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MODEL REDUCTION AND STOCHASTIC UPDATING OF A VIBRATING SYSTEM

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Abstract

A two-level framework is demonstrated for stochastic model updating. At the first level, variance-based global sensitivity analysis is carried out with the purpose of identifying those parameters with significant uncertainty and those that might be considered deterministic and can be eliminated as inputs to the metamodel. Then, at the second level, an inverse problem is solved to determine the statistics of the parameters of a modification that causes numerical metamodel results to converge on experimental data. Model updating is carried on a second metamodel with only the significant parameters retained. The methodology makes use of the Woodbury formula, resulting in a set of nonlinear characteristic equations in the unknown terms. The framework methodology is applied to a simulated three degrees of freedom representation of an experimental rig. Complex-eigenvalue data is generated from known parameter distributions and bivariate output probability density functions are produced using kernel density estimation. By sampling from this data, estimates of the generating parameters and their distributions are recovered.

Keywords: Stochastic model updating, polynomial chaos expansion, global sensitivity analysis, kernel density estimation.

1. Introduction

Aircraft structures are becoming lighter and more flexible as a consequence of the extended use of composite materials. The adoption of such materials is beneficial in terms of both the aerodynamics and the fuel efficiency, but their application leads to new problems of structural safety in assembled systems with unexpected dynamics [9]. The structural dynamics of overall systems and individual components is greatly influenced by the behaviour of joints and interfaces; their behaviour, often nonlinear, is intrinsically uncertain, therefore the dynamic response of an assembled structure is probabilistic [4]. However, the aerospace industry is still to a large extent reliant on linear and deterministic design methods that may not be accurate enough in representing the dynamic behaviour of real-world aero-structural systems.

During the design, the models are updated (or calibrated) to match experimental measurements. Deterministic model updating techniques [6] are widely used in the industry, but more recently probabilistic techniques have been introduced. Khodaparast et al. [8, 7] used perturbation methods, whereas Beck and his colleagues [2, 5] developed and applied the Bayesian methodology in model updating. The significant advantage of the probabilistic approach is that it provides a measure of confidence in the updated model. An important issue in any model updating scheme is the selection of parameters considered to be erroneous and in need of correction. Deterministic model updating is often based on local sensitivities computed from finite element models [6] whereas stochastic model updating makes use of global sensitivities [12, 11] that may be computed efficiently using the polynomial chaos (PC) expansion [13]. The outputs, often termed quantities of interest (QoIs), are expressed in terms of a truncated system of orthogonal polynomials in the inputs, i.e. the uncertain parameters, according to an assumed probabilistic distribution. The global sensitivity is essentially a

measure of the fractional contribution of each input to variance of the output. Yuan et al. [18] formulated a global-sensitivity based parameter selection method that was able to discriminated between correctly modelled probability distributions on parameters not in need of updating and incorrectly modelled ones that should be updated.
This work proposes a PC and global-sensitivity based framework that includes metamodeling (for model reduction) and the selection of updating parameters. Use is made of the Woodbury formula [16] in updating to obtain characteristic equations at known eigenvalues. Probabilistic distributions on such eigenvalues with sampling by Sobol sequence and kernel density estimation enables the updating of selected system parameters.

2. Framework

The proposed framework, shown in Figure 1, consists of two levels. The purpose of the first level is to reduce the number of model inputs (uncertain physical parameters) by the application of global sensitivity analysis (GSA). This involves choosing the quantities of interest (measurable outputs) and making a GSA-informed judgement on the significance of the inputs. Initially, one should include all feasible inputs and ultimately reduce this number by removing those inputs that are shown to make only very small contributions to the variances of the QoIs.
In the second level those inputs deemed by GSA to be insignificant are eliminated. This results in a reduced-order surrogate model (or metamodel). Then using the metamodel, model updating may be carried out with the purpose of converging the model outputs onto experimentally observed results.

