ADAPTIVE CONTROL INVERSION AND ACTUATOR FAILURE COMPENSATION FOR UAVS

Chris Raabe
The University of Tokyo
christopher.t.raabe@gmail.com

Keywords: *adaptive control, control inversion, fault tolerant control, failure tolerant control*

Abstract

For MIMO systems where it is desired to control n state rates with n actuators, this paper proposes a method to: 1) invert the design system's control dynamics to construct a control signal, 2) simultaneously estimate and adapt to differences between real and design actuator effectiveness, and 3) automatically switch control goals to attempt to maintain critical control upon detection of (an) ineffective actuator(s). The estimation and adaptation is based on indirect model reference adaptive control (indirect MRAC) similar to what is discussed in [1].

1 Introduction

1.1 Motivating Example

Consider the example case of a simple v-tail UAV with only four actuators: control surfaces on each of the tails, and two counter-rotating propellers under the wings. This configuration is shown in figure 1.

The primary goal is to control the following state-rates: roll acceleration, pitch acceleration, yaw acceleration, and acceleration in the forward direction (annotated $\dot{p}$, $\dot{q}$, $\dot{r}$, and $\dot{u}$ respectively). However, each of the four actuators affects several of the state-rates, so it is desired to invert the system dynamics so that the necessary actuation that will result in the desired state-rates can be solved for directly. Next it is desired to maintain this control inversion in spite of unknown, or an unforeseen change in, actuator effectiveness.

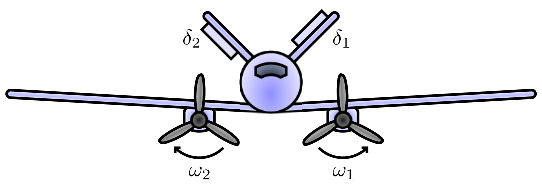

Fig. 1 Motivating Example: Simple V-Tail UAV

Finally it is clear that critical control of this kind of system (i.e. altitude and heading) should be possible even if one or two actuators become ineffective. Therefore, it is also desired to detect such failures and switch to a critical control mode in such cases.

1.2 Overview of Proposed Solution

The control method that is proposed to achieve the above goals is outlined in block form in figure 2. The essence of the control method can be described in three parts. First, a reference actuator state is constructed by inverting the dynamics of a model of the system in attempts to achieve some desired state-rates. Second, the model is reused in a kind of indirect model reference adaptive controller (indirect MRAC) to estimate changes in the effectiveness of the actuators and modulate the control signal appropriately to compensate. Third, an effectiveness monitor is used to detect ineffective actuators and switch the control inversion to maintain critical control by abandoning lower priority control goals as necessary.

This approach is especially useful for situations where it is unfeasible or uneconomical to

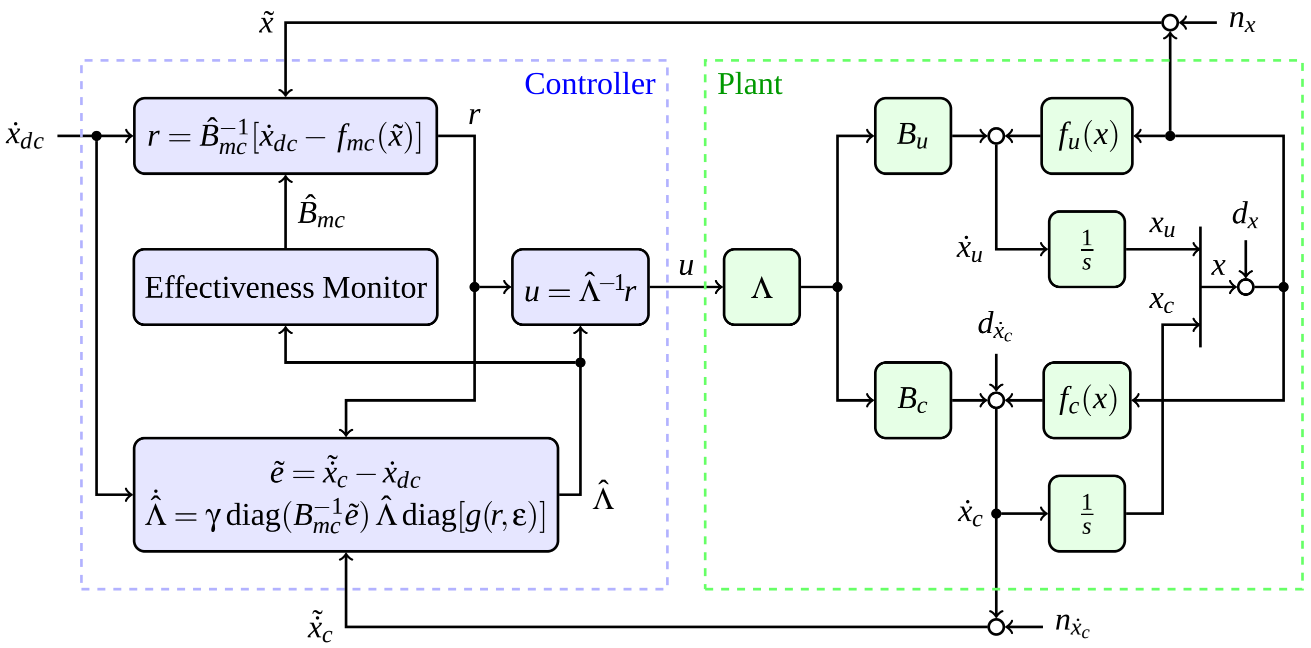

Fig. 2 Block Diagram of Proposed Control Scheme

have dedicated sensors to directly measure actuator effectiveness. In the case of the simple v-tail UAV described in section 1.1, such scenarios could be a failure of the linkage between a servo and a control surface or a propeller that becomes unsecured from the motor. In both cases, the actuators may continue to faithfully track commands but, unbeknown to a typical controller, would be completely ineffective.

