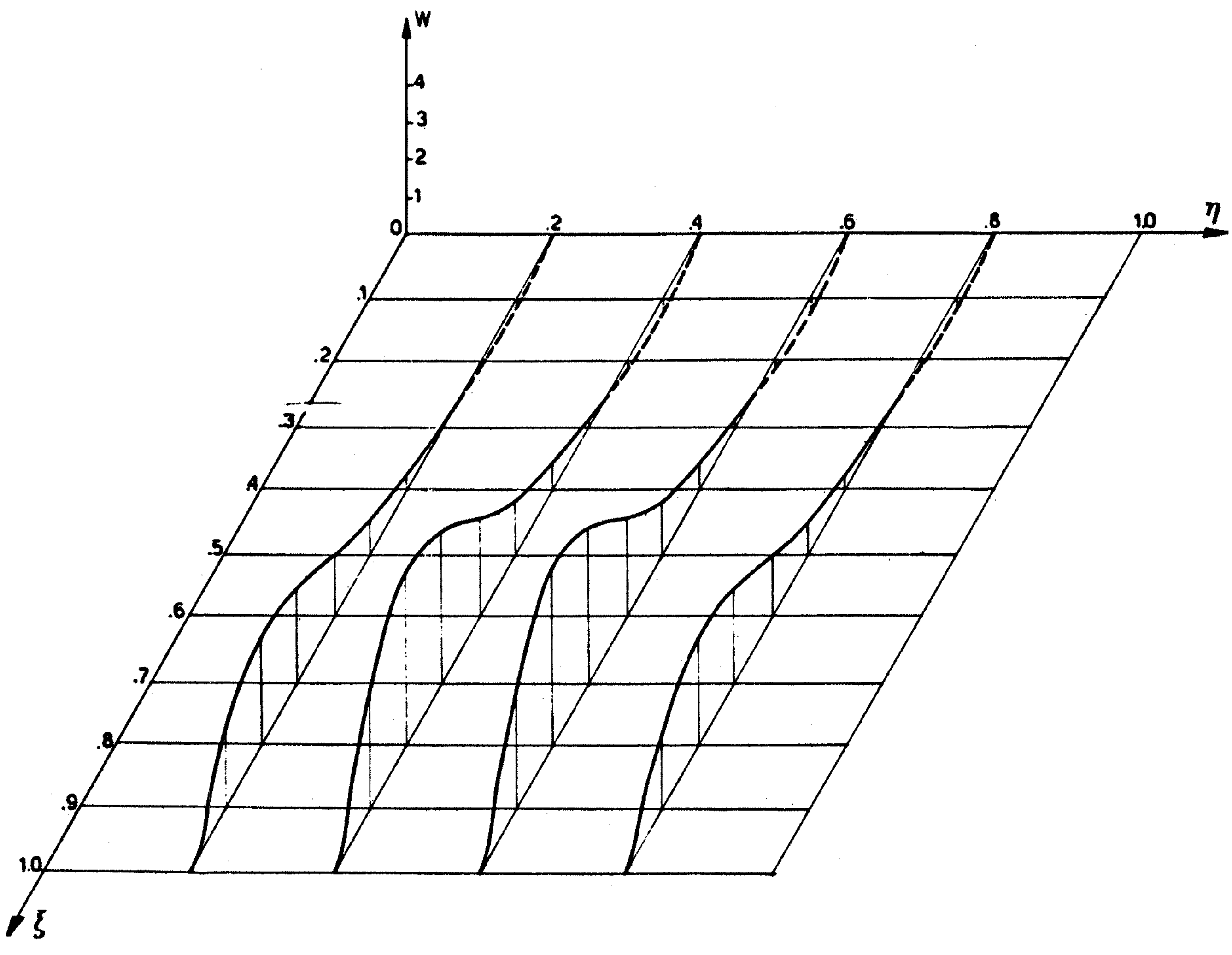

FIG. 2 DISPLACEMENT DISTRIBUTION FOR $\rho = 0, \sigma = 10, \theta = 0.1$ _

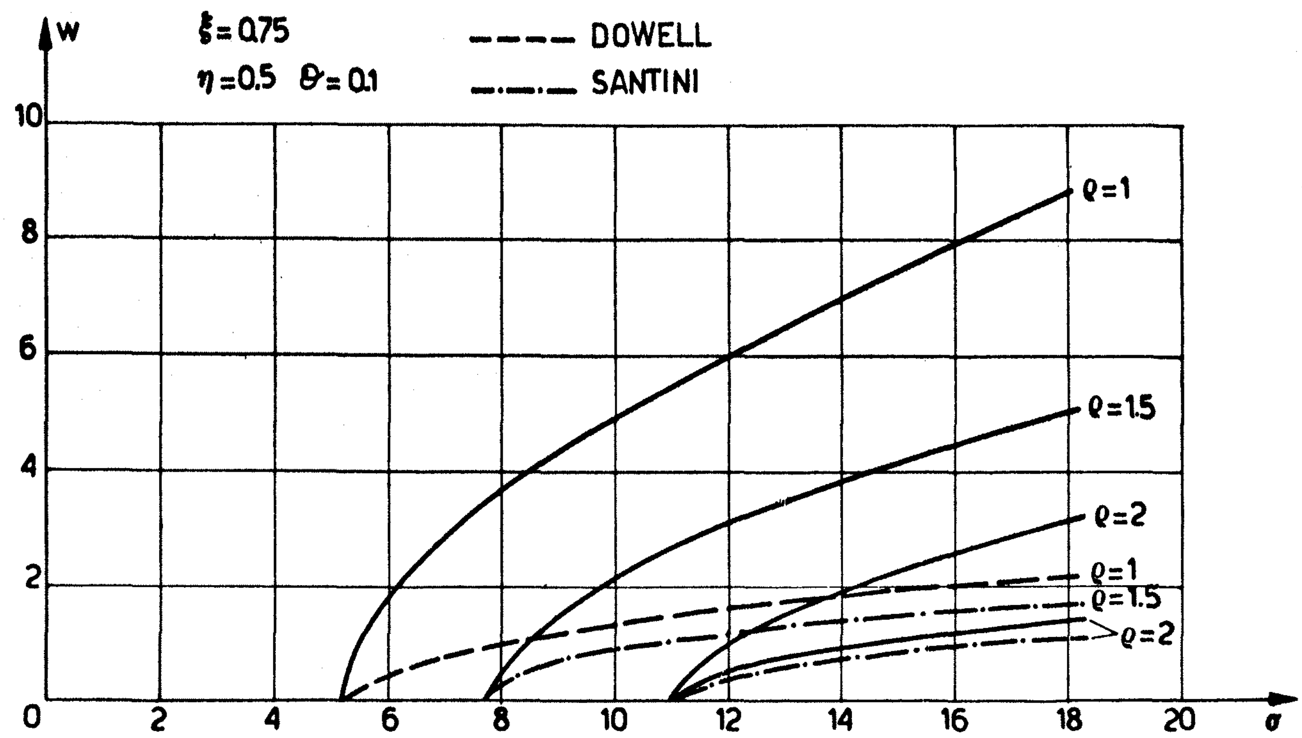