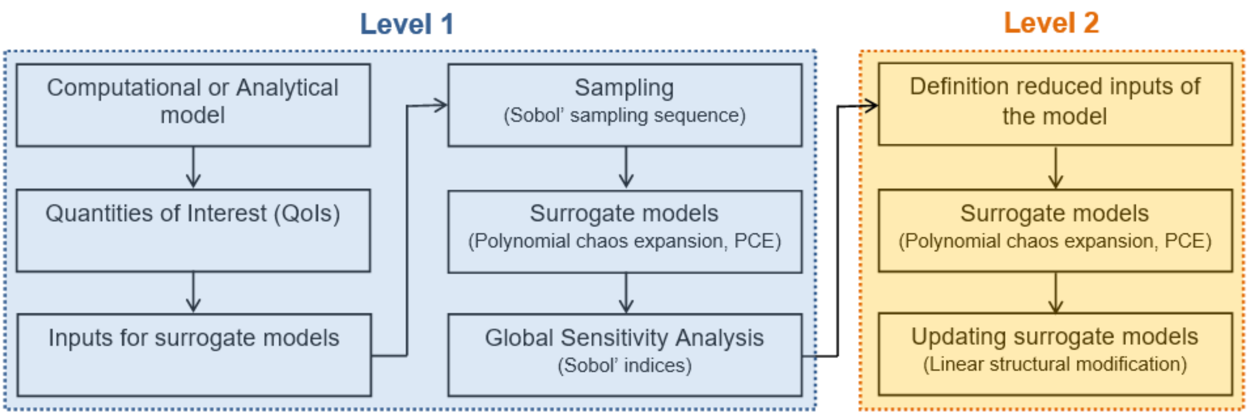


Figure 1 – Framework scheme.

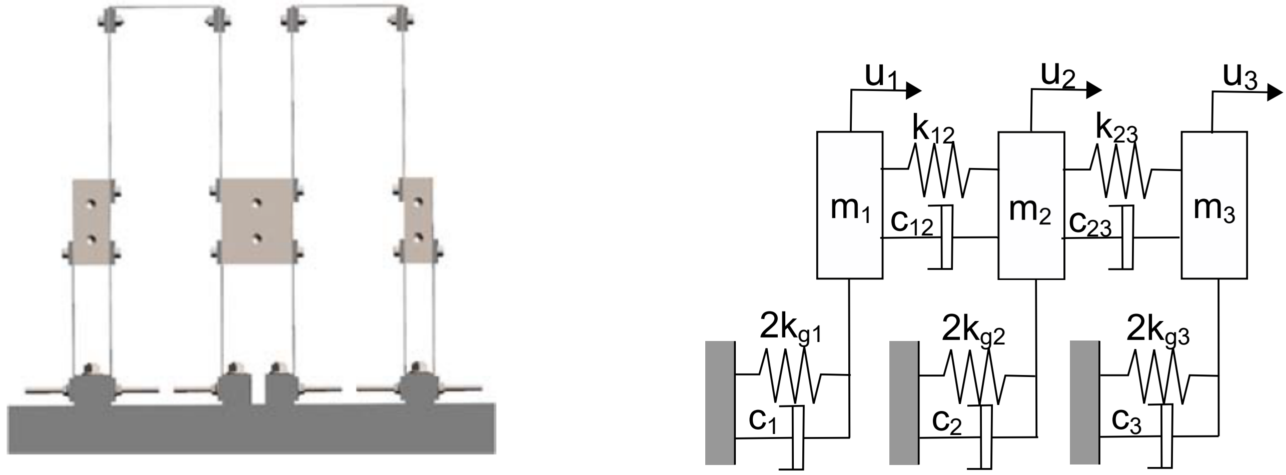
3. Experimental setup

The research described in this article is based upon the experimental three degree of freedom system shown in Figure 2a. However, for the purposes of illustrating the procedure, the present work is restricted to simulated experimental data.
The system consists in three masses supported and connected by a set of thin plate-like springs: three springs link each mass to the ground, two coupling springs join them, as shown in Figure 2a. A coupling spring is made by two plate springs connected by a top (almost rigid) element that can be moved up and down to change the stiffness of the coupling spring. The higher the top connector the softer the spring. A mass-damper-spring model is shown in Figure 2b, defined by the equation of motion,