2 Theory

2.1 Control Inversion

Consider a MIMO system that can be described by:

$$\dot{x}(t) = f(x(t)) + Br(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system; $f : \mathbb{R}^n \to \mathbb{R}^n$ is a possibly non-linear state function that describes how the system state effects the state derivatives; $B \in \mathbb{R}^{n\times m}$ is a constant matrix that describes how ideally effective actuators affect the state derivatives; and $r(t) \in \mathbb{R}^m$ is a reference actuator state.

Let a corresponding model of this system be given by:

$$\dot{x}_m(t) = f_m(x_m(t)) + B_m r(t) \tag{2}$$

with dimensions corresponding to those of the real system.

Assume that it is desired to directly control exactly m state derivatives with the m actuators. Then, without loss of generality, the system description in equation 1 can be divided as follows:

$$\begin{bmatrix}\dot{x}_c(t)\\ \dot{x}_u(t)\end{bmatrix} = \begin{bmatrix}f_c(x(t))\\ f_u(x(t))\end{bmatrix} + \begin{bmatrix}B_c\\ B_u\end{bmatrix} r(t) \tag{3}$$

where $x_c(t) \in \mathbb{R}^m$, $x_u(t) \in \mathbb{R}^{n-m}$, $f_c : \mathbb{R}^n \to \mathbb{R}^m$, $f_u : \mathbb{R}^n \to \mathbb{R}^{m-n}$, $B_c \in \mathbb{R}^{m\times m}$, and $B_u \in \mathbb{R}^{(n-m)\times m}$. The subscript "c" indicates the grouping of state derivatives that are desired to be directly controlled and the subscript "u" indicates the grouping of state derivatives that are to be uncontrolled.

The model of the system can be divided in a similar fashion:

$$\begin{bmatrix}\dot{x}_{mc}(t)\\ \dot{x}_{mu}(t)\end{bmatrix} = \begin{bmatrix}f_{mc}(x_m(t))\\ f_{mu}(x_m(t))\end{bmatrix} + \begin{bmatrix}B_{mc}\\ B_{mu}\end{bmatrix} r(t) \tag{4}$$

Let us now take the top row of equation 4 and attempt to invert the dynamics to construct a reference actuator state, r, that will result in some desired state rates, $\dot{x}_{dc}$. To do so, let us assume that the real system's state is measurable, so that we can set the model's state, x_m to the measured real state, $\tilde{x}$, and set the model's state derivatives, $\dot{x}_{mc}$, to the desired values, $\dot{x}_{dc}$. Then the description becomes:

$$\dot{x}_{dc}(t) = f_{mc}(\tilde{x}(t)) + B_{mc}r(t) \tag{5}$$

In the above equation, r is the only unknown and it can be solved for if the model's input matrix, B_{mc}, is invertible. In that case, equation 5 can be rewritten as:

$$r(t) = B_{mc}^{-1}[\dot{x}_{dc}(t) - f_{mc}(\tilde{x}(t))] \tag{6}$$

Finally, by comparing equation 5 to the top row of equation 3, it is clear that, given r from equation 6, the real state rates that are desired to be controlled, $\dot{x}_c$, will match the desired values, $\dot{x}_{dc}$, if the model perfectly describes the system (i.e. $f_{mc} = f_c$ and $B_{mc} = B_c$) and the system state is perfectly measurable (i.e. $\tilde{x} = x$).

2.2 Adaptive Compensation for Actuator Effectiveness

2.2.1 Unknown Actuator Effectiveness

Now suppose that in reality, the effectiveness of the actuators is either unknown, or has changed due to some damage, failure, etc. In that case, the description of the system can be written as:

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_u(t) \end{bmatrix} = \begin{bmatrix} f_c(x(t)) \\ f_u(x(t)) \end{bmatrix} + \begin{bmatrix} B_c \\ B_u \end{bmatrix} \Lambda(t)u(t) \tag{7}$$

where $\Lambda(t) \in \mathbb{R}^{+m\times m}$ is a positive diagonal matrix representing actuator effectiveness, and $u(t) \in \mathbb{R}^m$ is the real actuator state.

It is desired to construct an actuator state, u, such that the response of the real system to this actuator state will match the response of the ideal system to the reference actuator state, r. By inspection of equations 3 and 7, the response of the real system to u and the response of the ideally effective system to r, will match if $u = \Lambda^{-1}r$.

However, u cannot be constructed from $\Lambda^{-1}r$ since the diagonal matrix of actuator effectiveness, Λ, is not known a priori. Rather, let us define $\hat{\Lambda}$ as the estimate of Λ, and construct u such that $u = \hat{\Lambda}^{-1}r$. Then the real system dynamics become:

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_u(t) \end{bmatrix} = \begin{bmatrix} f_c(x(t)) \\ f_u(x(t)) \end{bmatrix} + \begin{bmatrix} B_c \\ B_u \end{bmatrix} \Lambda(t)\hat{\Lambda}^{-1}(t)r(t) \tag{8}$$

Note that if the system state, x, and the state derivatives that are desired to be controlled, $\dot{x}_c$, are perfectly measurable, then Λ is the only unknown in the top row of equation 8. However, Λ often cannot be solved for directly since one or more elements of r may often be zero. Rather, let us attempt to define an error and a derivative of $\hat{\Lambda}$ such that $\hat{\Lambda}$ approaches Λ (i.e. an improved estimate) whenever r is not (close to) zero.

2.2.2 Adaptive Law

First, let us define error as the difference between the state derivatives that are desired to be controlled and their desired values:

$$e(t) = \dot{x}_c(t) - \dot{x}_{dc}(t) \tag{9}$$

Similarly, measured error can be written as:

$$\tilde{e}(t) = \tilde{\dot{x}}_c(t) - \dot{x}_{dc}(t) \tag{10}$$

where a tilde indicates a measured value.

