

THE APPLICATION OF EMPIRICAL STRESS STRAIN FUNCTIONS
TO STRUCTURAL OPTIMISATION PROBLEMS

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ABSTRACT

Empirical stress-strain functions have been widely utilised to determine the effects of plasticity on the response of given structures to the application of load. A less common use has been in the design and optimisation of structures, when structural form is defined and it is required to find detail dimensions so that a given load may be safely and efficiently equilibrated.

The Ramberg-Osgood stress-strain function is used here to obtain solutions in closed form to a number of compression surface optimisation problems, including rectangular plates, honeycomb core sandwich panels, and surfaces with wide column type stiffening. On this basis a systematic evaluation of the relationship between material properties and structural efficiency becomes possible.

INTRODUCTION

In order to accurately account for the effects of plasticity in the analysis of structural behaviour, an adequate characterisation of realistic material stress-strain properties is essential. It is an advantage if this information can be made available in continuous functional form, as opposed to numerical data requiring interpolation since the rate of change of stress with strain is of importance, especially in stability calculations.

This feature is of particular significance when the process of analysis is extended into the field of design and optimisation as in the present paper. Historically, a number of function forms have been proposed for specific purposes.^{1,3,23} However, the relationship first proposed by Ramberg-Osgood³ has since its introduction continued to find wide and varied application in structural analyses of all kinds, and forms the basis of the work described here.

Solutions to a number of optimisation problems concerned with the stability of compression surfaces are obtained which may be useful to designers. These solutions are used to illustrate directly the relationship between basic material stress-strain properties, and structure weight for this class of structures.

STRESS-STRAIN RELATIONSHIPS

Ramberg and Osgood proposed the following relationship between uniaxial stress and strain.

$$\epsilon = \frac{f}{E} + \alpha \left(\frac{f}{E}\right)^n \quad (1)$$

where ϵ = engineering strain

f = uniaxial stress

E = Young's modulus

The first term in (1) may be identified as the elastic (recoverable) strain, and the second term as the plastic (irrecoverable) strain.

The shape of the stress-strain curve is determined by the exponent n . As n becomes large, yielding occurs more sharply until in the limit, as n tends to infinity the function represents elastic-perfectly plastic behaviour.

The coefficient α is of less obvious physical significance, but may be conveniently expressed in terms of f_2 , the 0.2% proof stress (usually taken as the maximum allowable stress in compression), so that

$$\alpha = \frac{0.002}{(f_2/E)^n} \quad (2)$$

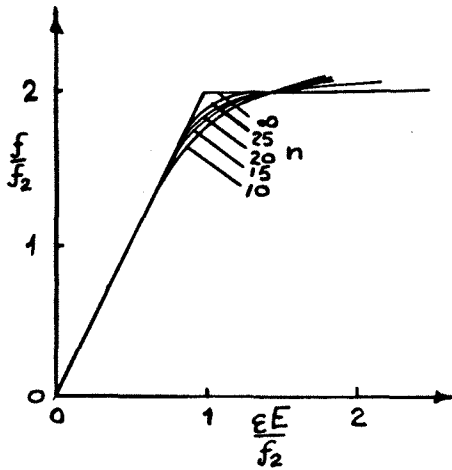


Figure 1. Ramberg-Osgood stress-strain curves

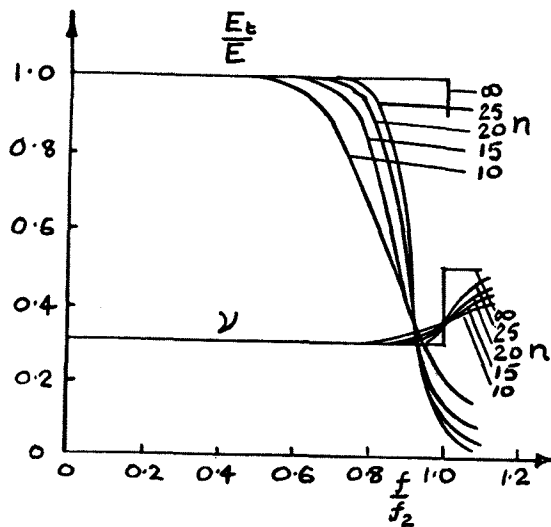


Figure 2. Stress-strain functions

A number of stress-strain relations may now be conveniently derived, including secant and tangent moduli given by