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{F}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K} \in \mathbb{R}^{3\times 3}$ are the structural mass, damping and stiffness matrices, respectively. $\mathbf{u}(t) \in \mathbb{R}^3$ is the vector of displacement. $\mathbf{F}(t) \in \mathbb{R}^3$ is the external force applied to the system. The structural matrices may be written as,

$$\mathbf{M}=\text{diag}\begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} \quad \mathbf{K}=\begin{bmatrix} 2k_{g1}+k_{12} & -k_{12} & 0 \\ -k_{12} & 2k_{g2}+k_{12}+k_{23} & -k_{23} \\ 0 & -k_{23} & 2k_{g3}+k_{23} \end{bmatrix} \quad \mathbf{C}=\beta\mathbf{K} \tag{2}$$



(a) Test-rig. (b) Schematic of the system.

Figure 2 – Three DOFs lumped mass-dumper-spring system.

where β is the damping coefficient, defined in Table 1.
The eigenvectors are considered to be mass-normalised such that,

$$\mathbf{I} = \Phi^T \mathbf{M} \Phi \qquad \Lambda = \Phi^T \mathbf{K} \Phi \tag{3}$$

Λ is the diagonal spectral matrix of eigenvalues and Φ is the modal matrix. The system described in Eq. (3) may be expressed in modal coordinates q by means of the transformation,

$$\mathbf{u} = \Phi \mathbf{q} \tag{4}$$

4. Polynomial chaos expansion – metamodel and global sensitivity

The use of a metamodel is essential in the uncertainty quantification of large scale engineering systems. One such metamodel is provided by the PC expansion and is chosen in the present case, not for reasons of large size (the system has only three degrees of freedom), but because the PC expansion provides for the efficient computation of global sensitivities.
The PC expansion provides a functional approximation of a computational model through its spectral representation on a basis of polynomial functions [15]. The random variables must be independent while the basis consists of univariate polynomials that are orthonormal to the marginal distributions of the inputs. The univariate polynomial families that are orthonormal to the most common statistical distributions are listed in several papers in literature, e.g. Xiu [17].
A random input $\mathbf{x} = (\ x_1, x_2, \ldots, x_p)\ ^T$, p is the dimension of the set that is usually scale to unit cube $[0,1]^p$ to simplify the mathematics. The ist output sample (QoI sample) may be represented as a truncated sum,

$$y_i = \sum_{j=0}^{n-1} \alpha_j \Psi_j(\mathbf{x}) \quad i = 1, 2, \ldots m \tag{5}$$

where α_j is a deterministic coefficient and Ψ_j an orthonormal polynomial in the random input $\mathbf{x}$, thereby forming a basis for the random y_i (the QoI). The Least Angle Regression (LAR) algorithm is applied to estimate the coefficients α_j of the PCE for the given basis. Recasting the terms in Eq. (5) in matrix-vector form, the sparse L1 norm solution is determined as,

$$\hat{\alpha} = \arg\min_{\alpha} \mathsf{E}\left[\left(\alpha^T \Psi(\mathbf{x}) - Y(\mathbf{x})\right)^2\right] + \eta \|\alpha\|_1 \tag{6}$$

where $\Psi \in R^{m \times n}$ in the matrix that assemble the values of all orthonormal polynomials in $\mathbf{x}$, $\alpha \in R^{n \times 1}$ is a containing the deterministic coefficients $\mathbf{y} \in R^{m \times 1}$ in the vector of the outputs. m is the number of QoI output and $n = (m+q)!/(m!q!)$ is the number of coefficients, q is the truncated order of the PC expansion. The first term in Eq. (6) is the least squares minimisation while $\eta \|\alpha\|_1$ favours the sparse L1 solution. Several algorithms exist, in particular UQLab applies the algorithm developed by Blatman and Sudret [3]. The mean and variance of the resulting metamodel may be expressed as,