Substituting equations 5 and 8 into the the equation 9 gives:

$$\begin{aligned} e(t) = &[f_c(x(t)) + B_c\Lambda(t)\hat{\Lambda}^{-1}(t)r(t)] \\ &- [f_{mc}(\tilde{x}(t)) + B_{mc}r(t)] \end{aligned} \tag{11}$$

Making the assumptions that the real system perfectly matches the model (i.e. $f_c = f_{mc}$ and $B_c = B_{mc}$) and and the system state is perfectly measurable (i.e. $\tilde{x} = x$), equation 11 can be rewritten as:

$$\begin{aligned} e(t) &= [\cancel{f_{mc}(x(t))} + B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)r(t)] \\ &\quad - [\cancel{f_{mc}(x(t))} + B_{mc}r(t)] \\ &= B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]r(t) \end{aligned} \tag{12}$$

Notice that if the estimate $\hat{\Lambda}$ approaches Λ, then error will approach zero.

Next, let us propose the definition for the derivative of $\hat{\Lambda}$ as:

$$\dot{\hat{\Lambda}}(t) = \gamma\,\mathrm{diag}(B_{mc}^{-1}\tilde{e}(t))\,\hat{\Lambda}(t)\,\mathrm{diag}[g(r(t),\varepsilon)] \tag{13}$$

where $\gamma, \varepsilon \in \mathbb{R}^+$ are constants to be defined by the control designer, and $g(r(t), \varepsilon)$ is a modified inverting function given by:

$$(g(\alpha, \varepsilon))_i = \begin{cases} {}^1/_{\alpha_i} & |\alpha_i| > \varepsilon \\ {}^{\alpha_i}/_{\varepsilon^2} & |\alpha_i| \leq \varepsilon \end{cases} \tag{14}$$

Assuming perfect measurement, equation 12 can be substituted into equation 13, and properties of diagonal matrices can be employed to simplify the estimation dynamics to:

$$\begin{aligned} \dot{\hat{\Lambda}}(t) &= \gamma \operatorname{diag}[\cancel{B_{mc}^{-1} B_{mc}} (\Lambda(t)\hat{\Lambda}^{-1}(t) - I) r(t)] \\ &\quad \hat{\Lambda}(t) \operatorname{diag}[g(r(t), \varepsilon)] \\ &= \gamma\, (\Lambda(t)\hat{\Lambda}^{-1}(t) - I) \operatorname{diag}(r(t)) \\ &\quad \hat{\Lambda}(t) \operatorname{diag}[g(r(t), \varepsilon)]^{\S} \\ &= \gamma\, (\Lambda(t)\hat{\Lambda}^{-1}(t) - I) \hat{\Lambda}(t) \operatorname{diag}(r(t)) \\ &\quad \operatorname{diag}[g(r(t), \varepsilon)]^{\dagger} \\ &= \gamma\, (\Lambda(t)\cancel{\hat{\Lambda}^{-1}(t)\hat{\Lambda}(t)} - \hat{\Lambda}(t)) \operatorname{diag}(r(t)) \\ &\quad \operatorname{diag}[g(r(t), \varepsilon)] \\ &= \gamma\, (\Lambda(t) - \hat{\Lambda}(t)) \operatorname{diag}(r(t)) \operatorname{diag}[g(r(t), \varepsilon)] \end{aligned} \tag{15}$$

In order to simplify future notation, let us define a few new matrices as follows:

$$\begin{aligned} R(t) &= \operatorname{diag}(r(t)) \\ G(t, \varepsilon) &= \operatorname{diag}[g(r(t), \varepsilon)] \\ \mathrm{K}(t, \varepsilon) &= R(t) G(t, \varepsilon) \end{aligned} \tag{16}$$

Upon inspection of K, we find that the elements are:

$$\kappa_{ii}(t, \varepsilon) = \begin{cases} 1 & |r_i(t)| > \varepsilon \\ {}^{r_i^2(t)}/_{\varepsilon^2} & |r_i(t)| \leq \varepsilon \end{cases} \tag{17}$$

which indicates that K is a positive-diagonal matrix.

Substituting matrix K into equation 15 gives:

$$\dot{\hat{\Lambda}}(t) = \gamma\, (\Lambda(t) - \hat{\Lambda}(t))\, \mathrm{K}(t, \varepsilon) \tag{18}$$

$^{\dagger}\operatorname{diag}(\vec{a})\operatorname{diag}(\vec{b}) = \operatorname{diag}(\vec{b})\operatorname{diag}(\vec{a})$

$^{\ddagger}\operatorname{diag}(\vec{a})\vec{b} = \operatorname{diag}(\vec{b})\vec{a}$

$^{\S}\operatorname{diag}(\operatorname{diag}(\vec{a})\vec{b}) = \operatorname{diag}(\vec{a})\operatorname{diag}(\vec{b})$

$^{\P}\operatorname{diag}(\vec{a} + \vec{b}) = \operatorname{diag}(\vec{a}) + \operatorname{diag}(\vec{b})$

Note that the above equation is strictly diagonal, so each diagonal element can be considered individually as follows:

$$\dot{\hat{\lambda}}_{ii}(t) = \gamma\, (\lambda_{ii} - \hat{\lambda}_{ii}(t))\, \kappa_{ii}(t, \varepsilon) \tag{19}$$

Now let us consider the above equation in 3 cases at some time t:

1. $r_i(t) = 0$
2. $|r_i(t)| > \varepsilon$
3. $0 < |r_i(t)| \leq \varepsilon$

2.2.3 Analysis Case 1

In the first case, $r_i = 0$ makes $\kappa_{ii} = 0$, thereby simplifying equation 19 to:

$$\dot{\hat{\lambda}}_{ii}(t) = 0 \tag{20}$$

So at time t, $\hat{\lambda}_{ii}$ is neither converging toward nor diverging from the true value, λ_{ii} (i.e. critically stable).