$$\frac{E_s}{E} = \frac{f/E}{\epsilon} = \frac{1}{1 + \phi} \quad (3)$$

where $\phi = \alpha(f/E)^{n-1} = \frac{0.002}{f_2/E} \cdot \left[\frac{f/E}{f_2/E} \right]^{n-1}$
from (2) (4)

$$\frac{E_t}{E} = \frac{\partial(f/E)}{\partial \epsilon} = \frac{1}{1 + n\phi} \quad (5)$$

The separate identification of elastic and plastic strains also allows the formulation of transition values of Poisson's ratio, giving

$$\nu = \frac{\nu_e + \nu_p \phi}{1 + \phi} \quad (6)$$

where ν_e and ν_p are elastic and fully plastic values of Poisson's ratio respectively (usually about 0.3 and 0.5)

DESIGN APPLICATIONS

Uniform isotropic plate

It is required to design a flat, uniform, rectangular plate of length a and width b , to carry a uniformly applied axial loading ω . The plate has simply supported edges and is to be made from a material which may be characterised by equation (1) and which may therefore be identified by three parameters E , n and f_2 .

Stowell⁷ gives for the buckling load intensity ω of such a plate an expression similar to the following, which correlates well with experimental observation

$$\omega = \left[\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s} \left(\frac{mb}{a} \right)^2 + \left(\frac{a}{mb} \right)^2 + 2 \right] \frac{\pi^2 E_s}{12(1-\nu^2)} \cdot \frac{t^3}{b^2} \quad (7)$$

The analysis is based on an approximate plasticity theory, giving lower loads than the more precise theory of Ilyushin⁶. However, the good agreement with experiment may well be accounted for by the presence of small imperfections as in the case discussed by Hutchinson and Budiansky²⁶

In his paper, Stowell assumes $\nu = 0.5$, which is an unnecessary restriction since equation (6) is available.

Equations (3)(4)(5) and (6) may be used to transform (7) into a design equation, when the load intensity ω is given and the plate thickness t or stress $f(=\omega/t)$ is required. This procedure gives

$$\frac{\omega}{Eb} = \frac{(f/E)^{\frac{3}{2}} \left\{ 12(1+\phi) \left[1 - \left(\frac{\nu_e + \phi \nu_p}{1 + \phi} \right)^2 \right]^{\frac{3}{2}} \right\}}{\pi \left\{ \frac{1}{4} \left[1 + \frac{3(1-\phi)}{(1+n\phi)} \right] \left(\frac{mb}{a} \right)^2 + \left(\frac{a}{mb} \right)^2 + 2 \right\}^{\frac{3}{2}}} \quad (8)$$

$$\frac{t}{b} = \left[\frac{\omega}{Eb} \right] / \left(\frac{f}{E} \right) \quad (9)$$

For given values of $\frac{\omega}{Eb}$ and $\frac{a}{b}$, a value of stress may always be found which satisfies (8), and then plate thickness follows from (9). Note that for any given value of stress, the integer n must be chosen which maximises the right hand side of (8).

The results for plates of finite aspect ratio shown in figure 3 indicate that the onset of plasticity, illustrated by reducing proof stress, increases plate thickness and reduces buckle wavelength below the ideal elastic value.