$$\mathrm{E}\left[y(X)\right] = \alpha_0 \tag{7}$$

$$\mathrm{Var}\left[y(X)\right] = \sum_{n=0}^{n-1} \alpha_j^2 \tag{8}$$

where y_0 is the coefficient of the constant basis term $\Psi_0 = 1$ and the sum is over the coefficients of the non-constant basis element only.
The first-order global sensitivity of an output with respect to a specific input is then given by bringing together as a fraction of the total variance, all the variance terms in Eq. (7)-(8) that are polynomial functions of that input alone. The total global sensitivity of an output is given by the sum of all the variance contributions that include that specific input, both singly and in combination with other inputs, again as a fraction of the total variance.
The three eigenvalues of the system, $(\lambda_1, \ \lambda_2, \ \lambda_3)$, are known to be sensitive to small changes in the system parameters, whereas the eigenvectors, $(\phi_1, \ \phi_2, \ \phi_3)$, are usually quite insensitive. Thus the three eigenvalues are chosen as QoIs. The number of inputs should be inclusive of as many candidate parameters as possible, but care must be taken in defining their distributions. For example, the values of the three masses (and the masses of the plate-like springs) are very well known, so that the distributions on the masses should be very restricted. The stiffnesses of the supporting spring are also considered to be accurately modelled, but uncertainty in the connecting spring stiffnesses, which are adjustable, is significantly greater. The damping coefficient β, which is small is considered to be constant. Table 1 shows the input parameters, their statistical distributions and nominal values. Global sensitivity analysis was carried out with parameter uncertainties defined in Table 1 .

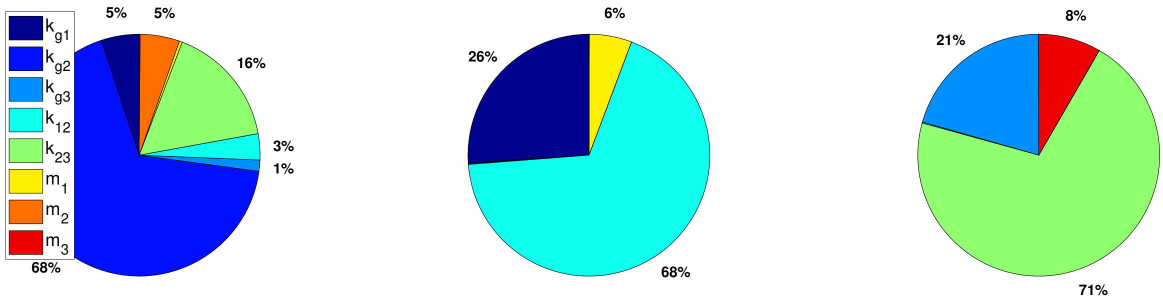
Table 1 – Statistical distribution of random variables

Random variable	Distribution		Nominal value
k_{g1} [N/m]	Uniform	U[2825 2875]	2853.2
k_{g2} [N/m]	Uniform	U[2915 2965]	2931.0
k_{g3} [N/m]	Uniform	U[2790 2940]	2817.0
k_{12} [N/m]	Uniform	U[1322.5 1422.5]	922.5
k_{23} [N/m]	Uniform	U[1475.5 1557.5]	1207.5
m_1 [kg]	Uniform	U[1.58 1.60]	1.59
m_2 [kg]	Uniform	U[2.66 2.67]	2.67
m_3 [kg]	Uniform	U[1.16 1.17]	1.16
β	Constant	$5.5508e-05$	

The polynomial of dimension 8 (the number inputs), truncated at the 3rd order, gives $n = 165$ as the number of coefficients in Eqs. (5) and (6). The number of samples was 200 and Sobol sampling applied. The moments of the three metamodels are shown in Table 2. The first order sensitivities with respect to the three eigenvalues of the system are shown in Table 2 and in the form of pie-charts in Figure 3. It is seen that the second and third eigenvalues are predominantly sensitive to springs k_{12} and k_{23} respectively. Uncertainty on the other parameters has only a small effect on these two eigenvalues. They are each predominantly sensitive to a different single parameter. Since the parameters are statistically independent, the second and third eigenvalues may be considered also to be statistically independent. The first-order and total sensitivities are found to be identical in the cases of all three of the eigenvalues, thereby confirming that there is no interaction between the parameters.