FIG. 3 LIMIT CYCLE AMPLITUDE VS. DYNAMIC PRESSURE _

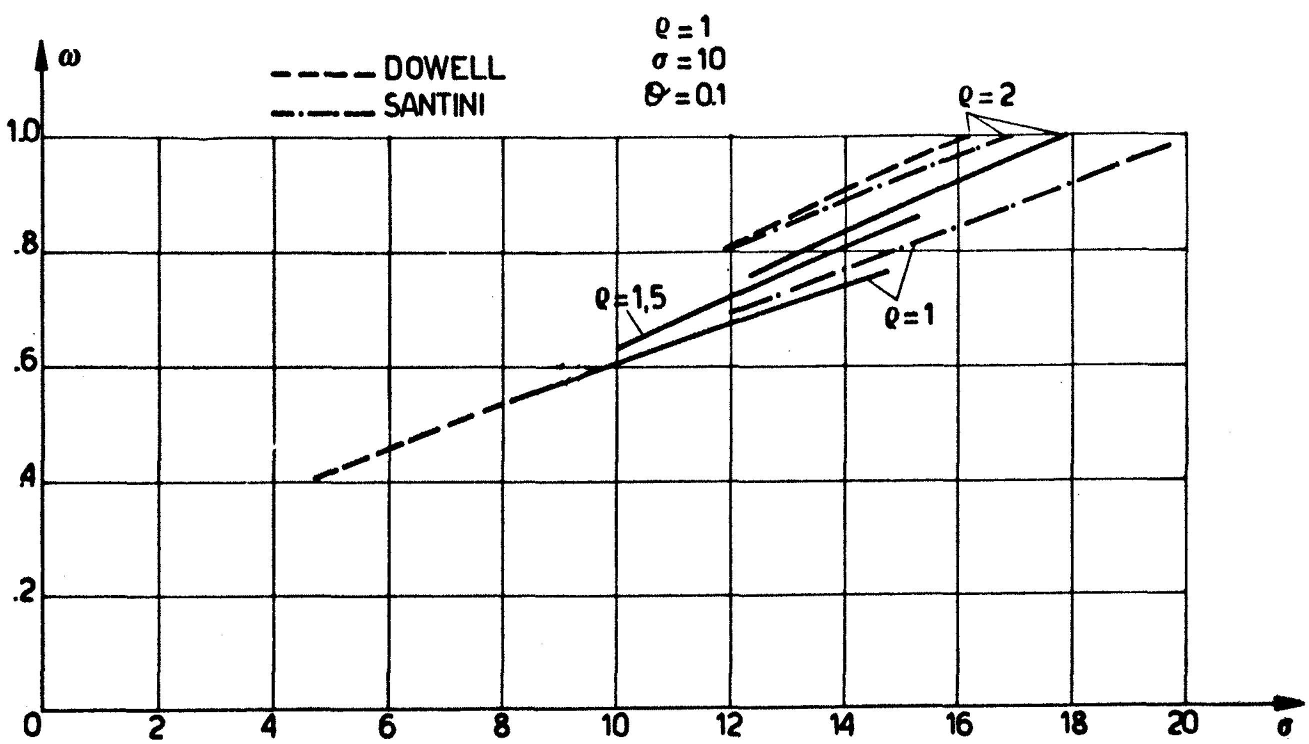

FIG. 4 LIMIT CYCLE VS. DYNAMIC PRESSURE _