2.2.4 Analysis Case 2

In the second case, $|r_i| > \varepsilon$ makes $\kappa_{ii} = 1$, thereby simplifying equation 19 to:

$$\dot{\hat{\lambda}}_{ii}(t) = \gamma\, (\lambda_{ii} - \hat{\lambda}_{ii}(t)) \tag{21}$$

Taking the Laplace transform of this differential equation gives the following transfer function:

$$\frac{\hat{\lambda}_{ii}(s)}{\lambda_{ii}(s)} = \frac{1}{\gamma^{-1} s + 1} \tag{22}$$

which is a simple first-order lag of λ_{ii} with time-constant γ^{-1}.

Figure 3 shows the positive and negative approach trajectories for $\hat{\lambda}_{ii}$ to a constant λ_{ii}, moving backwards in time from when $\hat{\lambda}_{ii}$ is within 1% of the value of λ_{ii}.

Notice that increasing γ increases the speed of the approach.

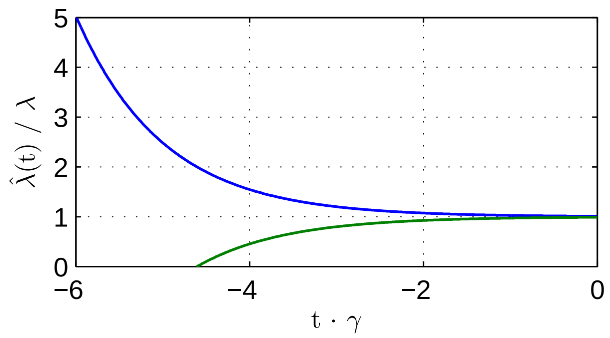

Fig. 3 Approach Trajectories of Actuator Effectiveness Estimates

2.2.5 Analysis Case 3

In the third case, $0 < |r_i(t)| \leq \varepsilon$ makes $\kappa_{ii}(t,\varepsilon) = r_i^2(t)/\varepsilon^2$, and equation 19 becomes:

$$\dot{\hat{\lambda}}_{ii}(t) = \gamma\,(\lambda_{ii} - \hat{\lambda}_{ii}(t))\,\frac{r^2(t)}{\varepsilon^2} \qquad (23)$$

The case statement $0 < |r_i(t)| \leq \varepsilon$ can be rephrased as $0 < r_i^2(t)/\varepsilon^2 \leq 1$. Therefore, the effect of $r_i^2(t)/\varepsilon^2$ in the above equations is equivalent to decreasing the value of γ, i.e. a slower asymptotic approach to λ_{ii}.

2.2.6 Error Dynamics

Sections 2.2.3 through 2.2.5 prove that, assuming perfect measurement and a perfect model of the system, the estimates of actuator effectiveness, $\hat{\Lambda}$, are guaranteed to asymptotically approach the true values, Λ, whenever $r \neq 0$. Furthermore, the speed of approach will be maximum (i.e. time constant of γ^{-1}) whenever $|r| \geq \varepsilon$.

Inspection of equation 12 has already shown that error, e, goes to zero when $\hat{\Lambda}$ approaches Λ. Now let us examine the error dynamics during that approach.

Taking the derivative of equation 12 with respect to time gives:

$$\begin{aligned}\dot{e}(t) = {} & B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]\dot{r}(t)\\ & + B_{mc}\dot{\Lambda}(t)\hat{\Lambda}^{-1}(t)r(t)\\ & - B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)\dot{\hat{\Lambda}}(t)\hat{\Lambda}^{-1}(t)r(t) \qquad (24)\end{aligned}$$

Let us make a few assumptions to simplify this analysis. First, let us assume that real actuator effectiveness does not change often (i.e. damage or failures are rare) so $\dot{\Lambda}$ is typically 0 and therefore the second term in equation 24 is typically 0. Second, we have just shown that the estimate $\hat{\Lambda}$ is guaranteed to approach a constant Λ, so the first term in this equation is also guaranteed to go to zero. Furthermore, since r is usually bounded and smooth for actuators, $\dot{r}$ is also bounded and cannot be sustained away from zero. Therefore, we can ignore the first two terms while being aware that they may occasionally perturb the remaining dynamics.

Ignoring the first two terms of equation 24 and substituting in equation 12 and the simplified notation of equations 16 gives:

$$\begin{aligned}\dot{e}(t) = {} & -B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)[\gamma\,\mathrm{diag}(B_{mc}^{-1}e(t))\,\cancel{\hat{\Lambda}(t)}\\ & G(t,\varepsilon)]\cancel{\hat{\Lambda}^{-1}(t)}r(t)^{\dagger}\\ = {} & -\gamma B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)G(t,\varepsilon)\\ & \mathrm{diag}(r(t))B_{mc}^{-1}e(t)^{\dagger\ddagger}\\ = {} & -\gamma\,\underbrace{B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)\mathrm{K}(t,\varepsilon)B_{mc}^{-1}}_{A}e(t) \qquad (25)\end{aligned}$$