Table 2 – Statistical moments and Sobol indices of the quantities of interest.

Parameter	Result	λ_1	λ_2	λ_3
	E(Y) $[rad/s]$	52.83	68.34	79.79
	Var(Y)	10.69	30.45	49.80
k_{g1}	$S_1 = S_{T1}$	0.05	0.26	–
k_{g2}	$S_2 = S_{T2}$	0.68	–	–
k_{g3}	$S_3 = S_{T3}$	0.01	–	0.21
k_{12}	$S_4 = S_{T4}$	0.03	0.68	–
k_{23}	$S_5 = S_{T5}$	0.16	–	0.71
m_1	$S_6 = S_{T6}$	–	0.06	–
m_2	$S_7 = S_{T7}$	0.05	–	–
m_3	$S_8 = S_{T8}$	–	–	0.08



(a) First order sensitivity pie - λ_1. (b) First order sensitivity pie - λ_2. (c) First order sensitivity pie - λ_3.

Figure 3 – Pie charts Sobol total indices of the quantities of interest.

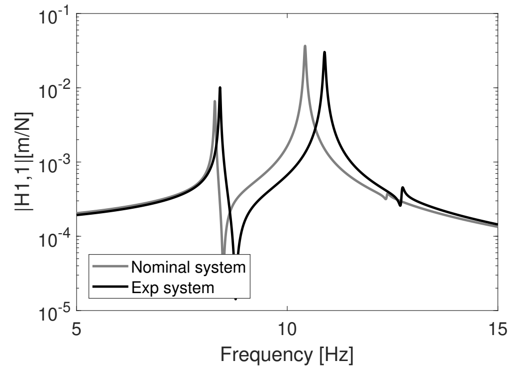
5. Data

Data in the form of simulated ‘experimental’ results, takes the form of the complex eigenvalues of the system and are generated from the system described in Eq. 2 with uncertainty distributions on k_{12} and k_{23} are $\Delta k_{12} = U\,[-50\ \ 50]\,[N/m]$ and $\Delta k_{23} = U\,[-50\ \ 50]\,[N/m]$ - the other parameters (masses and grounded stiffnesses) are considered to be deterministic with nominal values. The purpose is to reconstruct the distributions on the two uncertain parameters using complex eigenvalue data (in the form of PDFs) generated by the same parameters. It is the simplest exercise possible, but more complicated than might be thought initially.

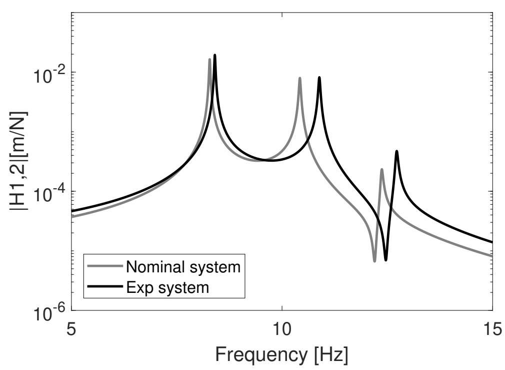
Figure 3 shows the three frequency response functions (FRFs) generated by (1) the nominal system, grey line, and (2) the ‘experimental’ system with the means of the two uncertain parameters ($k_{12} = 1372.5$ N/m and $k_{23} = 1507.5$ N/m), black line. It is seen from the figures and from Table 3 that the second and third eigenvalues are in error to a greater extent than the first eigenvalue. Recall from the sensitivity analysis that the second eigenvalue is sensitive to k_{12} and the third to k_{23}. Therefore, complex eigenvalue data was produced by Sobol sampling from Δk_{12} and Δk_{23}. The number of samples was 500, these being used to generate independent bivariate PDFs for each of the two eigenvalues, bivariate because the real and imaginary parts are considered as two correlated variables, using a kernel density estimator [14].