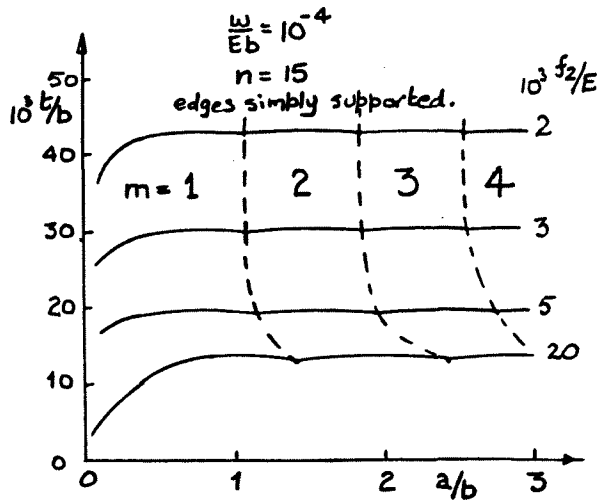


Figure 3. Design of simply supported rectangular plates in compression.

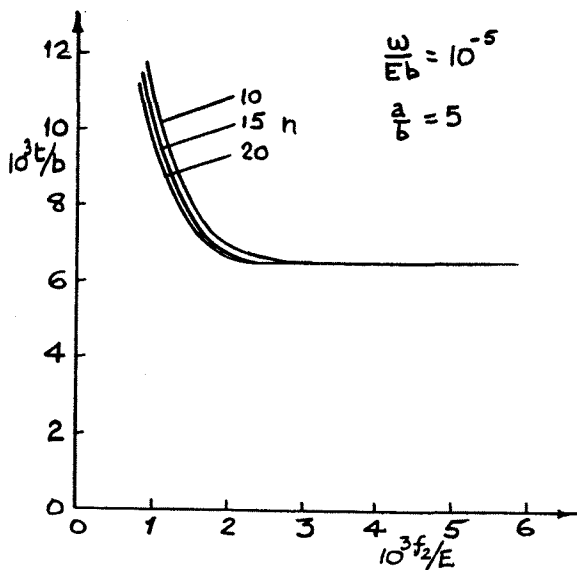


Figure 4. Long plates in compression: material property effects

Figure 4 utilises long plates ($a/b = 5.0$) to show how improvements in proof stress and changes in plastic strain exponent n affect the required thickness, and hence the weight of plates of this type.

Honeycomb sandwich panel

The analysis of sandwich panel behaviour has received extensive treatment especially during the period 1950-65, when many high speed aeroplane projects were being developed.^{5, 9, 10, 15}

The work has been summarised comprehensively by Plantema.¹⁷ Several authors have considered the problems of design and optimisation including Wittrick,⁵ Williams,¹⁰ Bijlaard⁹ and Kaechele.¹⁵ However no systematic introduction of plasticity effects has been attempted.

The structure considered here is a rectangular, simply supported panel with isotropic faceplates and core. The core is honeycomb, fabricated from the same material as the faceplates. The panel is required to carry a uniform uniaxial compressive loading ω without buckling, and the design process is required to specify face thickness, panel depth and core stiffness. All elements are assumed to be perfectly manufactured, and no allowance is made for adhesive.

Two modes of instability must be considered, wrinkling and overall panel buckling.

The faceplate wrinkling stress is given by⁴

$$f = K [E_f E_c G_c]^{1/3} \quad (10)$$

where G_c = core transverse shear modulus

E_c = core direct modulus under loads normal to panel surface

for an ideal hexagonal honeycomb core¹⁵

$$E_c = \frac{64(1+\nu_e)}{15} G_c \quad (11)$$

E_f = faceplate modulus.