According to [2] The stability of the above equation can be determined by analyzing the eigenvalues of matrix A, where eigenvalues are all of the solutions, ν, to the equation:

$$\det\{A - \nu I\} = 0 \qquad (26)$$

Expanding matrix A and using properties of determinants, equation 26 becomes:

$$\begin{gathered}\det\{-\gamma B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)\mathrm{K}(t,\varepsilon)B_{mc}^{-1} - \nu I\} = 0\\ \Downarrow\\ \det\{B_{mc}[-\gamma\Lambda(t)\hat{\Lambda}^{-1}(t)\mathrm{K}(t,\varepsilon) - \nu I]B_{mc}^{-1}\} = 0\\ \Downarrow\\ \cancel{\det(B_{mc})}\det\{-\gamma\Lambda(t)\hat{\Lambda}^{-1}(t)\mathrm{K}(t,\varepsilon)\\ -\nu I\}\cancel{\det(B_{mc}^{-1})} = 0^{*\|}\\ \Downarrow\\ \det\{-\gamma\Lambda(t)\hat{\Lambda}^{-1}(t)\mathrm{K}(t,\varepsilon) - \nu I\} = 0 \qquad (27)\end{gathered}$$

$^*\det(AB) = \det(A)\det(B)$

$^{\|}\det(A^{-1}) = \det(A)^{-1}$

What remains is purely diagonal, so the eigenvalues are simply each of the diagonal elements of the inner left part:

$$\nu_i(t) = -\gamma \frac{\lambda_{ii}(t)}{\hat{\lambda}_{ii}(t)} \kappa_{ii}(t) \tag{28}$$

Since γ, λ_{ii}, $\hat{\lambda}_{ii}$, and κ_{ii} are positive, then it is clear that the i^{th} eigenvalue is real and negative when r_i is not zero (and zero when it is). Negative eigenvalues indicate that the error will asymptotically approach zero with speed proportional to the size of the eigenvalue. Therefor, increasing γ and decreasing ε will generally result in an increase the speed of the approach.

2.3 Automatic Compensation for Failed Actuators

Section 2.2 showed that the effectiveness of an actuator can be quickly and accurately estimated. This includes the case where an actuator becomes ineffective, i.e. effectiveness near zero.

When one or more of m actuators become ineffective, the ability to directly control m state derivatives is no longer achievable. However, the control inversion process presented in section 2.1 can be modified to maintain control of higher-priority state derivatives, while releasing control of lower-priority state derivatives. Specifically, the model of the input matrix, B_{mc}, used to build the reference actuator state, r, in equation 6 can be manipulated such that effective actuators should not attempt to affect lower-priority state derivatives, and contributions from ineffective actuators to higher-priority states should be ignored.

For example, consider a system with four actuators. The control designer decides that, in the case of failure of actuator 2, control of state derivatives 1, 2 and 3 should be maintained, and state derivative 4 should be released. This is achieved by placing zeros in the elements of B_{mc} that correspond to actuator 2's contribution to state derivatives 1, 2 and 3; and actuator 1, 3 and 4's contribution to state derivative 4, as follows:

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} \rightarrow \begin{bmatrix} B_{11} & 0 & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & 0 & B_{33} & B_{34} \\ 0 & B_{42} & 0 & 0 \end{bmatrix}$$

When the estimate of the effectiveness of actuator 2 drops below a threshold that is predetermined by the control designer, the controller should automatically switch B_{mc} in the formulation of r, i.e. equation 6, to the above predetermined modified B_{mc}.

Let us then rephrase equation 6 with this behavior in mind:

$$r(t) = \hat{B}_{mc}^{-1}[\dot{x}_{dc}(t) - f_{mc}(\tilde{x}(t))] \tag{29}$$

where $\hat{B}_{mc}$ is the nominal B_{mc} when no actuators are estimated to be ineffective, but switches to a modified version of B_{mc} when an actuator is estimated to be ineffective. Note that there should be a different modified version of B_{mc} for each combination of failures that can remain controllable. The control designer must exercise some caution when choosing which state derivative(s) to release in each failure scenario to ensure that $\hat{B}_{mc}$ remains invertible and the overall system remains controllable.

3 Simulation Results

The theory described in section 2 was applied to a simplified simulation of the small v-tail UAV described in section 1.1.

Given four actuators, two counter-rotating propellers and control surfaces on each of the two tails, it was desired to control the following four state rates: $\dot{p}$, $\dot{q}$, $\dot{r}$, and $\dot{u}$. The control inversion described in section 2.1 was successfully applied to this model to isolate each of those rates for independent control.

An example simulation result is presented in the following figures. In this example, the simulation begins trimmed in straight and level flight. The tail control surfaces are deflected slightly to produce a pitch-down moment to counteract a pitch-up moment from the engines.

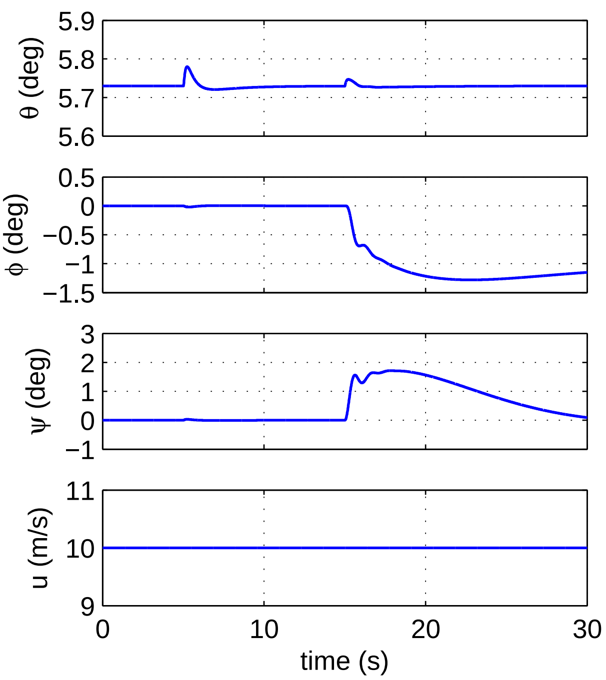

Fig. 4 Aircraft States

Figure 4 shows the result to the aircraft attitude and speed. Figure 5 shows the actuator state. Figure 6 shows the estimation of actuator effectiveness.