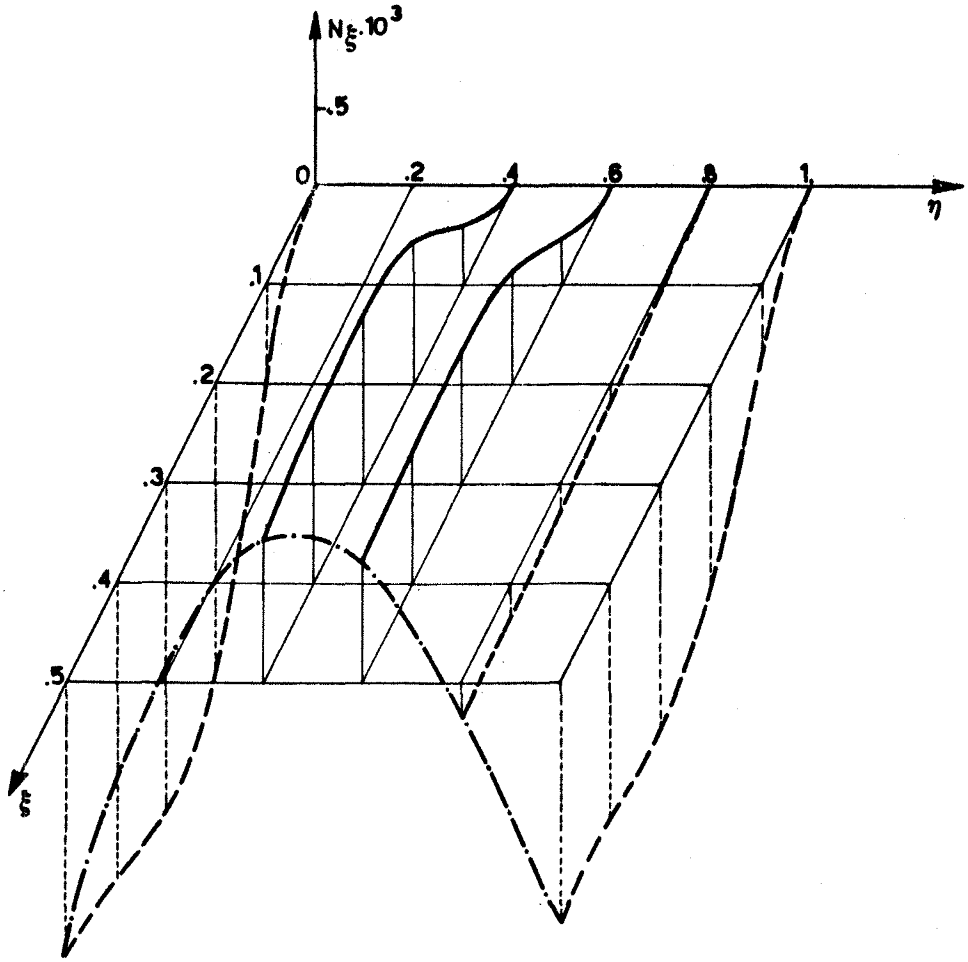

FIG. 5 STRESS DISTRIBUTION FOR $\varrho = 1$, $\sigma = 10$, $\vartheta = 0.1$ –