Plantema¹⁷ suggests that the effects of faceplate plasticity are accounted for by using the so-called geometric modulus, so that

$$\frac{E_f}{E} = \frac{4E_t}{[E^{1/2} + E_t^{1/2}]^2} \quad (12)$$

The buckling coefficient K may be conservatively taken to be 0.5. Equation (10) may now be solved for the core stiffness required to prevent face wrinkling under the stress f , giving

$$\frac{G_c}{E} = \frac{1}{16} \left[\frac{15}{(1+\nu)K^3} \right]^{1/2} \left(\frac{f}{E} \right)^3 [1+(1+n\phi)^{1/2}] \quad (13)$$

For the loading at which overall panel buckling occurs Plantema gives

$$\frac{\omega}{Eb} = \frac{\frac{h}{b} \cdot \frac{E_r}{E} \cdot \frac{G_c}{E} (1+\lambda)^2}{1+\lambda \left[1 + \frac{2(1-\nu_e^2)}{\pi^2} \frac{G_c/E}{t/b \cdot h/b} \right]} \quad (14)$$

which takes account of transverse shear flexibility.

t = face thickness = $\frac{\omega}{2f}$

h = panel depth

b = panel width transverse to loading direction

$$E_r = \frac{2EE_t}{E + E_t} = \frac{2}{2 + n\phi}$$

λ = wavelength parameter = $\left(\frac{a}{mb}\right)^2$
 a = panel length
 m = number of half-waves in direction of loading which for a given panel must be chosen to minimise w .

When loading ω is given, (14) may be solved for the panel depth required to prevent buckling, giving

$$\frac{h}{b} = \frac{\frac{\omega}{Eb} \left[1 + \frac{16(1-\nu_e^2) \cdot f \cdot E_r}{\pi^2 \cdot E \cdot E} \left[\frac{G_c/E}{\omega/Eb} \right]^2 \right]^{\frac{1}{2}}}{2(1+\lambda) \frac{G_c \cdot r}{E \cdot E}} \quad (15)$$

Note that for given ω , m and hence λ , must be chosen to maximise panel depth h .

Panel equivalent thickness t^* , which is proportional to weight is given by

$$t^* = 2t + \frac{\rho_c}{\rho} \cdot h$$

where ρ_c = core density
 ρ = faceplate density

For a honeycomb core made of the same materials as the faces,

$$\frac{\rho_c}{\rho} = \frac{64(1+\nu_e)}{15} \cdot \frac{G_c}{E}$$

so that finally

$$\frac{t^*}{b} = \left(\frac{\omega}{Eb}\right) \left(\frac{f}{E}\right) + \frac{64}{15} (1+\nu_e) \cdot \frac{G_c}{E} \cdot \frac{h}{b} \quad (16)$$

Equations (13), (15) and (16) are functions of face stress alone so that the value of stress which minimises panel equivalent thickness may be found by a simple numerical procedure, as indicated by figure 5.

Typical results for short panels are shown in Figure 6. which indicates how plasticity reduces buckle wavelength and increases panel weight. The effect of material parameters n and f_2/E on panel weight is illustrated in figure 7 for long panels ($a/b = 5$)