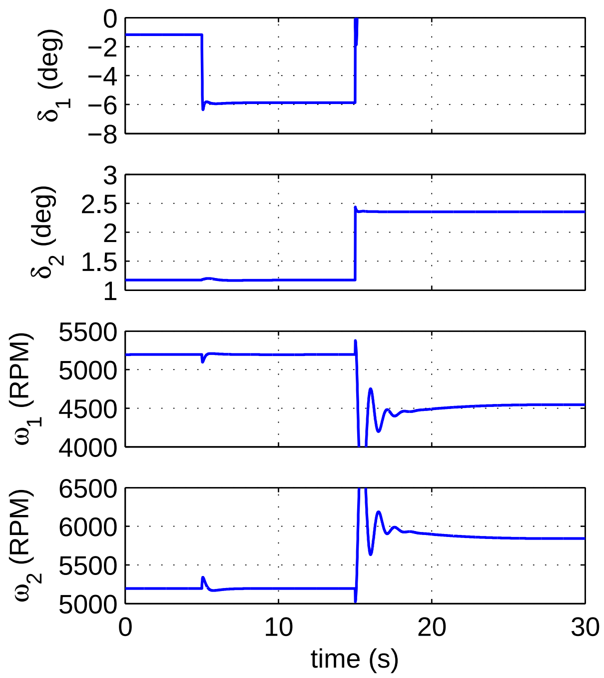

Fig. 5 Actuator Commands

At $t = 5$ seconds, the left tail control surface loses 80% of its effectiveness. The adaptive controller proposed in section 2.2 quickly and accurately estimates the reduced effectiveness of the actuator and effectively compensates by increasing deflection.

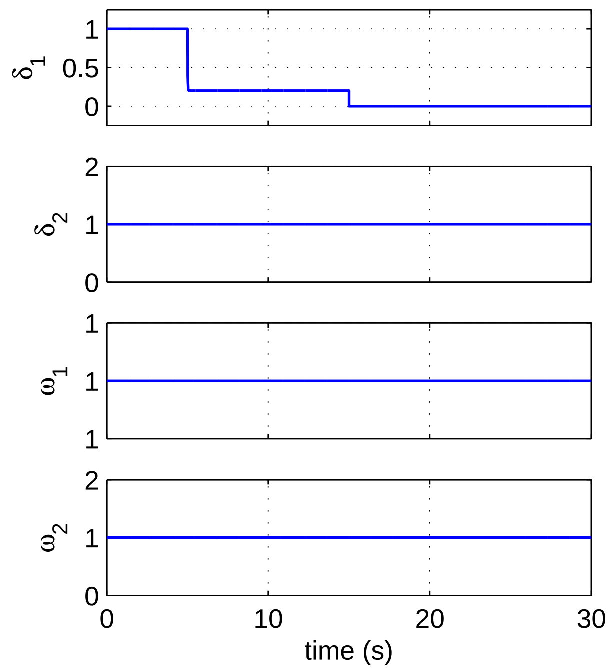

Fig. 6 Actuator Effectiveness Estimates

At $t = 15$ seconds, the left control surface becomes completely ineffective. The ineffectiveness is quickly detected and the scheme proposed in section 2.3 is stimulated in order to abandon control of $\dot{r}$ in favor of the higher priority state rates: $\dot{p}$, $\dot{q}$, and $\dot{u}$.

In this simulation, simple PID controllers were used in the outer-loop to control airspeed through $\dot{u}$, altitude through $\dot{q}$, heading through $\dot{p}$, and remove sideslip through $\dot{r}$. After abandoning control of $\dot{r}$, the outer-loop controller was able to recover the desired heading, but resulted in crabbed flight (non-wings-level sideslip) since there was no longer the ability to remove sideslip through $\dot{r}$.

These simulation results effectively demonstrate the intended use for the control method proposed in this paper. Loss of actuator effectiveness is effectively compensated for using indirect model reference adaptive control until the effectiveness drops below some predetermined minimum threshold. At that point the actuator is considered ineffective and the controller automatically abandons control of some low priority state derivative in order to maintain critical control of the higher priority state derivatives.

4 Sensitivity Analysis

Section 2 introduced the theory and stability proofs for the adaptive control inversion method presented in this paper. However, the stability proofs often relied on strict assumptions of perfect measurement and incorporation of a perfectly accurate model of the system. This section will consider various deviations from those assumptions, similar to the analysis performed in [3].

4.1 Measurement Noise and Disturbances

For this analysis, let us assume that the real system perfectly matches the model of the system (i.e. $f_c = f_{mc}$ and $B_c = B_{mc}$), except that it is subject to some disturbances to both the state and state derivatives. In addition, let us assume that the measurements of both the state and state derivatives contain some noise. Then the system can be described by:

$$\begin{aligned}
\dot{x}_c(t) &= f_{mc}(x(t) + d_x(t)) \\
&\quad + B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)r(t) + d_{\dot{x}_c}(t) \\
\tilde{x}(t) &= x(t) + d_x(t) + n_x(t) \\
\tilde{\dot{x}}_c(t) &= \dot{x}_c(t) + n_{\dot{x}_c}(t)
\end{aligned} \tag{30}$$

In this case, the measured error, $\tilde{e}$, will differ from true error since it will contain measurement noise. Substituting the above system descriptions into the definition of measured error, equation 10, gives:

$$\begin{aligned}
\tilde{e}(t) &= [f_{mc}(x(t) + d_x(t)) + B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)r(t) \\
&\quad + d_{\dot{x}_c}(t) + n_{\dot{x}_c}(t)] - [f_{mc}(x(t) + d_x(t) \\
&\quad + n_x(t)) + B_{mc}r(t)] \\
&= B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]r(t) + \\
&\quad \left.\begin{array}{l} [d_{\dot{x}_c}(t) + n_{\dot{x}_c}(t) + f_{mc}(x(t) + d_x(t)) \\ - f_{mc}(x(t) + d_x(t) + n_x(t))] \end{array}\right\} \tilde{n}(t) \\
&= B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]r(t) + \tilde{n}(t)
\end{aligned} \tag{31}$$

where all of the noise and disturbance terms have been grouped into a single term called $\tilde{n}$.