Table 3 – Confront between computational and experimental poles before the structural modification.

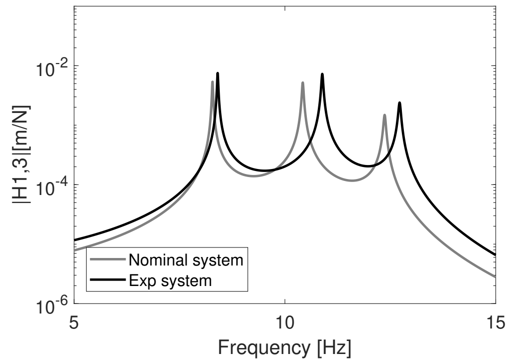
	Nominal parameter values	Experimental mean parameter values
$\lambda_{1,2}$	$-0.337 \pm 52.04i$	$-0.324 \pm 52.81i$
$\lambda_{3,4}$	$-0.128 \pm 65.49i$	$-0.093 \pm 68.40i$
$\lambda_{5,6}$	$-0.0186 \pm 77.70i$	$-0.0118 \pm 79.93i$



(a) FRF $|H_{1,1}|$



(b) FRF $|H_{1,2}|$



(c) FRF $|H_{1,3}|$

Figure 4 – Frequency response functions – nominal and mean 'experimental' parameters.

Let $X = \mathbf{x}_1, \ldots, \mathbf{x}_n$ be a sample of bivariate random vectors drawn from a density function f. The KDE $\hat{f}$, for $x \in \mathbb{R}^2$, is defined as

$$\hat{f}(\mathbf{x}, \mathbf{H}) = \frac{1}{n} \sum_{i=1}^{n} K_{\mathbf{H}}(\mathbf{x} - \mathbf{X}_i) \tag{9}$$

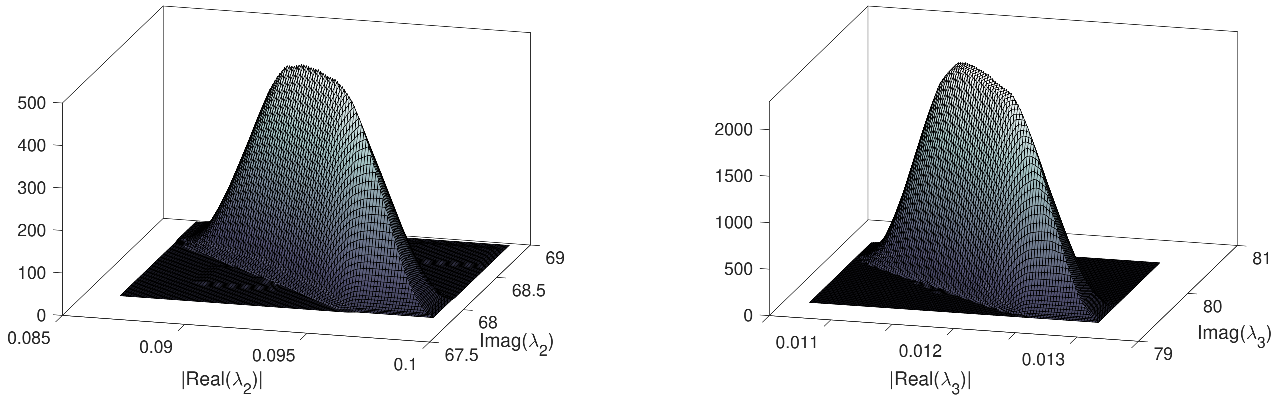
where n denotes the number of samples and K is a kernel function, which is often defined as a multivariate symmetric probability function, $\mathbf{H}$ is a positive definite 2×2 symmetric matrix, it is called bandwidth matrix, that controls the amount and orientation of smoothing induced. $K_{\mathbf{H}}$ is the scaled kernel function, its choice does not influence the accuracy of the estimation as the bandwidth matrix does. For this reason, the standard multivariate normal kernel is usually used