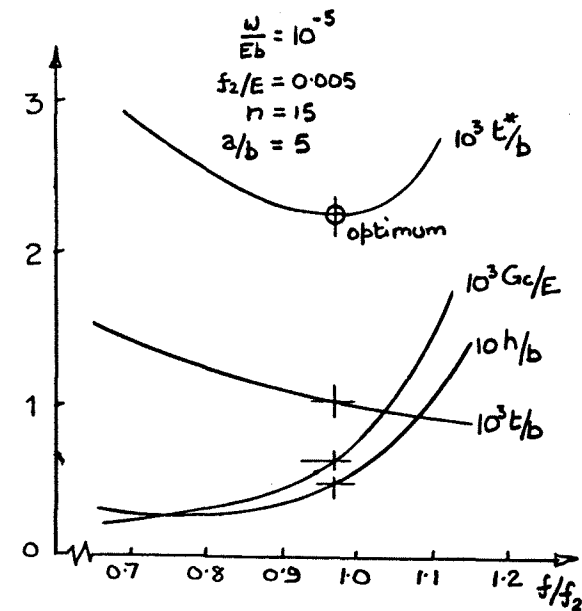


Figure 5. Sandwich panel optimisation: faceplate stress variation

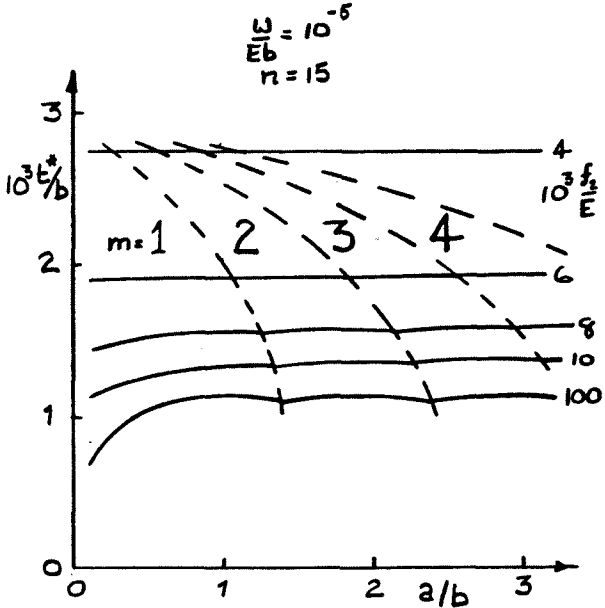


Figure 6. Design of simply supported rectangular sandwich panels in compression

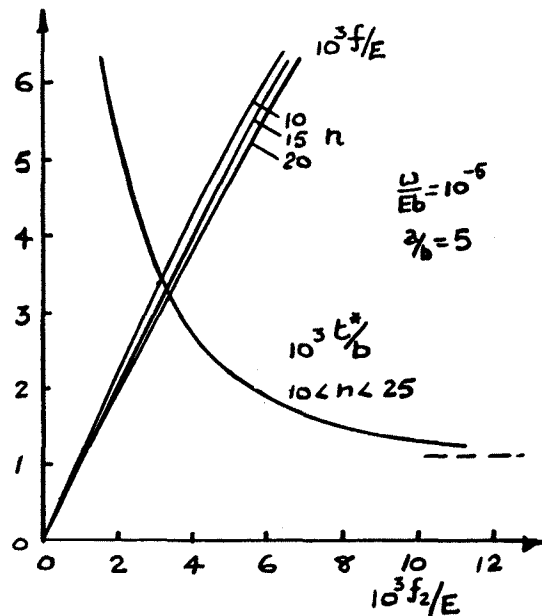


Figure 7. Design of long sandwich panels loaded in compression

Compression panels with wide-column type reinforcement

Perhaps the most common type of compression surface utilised in shell structures for aerospace applications is of the uniaxially stiffened wide column type.

The optimisation of such surfaces has been considered in detail by many authors including Farrar,⁸ Catchpole,⁶ Cox,¹⁶ Emero and Spunt, Gerard²³ and Gallagher²¹. Their work has shown that theoretically the lightest designs are obtained when local and flexural modes coincide at the design load. More recently a number of authors have shown in principle that these procedures may lead to designs with significant reductions in strength, coupled with undesirable failure characteristics due to manufacturing imperfections.^{15 19 20 24 26} However, there is as yet little experimental evidence of these effects. This poses certain difficulties for the practical designer in that not only are these investigations incomplete but also the results which are available are not easily generalised especially with regard to the data on imperfection magnitude and distribution required when considering a spectrum of possible designs. In the circumstances, a sensible course would seem to require that a rather conservative view be taken of theoretical buckling strengths, together with a continuation of the traditional procedure of thorough ad hoc testing.

With these reservations in mind, we may proceed to consider the introduction of plasticity effects into the design process.

Short wave (local) buckling stress is given by

$$f = K_1 E_t \left(\frac{t}{b}\right)^2 \quad (17)$$

where t = skin thickness

b = stiffener spacing

K_1 = local buckling coefficient usually available in numerical form.