The above description of measured error can then be substituted into the definition of the estimate derivative, equation 13, and simplified as follows:

$$\begin{aligned}
\dot{\hat{\Lambda}}(t) &= \gamma\,\mathrm{diag}\{B_{mc}^{-1}[B_{mc}(\Lambda(t)\hat{\Lambda}^{-1}(t) - I)r(t) \\
&\quad + \tilde{n}(t)]\}\hat{\Lambda}(t)G(t,\varepsilon) \\
&= \gamma\,\hat{\Lambda}(t)\{\mathrm{diag}[(\Lambda(t)\hat{\Lambda}^{-1}(t) - I)r(t)] \\
&\quad + \mathrm{diag}(B_{mc}^{-1}\tilde{n}(t))\}\,G(t,\varepsilon)^{\dagger\P} \\
&= \gamma\,\hat{\Lambda}(t)\{(\Lambda(t)\hat{\Lambda}^{-1}(t) - I)\mathrm{diag}(r(t)) \\
&\quad + \mathrm{diag}(B_{mc}^{-1}\tilde{n}(t))\}\,G(t,\varepsilon)^{\S} \\
&= \gamma\,\{(\Lambda(t) - \hat{\Lambda}(t)) + \hat{\Lambda}(t)\mathrm{diag}(B_{mc}^{-1}\tilde{n}(t)) \\
&\quad R^{-1}(t)\}R(t)\,G(t,\varepsilon) \\
&= \gamma\,\{\Lambda(t)\mathrm{K}(t,\varepsilon) - \hat{\Lambda}(t)[\mathrm{K}(t,\varepsilon) \\
&\quad - \mathrm{diag}(B_{mc}^{-1}\tilde{n}(t))G(t,\varepsilon)]\}
\end{aligned} \tag{32}$$

Upon inspection, we find that the resulting estimate dynamics are identical to the ideal case, equation 18, except that they include an extra term containing $\tilde{n}$. Here are few observations that can be made from the presence of this extra term:

- $(B_{mc}^{-1}\tilde{n})_i \cdot g(r_i,\varepsilon)$ perturbs $\hat{\lambda}_{ii}$ either toward zero if negative or toward positive infinity if positive
- This perturbation will occur with a "gain" of $\gamma \cdot g(r_i,\varepsilon)$
- A sustained $(B_{mc}^{-1}\tilde{n})_i \cdot g(r_i,\varepsilon) > \hat{\lambda}_{ii} \cdot \kappa_{ii}$ will cause $\hat{\lambda}_{ii}$ to diverge toward infinity at an exponentially increasing rate
- A large enough excursion toward zero will cause the effectiveness monitor to believe that an actuator has become ineffective

Now let us examine the effects of these estimation dynamics on the real error dynamics.

Substituting the system description from equations 30 into the definition of real error, equation 9, gives:

$$\begin{aligned} e(t) &= [f_{mc}(x(t)+d_x(t)) + B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)r(t) \\ &\quad + d_{\dot{x}_c}(t)] - [f_{mc}(x(t)+d_x(t)+n_x(t)) \\ &\quad + B_{mc}r(t)] \\ &= B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]r(t) + \\ &\quad \left.\begin{matrix} [d_{\dot{x}_c}(t) + f_{mc}(x(t)+d_x(t)) \\ - f_{mc}(x(t)+d_x(t)+n_x(t))] \end{matrix}\right\} n(t) \\ &= B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]r(t) + n(t) \end{aligned} \tag{33}$$

which is identical to the measured error, but without the influence of noise from the measurement of the state rates.

Taking the derivative of the above equation with respect to time gives:

$$\begin{aligned} \dot{e}(t) &= B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]\dot{r}(t) \\ &\quad + B_{mc}\dot{\Lambda}(t)\hat{\Lambda}^{-1}(t)r(t) \\ &\quad - B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)\dot{\hat{\Lambda}}(t)\hat{\Lambda}^{-1}(t)r(t) \\ &\quad + \dot{n}(t) \end{aligned} \tag{34}$$

Let us analyze this equation one term at a time as had been done for the ideal case, equation 12, in section 2.2.6. Let us again assume that real actuator effectiveness does not change often (i.e. damage or failures are rare) so $\dot{\Lambda}$ is typically 0 and therefore the second term in equation 34 is typically 0. It was also shown in section 2.2.6 that the third term is stabilizing and will drive the error asymptotically toward zero. The fourth term is the direct effect of the disturbances and noise on the error dynamics and is not affected by the designers choice of γ and ε. While this term will perturb error away from zero, the third term's stabilizing effect will bring the error back toward zero.

In section 2.2.6, it had been argued that the first term in equation 12 would quickly go to zero since $\hat{\Lambda}$ was guaranteed to approach Λ. In this case however, equation 32 shows that noise and disturbances can perturb $\hat{\Lambda}$ away from Λ, and that those perturbations have a "gain" of $\gamma \cdot g(r,\varepsilon)$. A large deviation from Λ, especially toward zero, can cause error to become very sensitive to actuator transients.

In order to mitigate the negative effects of noise and disturbances that have been exposed in this section, the "gain", $\gamma \cdot g(r_i,\varepsilon)$, should be made as small as possible. However, this exposes a design trade-off. It has been shown in section 2.2 that the speed of approach of the effectiveness estimates to the true values is proportional to $\gamma \cdot \kappa_{ii}$, which is $\gamma \cdot g(r_i,\varepsilon) \cdot r_i$. Clearly the designer must carefully choose γ and ε to get the fastest estimation possible while protecting against sensitivity to noise and disturbances.