$$K_{\mathbf{H}} = (2\pi)^{-1} \mathbf{H}^{-1/2} \exp\left(-\frac{1}{2} \mathbf{x}^T \mathbf{H}^{-1} \mathbf{x}\right) \tag{10}$$

The bandwidth matrices imposed for the kernel density estimate of the two sets of data are:

$$\mathbf{H}_{\lambda_2} = \begin{bmatrix} 8.77e-4 & 0 \\ 0 & 7.73e-2 \end{bmatrix} \qquad \mathbf{H}_{\lambda_3} = \begin{bmatrix} 1.60e-4 & 0 \\ 0 & 8.49e-2 \end{bmatrix}$$

The contribution of each sample of the set is smoothed into the region around it, then aggregating all the contributions, the bi-variate PDF of the data is defined. A disadvantage of the KDE is that variables are assumed to be independent, requiring the application of a copula [1] to describe the statistical interactions between the components of a bivariate distribution. In the present case a t-family (t-distribution) pair-cupola was applied with zero rotation and parameter $\theta_2 = [-0.999 \;\; 0.1001]$ for the λ_2 distribution and $\theta_3 = [0.9978 \;\; 0.6303]$ for the λ_3 distribution. The resulting bi-variate PDFs are shown in Figure 5. The inter-dependence between the real and imaginary parts is clearly visible, with rather 'uniform' appearance in the central section, as might be expected since the generating distributions on Δk_{12} and Δk_{23} were uniform.



(a) Second eigenvalue (λ_2). (b) Third eigenvalue (λ_3)

Figure 5 – Bivariate PDFs for λ_2 and λ_3.

The data is of exceptionally high quality, having been generated from precise knowledge of the uncertainty we intend to estimate. In real systems, in industry, only very vague knowledge of eigenvalue distributions would be expected, with somewhat better understanding of natural frequencies than of damping. In the following section the distributions on stiffness k_{12} and k_{23} will be estimated by stochastic model updating.

6. Stochastic Model Updating

There are numerous ways in which stochastic model updating may be carried out [10, 8], but the present case use is made of the Woodbury formula [16], for the modification of an inverse matrix. The matrix in question is the dynamic stiffness matrix, whose inverse is the matrix of receptances,

$$\mathbf{H}(s) = \mathbf{Z}^{-1}(s) = \left(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K} \right)^{-1} \tag{11}$$

The Woodbury formula takes the form,

$$\left(\mathbf{Z}(s) + \mathbf{V}\mathbf{P}(s)\mathbf{V}^T \right)^{-1} = \mathbf{Z}^{-1}(s) - \mathbf{Z}^{-1}(s)\mathbf{V}\left(\mathbf{I} + \mathbf{P}(s)\mathbf{V}^T\mathbf{Z}^{-1}(s)\mathbf{V} \right)^{-1}\mathbf{P}(s)\mathbf{V}^T\mathbf{Z}^{-1}(s) \tag{12}$$

where specifically,

$$\mathbf{P} = \begin{bmatrix} \Delta k_{12}(1+\beta s) & 0 \\ 0 & \Delta k_{23}(1+\beta s) \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$\mathbf{P}$ is a diagonal matrix whose terms are the parameters of the modification, $\mathbf{V}$ is a matrix whose terms are the coordinates of the modifications.
The characteristic equation of the modified system is then given by,

$$\det\left(\mathbf{I} + \mathbf{P}(\lambda_i)\mathbf{V}^T\mathbf{Z}(\lambda_i)^{-1}\mathbf{V} \right) = 0 \qquad i = 2,3 \tag{13}$$