The Euler buckling stress is

$$f = K_2 E_t \left(\frac{b}{L}\right)^2 \quad (18)$$

where L = frame spacing

K_2 = Euler buckling coefficient

The use of tangent modulus in equations (17) and (18) is an approximation which in each case is justified by experimental observation. For local buckling this represents a lower bound⁷ for the ideal structure whereas for Euler buckling the tangent modulus, although theoretically ill-founded, nevertheless qualitatively accounts for some small degree of imperfection of the order of that encountered in practice. Stress

and load intensity are related by

$$f = K_3 \frac{\omega}{t} \quad (19)$$

where K_3 is the area thickness coefficient.

Equations (17)(18) and (19) may be combined together to give

$$\begin{aligned} f &= F \left[\frac{\omega E_t}{L} \right]^{\frac{1}{2}} \\ b &= B \left[\frac{\omega L^3}{E_t} \right]^{\frac{1}{4}} \\ t &= T \left[\frac{\omega L}{E_t} \right]^{\frac{1}{2}} \end{aligned} \quad (20)$$

where coefficients are

$$\begin{aligned} F &= \left[\frac{K_1 K_2 K_3^2}{K_1 K_2 K_3^2} \right]^{\frac{1}{4}} \\ B &= \left[\frac{K_1 K_3^2}{K_2^3} \right]^{\frac{1}{4}} \\ T &= \left[\frac{K_3^2}{K_1 K_2} \right]^{\frac{1}{4}} \end{aligned} \quad (21)$$

These coefficients are functions only of the dimensionless ratios describing the proportions of the surface stiffening system and may be separately varied to maximise F and hence surface stress. Thus the coefficients achieve specific values for any particular form of stiffening system, and also reflect any reductions which may be made to account for the effects of imperfections mentioned above.

Equations (20) may be conveniently recast as functions of surface stress by using relationships already developed, giving

$$\frac{F^2 \omega}{EL} = \left(\frac{f}{E}\right)^2 (1 + n\phi) \quad (22)$$

$$\frac{Ft}{EL} = \left(\frac{f}{E}\right) (1 + n\phi) \quad (23)$$

$$\frac{F^{\frac{1}{2}} b}{BL} = \left(\frac{f}{E}\right) (1 + n\phi)^{\frac{1}{2}} \quad (24)$$

$$\frac{F^2 t^*}{L} = \left(\frac{f}{E}\right) (1 + n\phi) \quad (25)$$

where t^* = panel equivalent thickness.

These expressions provide a suitable basis for the preparation of data sheets specific to any given material as shown in Figure 8.

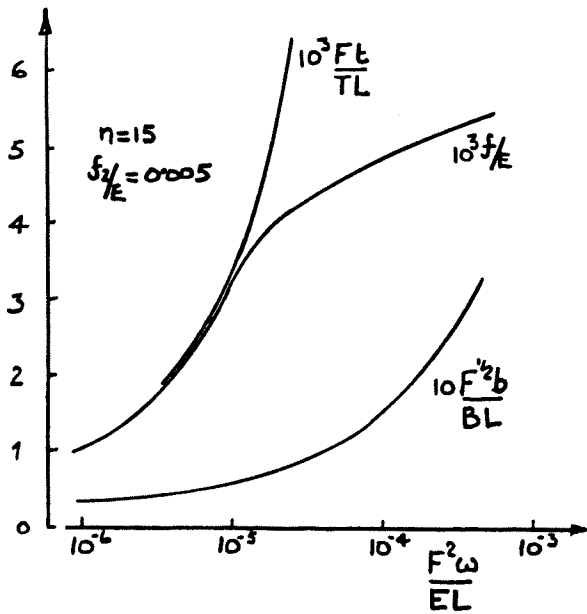


Figure 8. Ideal optimum data for wide column stiffened panels

Wide column surfaces with optimum support locations

The results obtained above may be utilised in the design of an array of such surfaces, supported at regular intervals measured in the direction of loading, by ribs or frames of cross section area A_r .