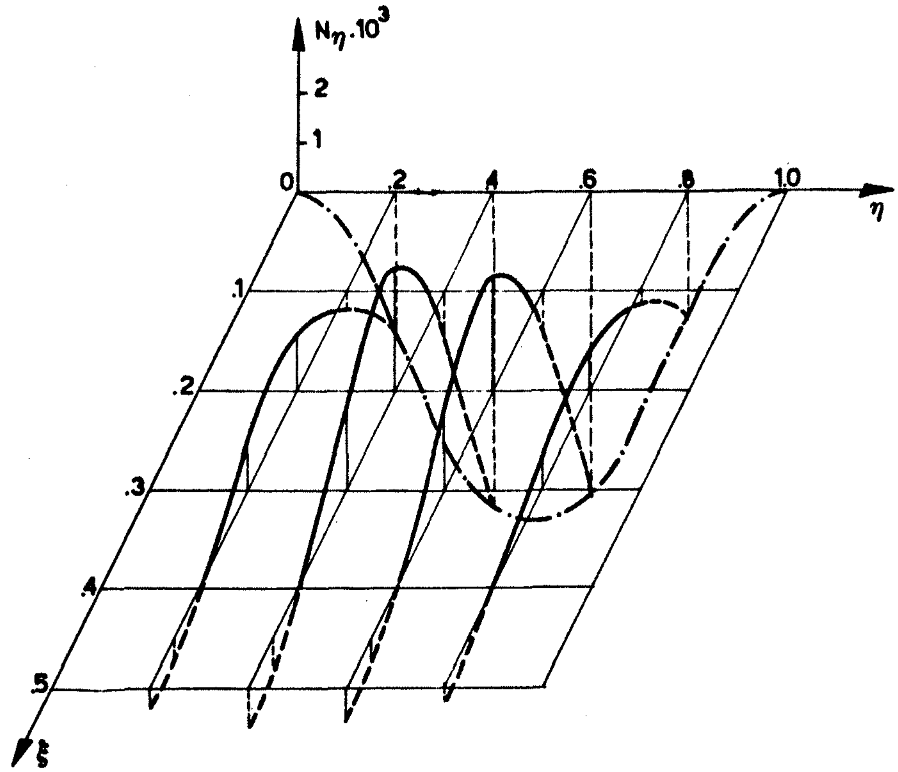

FIG. 6 STRESS DISTRIBUTION FOR $\varrho = 1$, $\sigma = 10$, $\vartheta = 0.1$ –

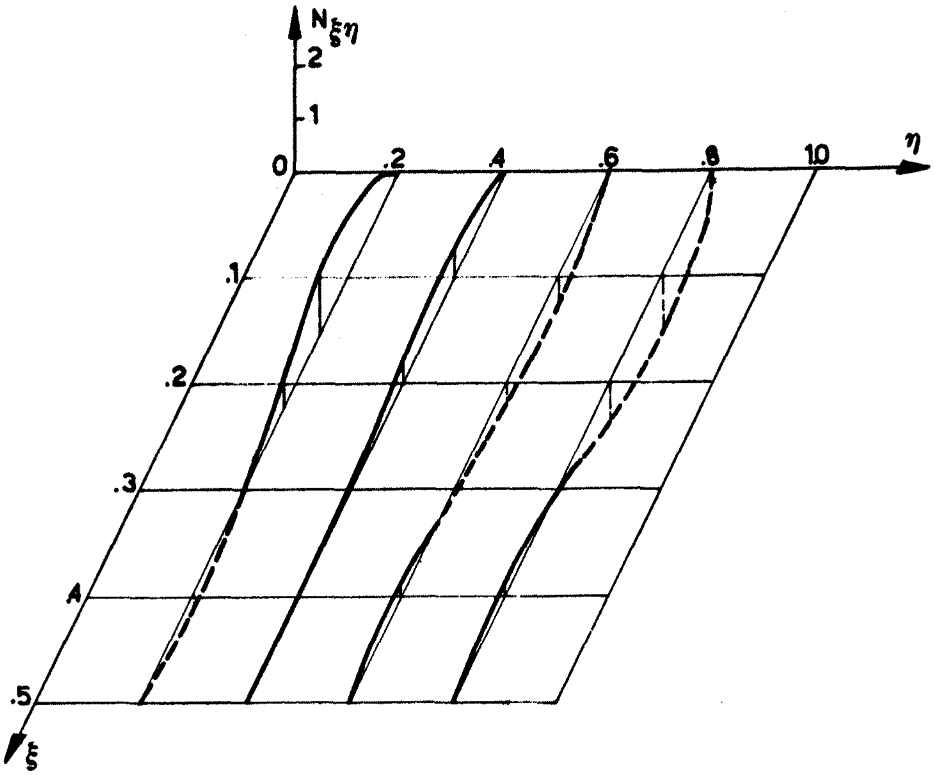