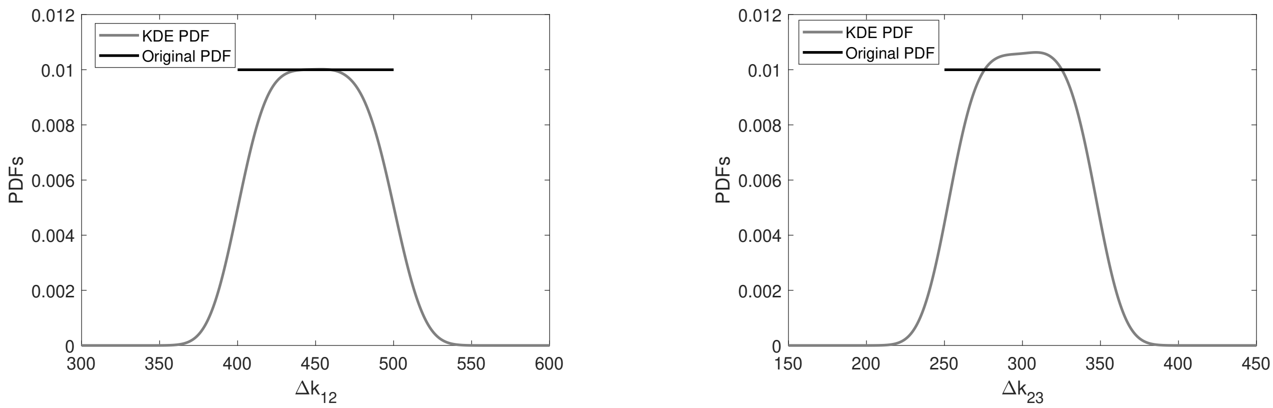
and may be solved for Δk_{12} and Δk_{23} when cast in modal coordinates, whereupon

$$\mathbf{V} = \begin{bmatrix} (\phi_1 - \phi_2) & (\phi_2 - \phi_3) \end{bmatrix} \tag{14}$$

The coordinate transform is necessary for the updating of a second metamodel with inputs, Δk_{12} and Δk_{23}, and outputs λ_2 and λ_3 – the eigenvectors are assumed constant.
The two eigenvalues are assigned to the characteristic equation and the nonlinear system is solved using a Gauss-Newton algorithm. Sobol sampling is applied with 200 points from the bivariant eigenvalue distributions developed (in Section 4). PDFs for the two stiffness k_{12} and k_{23} are produced by univariate KDE, as shown in Figure 6. As can be seen in the figure the estimated PDFs compare very favourably with original (generating) uniform distributions. The means and standard distributions are provided in Table 3.
Clearly the estimates of the means are perfect and there are small errors in the standard deviations. This may be due to inaccuracy in the bivariant eigenvalue PDFs obtained by KDE, resulting from the assumption of independence of the eigenvalue distributions and the choice of copula.

Table 4 – Statistics of the generating and estimated PDFs (N/m).

	Generating Distribution		Estimated Distribution	
Parameter	Mean	STD	Mean	STD
Δk_{12}	450	28.87	450	30.45
Δk_{23}	300	28.87	300	28.77



(a) Δk_{12}. (b) Δk_{23}.

Figure 6 – Generating and estimated PDFs for Δk_{12} and Δk_{23}.

7. Conclusion

A framework for stochastic model updating has been demonstrated in a simulated experiment of a three degree of freedom damped mass-spring system. The procedure makes use of the PC expansion, global sensitivity, KDE and Woodbury formula for structural modification. The methodology results in close agreement between the generating-parameter and estimated PDFs, especially the first two statistical moments. In a realistic application, such as an industrial structure, the eigenvalue distributions are likely to be known only vaguely, with better understanding of frequencies (perhaps within upper and lower bounds) than damping, which might be considered constant. Further development and proving of the methodology are planned, including an industrial application.

8. Copyright Statement

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