The total equivalent surface thickness of such an array is

$$t_e = t^* + \frac{A_r}{L}$$

If we assume that support area A_r is constant i.e. independent of surface stress and support spacing, this equation may now be expressed entirely in terms of surface stress, so that

$$\frac{Ft^e}{A_r^{1/2}} = \frac{Z}{(f/E)} + \frac{(f/E)^2}{Z}(1 + n\phi) \quad (26)$$

where $Z = \frac{F\omega}{EA_r^{1/2}}$

Equation (26) may be differentiated directly to find the surface stress which minimises total equivalent thickness, giving the following results.

$$\frac{F\omega}{EA_r^{1/2}} = \left(\frac{f}{E}\right)^{3/2} [2 + n(n+1)\phi]^{1/2} \quad (27)$$

$$\frac{L}{FA_r^{1/2}} = \frac{[2 + n(n+1)\phi]^{1/2}}{(f/E)^{1/2}(1+n\phi)} \quad (28)$$

$$\frac{t}{TA_r^{1/2}} = \left(\frac{f}{E}\right)^{1/2} [2 + n(n+1)\phi]^{1/2} \quad (29)$$

$$\frac{b}{B^{1/2}A_r^{1/2}} = \frac{[2 + n(n+1)\phi]^{1/2}}{(1 + n\phi)^{1/2}} \quad (30)$$

$$\frac{Ft^e}{A_r^{1/2}} = \frac{\left(\frac{f}{E}\right)^{1/2} [3+n(n+2)\phi]}{[2 + n(n+1)\phi]^{1/2}} \quad (31)$$

Again, for any given material, these functions may be plotted directly as a set of data sheets (figure 9) which may be used for initial design purposes or for comparative studies of different materials.

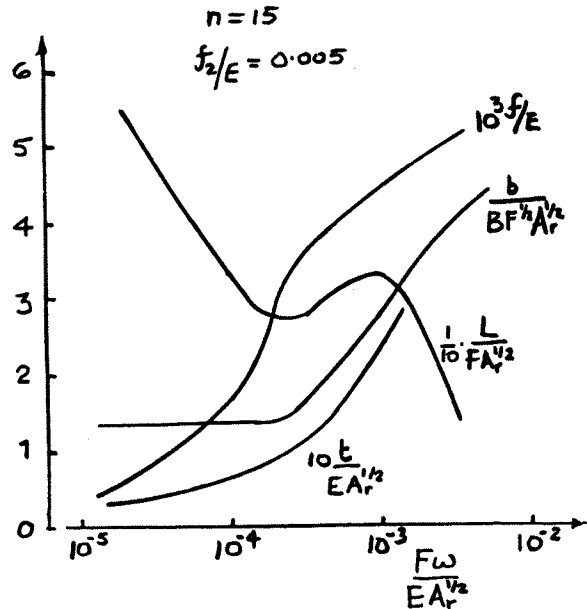


Figure 9. Ideal optimum data for panels with optimised support locations

Analogous expressions may be developed for the situation where frame size is related to the transverse support stiffness required to prevent general instability.

This analysis is afforded a more extensive treatment in reference 27.

Optimum reinforced circular shell

The maximum endload intensity in a thin-walled circular shell of diameter D required to transmit a bending moment M is given by

$$\omega = \frac{4M}{\pi D^2} \quad (32)$$

If the optimum frame spacings derived above are used, shell diameter and maximum surface stress may therefore be related using (27) and (32) to give

$$D \left[\frac{\pi}{4} \cdot \frac{EA r^{\frac{1}{2}}}{FM} \right]^{\frac{1}{2}} = \frac{1}{\left(\frac{f}{E}\right)^{\frac{1}{2}} [2+n(n+1)\phi]^{\frac{1}{2}}} \quad (33)$$

Total equivalent shell cross-section area is given by

$$A_s = \pi D t e$$

so that from (31) and (33)

$$A_s \left[\frac{FE}{4\pi M A r^{\frac{1}{2}}} \right]^{\frac{1}{2}} = \frac{[3 + n(n+2)\phi]}{\left(\frac{f}{E}\right)^{\frac{1}{2}} [2+n(n+1)\phi]^{\frac{3}{2}}} \quad (34)$$

For a perfectly elastic material $\phi = 0$, and equations (33) and (34) indicate that the optimum shell will have a diameter as small as possible, limited only by the maximum permitted surface stress.