FIG. 7 STRESS DISTRIBUTION FOR $\varrho=1, \sigma=10, \theta=0.1$

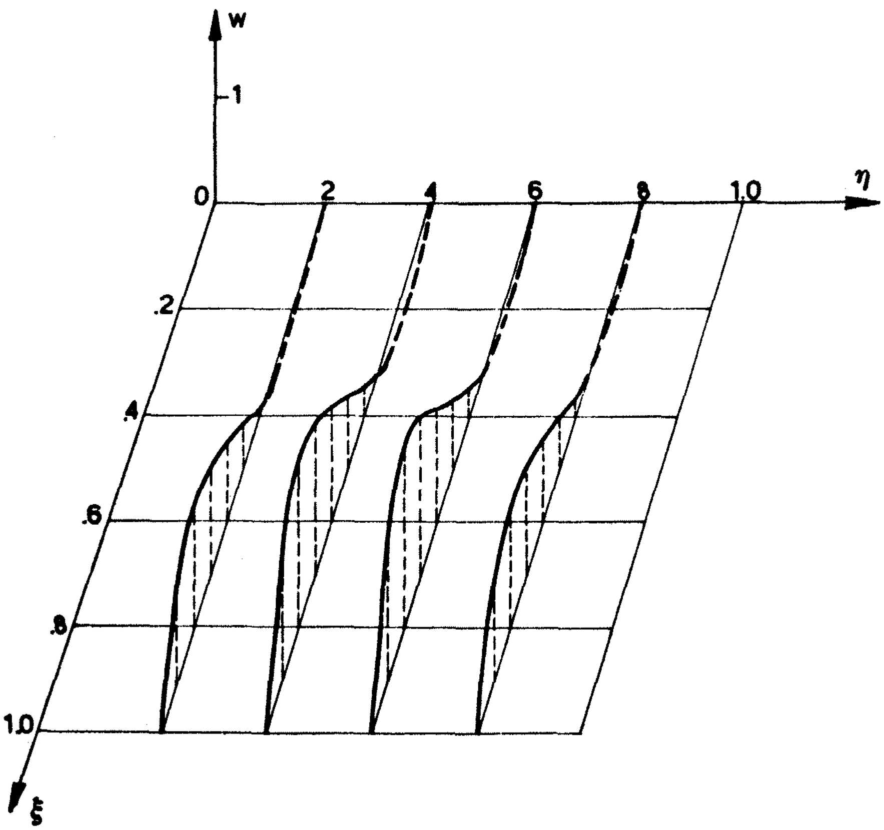

FIG. 8 LIMIT CYCLE DEFLECTION FOR $\sigma=10$, $\varrho=1$, $\theta=0.1$

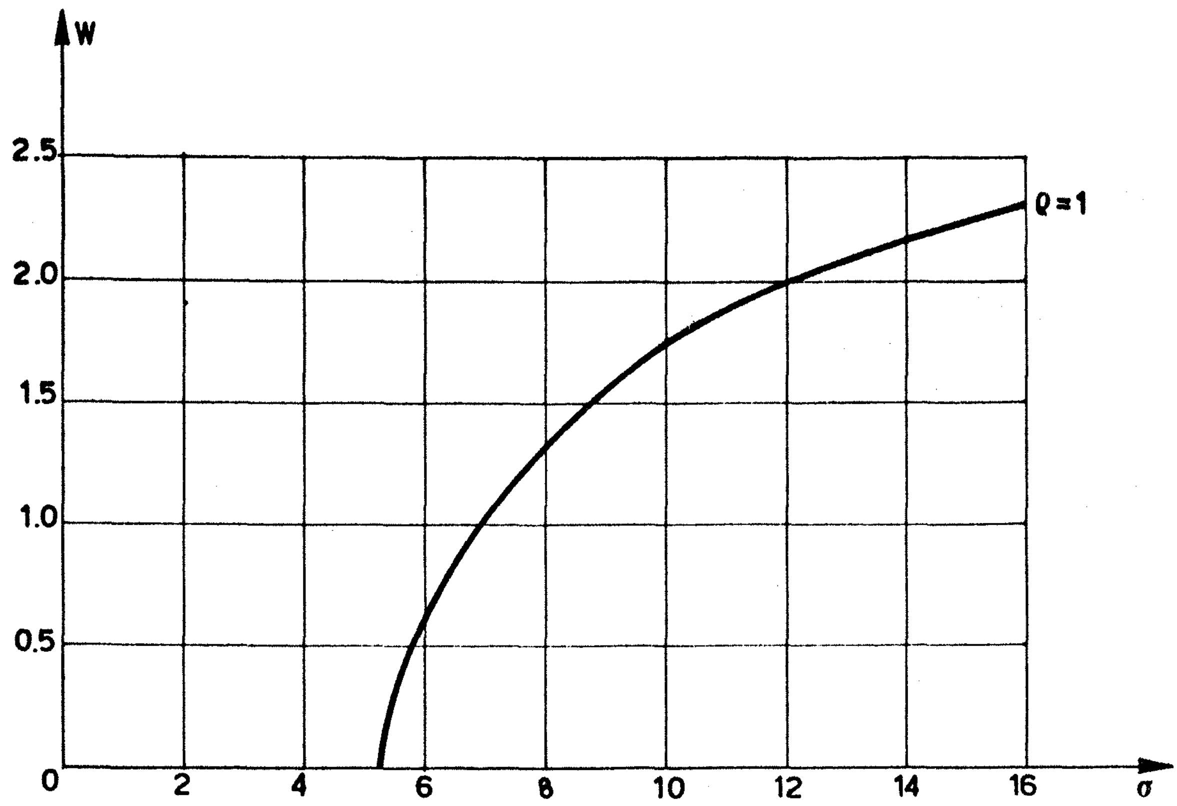

FIG. 9 LIMIT CYCLE AMPLITUDE VS. DYNAMIC PRESSURE $\theta = 0.1$