When plasticity is included, equation (34) (which is illustrated in figure 10) exhibits a minimum value at a stress which is therefore a characteristic property of the material under consideration, provided $n > 2$ (which is the case for all materials commonly used in aircraft construction). The optimum diameter then follows from (33).

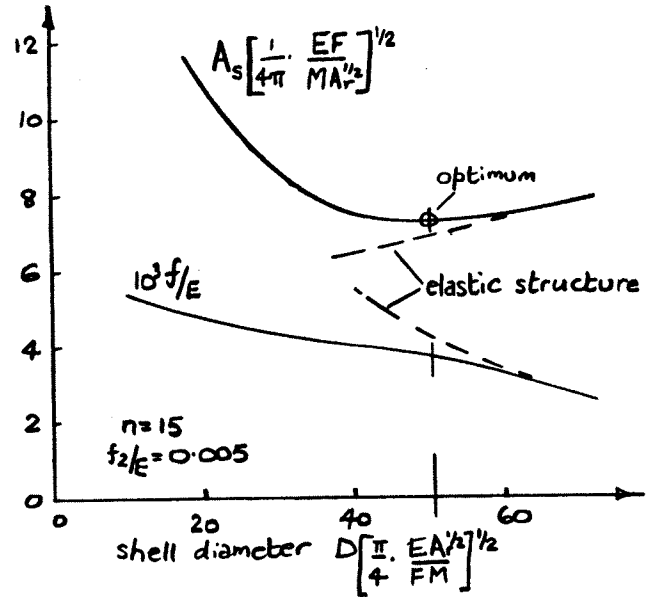


Figure 10. Wide column reinforced shell: variation of shell diameter

The relationships developed above may therefore be used to systematically explore the effect of basic material parameters on the potential weight efficiency of a wide class of structural forms as shown in Figure 11.

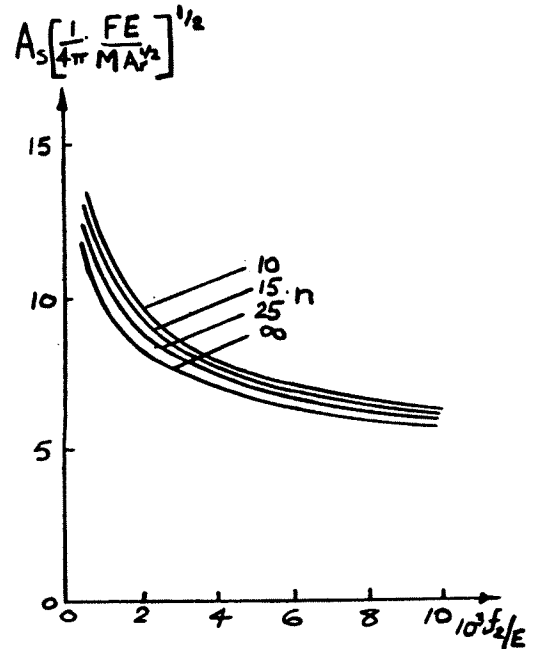


Figure 11. Material property effects on the weight of ideal optimum shells

CONCLUSION

A well established empirical stress strain relationship has been used to derive design equations for a number of structural elements whose strength is limited by buckling phenomena.

The results obtained serve as a basis not only for determining ideal structural dimensions, but also for the systematic parametric study of material properties in relation to structural efficiency. The structure weights indicated should be regarded as lower bound values, which will inevitably increase as practical constraints are imposed.

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