4.2 Uncertainty in the State Function

For this analysis, let us assume perfect measurements and that the real system input matrix perfectly matches the model (i.e. $B_c = B_{mc}$), but that the real state function, f_c, differs from the model, f_{mc}. Then the system can be described by:

$$\begin{aligned} \tilde{\dot{x}}_c(t) &= \dot{x}_c(t) = f_c(x(t)) + B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)r(t) \\ \tilde{x}(t) &= x(t) \end{aligned} \tag{35}$$

Substituting the above system descriptions into the definition of measured error, equation 10, gives:

$$\begin{aligned} \tilde{e}(t) &= [f_c(x(t)) + B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)r(t)] \\ &\quad - [f_{mc}(x(t)) + B_{mc}r(t)] \\ &= B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]r(t) + \\ &\quad \underbrace{[f_c(x(t)) - f_{mc}(x(t))]}_{\delta_f(x(t))} \end{aligned} \tag{36}$$

where the difference in contribution from the modeled and real state functions has been grouped into a single term called $\delta_f(x(t))$.

The above description of measured error can then be substituted into the definition of the estimate derivative, equation 13, giving:

$$\begin{aligned} \dot{\hat{\Lambda}}(t) &= \gamma\{\Lambda(t)\mathrm{K}(t,\varepsilon) - \hat{\Lambda}(t)[\mathrm{K}(t,\varepsilon) \\ &\quad - \mathrm{diag}(B_{mc}^{-1}\delta_f(x(t)))G(t,\varepsilon)]\} \end{aligned} \tag{37}$$

This result is strikingly similar to that for the previously investigated case of the system in the presence of disturbances and noise. In this case again, modeling error in the state function perturbs $\hat{\Lambda}$ away from Λ with a "gain" of $\gamma \cdot g(r, \varepsilon)$. The same observations that were made about the influence of noise and disturbances can be made about modeling error in the state function.

As for the error dynamics, since we have assumed perfect measurement, then real error, e, is identical to measured error, $\tilde{e}$, so the time derivative of e can be written by taking the derivative of equation 36, giving:

$$\begin{aligned}\dot{e}(t) = &B_{mc}[\Lambda(t)\hat{\Lambda}^{-1}(t) - I]\dot{r}(t)\\ &+ B_{mc}\dot{\Lambda}(t)\hat{\Lambda}^{-1}(t)r(t)\\ &- \gamma B_{mc}\Lambda(t)\hat{\Lambda}^{-1}(t)\mathrm{K}(t,\varepsilon)B_{mc}^{-1}e(t)\\ &+ J_{f_c - f_{mc}}(x(t))\,\dot{x}_c(t) \qquad (38)\end{aligned}$$

This result is also similar to to that for the previously investigated case of the system in the presence of disturbances and measurement noise. The only difference in this case is that the control designer may have some influence on the final term of the error dynamics by the choice of the model state function. It behooves the control designer to include a model that is as accurate as possible to mitigate the negative effects of state function modeling error.

4.3 Uncertainty in the Input Matrix

For this analysis, let us assume perfect measurements and that the real system state function perfectly matches the model (i.e. $f_c = f_{mc}$), but that the real input matrix, B_c, differs from the model, B_{mc}. Then the system can be described by:

$$\begin{aligned}\dot{\tilde{x}}_c(t) &= \dot{x}_c(t) = f_{mc}(x(t)) + B_c\Lambda(t)\hat{\Lambda}^{-1}(t)r(t)\\ \tilde{x}(t) &= x(t) \qquad (39)\end{aligned}$$

Substituting the above system descriptions into the definition of measured error, equation 10, gives:

$$\begin{aligned}\tilde{e}(t) &= [\cancel{f_{mc}(x(t))} + B_c\Lambda(t)\hat{\Lambda}^{-1}(t)r(t)]\\ &\quad - [\cancel{f_{mc}(x(t))} + B_{mc}r(t)]\\ &= [B_c\Lambda(t)\hat{\Lambda}^{-1}(t) - B_{mc}]r(t) \qquad (40)\end{aligned}$$

The above description of measured error can then be substituted into the definition of the estimate derivative, equation 13, giving:

$$\begin{aligned}\dot{\hat{\Lambda}}(t) &= \gamma\,\mathrm{diag}\{B_{mc}^{-1}[B_c(\Lambda(t)\hat{\Lambda}^{-1}(t) - I)r(t)]\}\\ &\quad\hat{\Lambda}(t)G(t,\varepsilon)\\ &= \gamma\,\hat{\Lambda}(t)[\mathrm{diag}(B_{mc}^{-1}B_c\Lambda(t)\hat{\Lambda}^{-1}(t)r(t))\\ &\quad - \mathrm{diag}(Ir(t))]\,G(t,\varepsilon)\\ &= \gamma\,[\mathrm{diag}(\hat{\Lambda}(t)B_{mc}^{-1}B_c\hat{\Lambda}^{-1}(t)\Lambda(t)r(t))G(t,\varepsilon)\\ &\quad - \hat{\Lambda}(t)\mathrm{K}(t,\varepsilon)] \qquad (41)\end{aligned}$$

Here it is found that the failure of B_{mc}^{-1} to cancel with B_c prevents much of the simplification that would have been possible otherwise. This equation shows that the estimate dynamics will act as a first-order lag filter of $\mathrm{diag}(\hat{\Lambda}B_{mc}^{-1}B_c\hat{\Lambda}^{-1}\Lambda r)G(r,\varepsilon)$, which will likely be different from the true values, Λ, as a function of r.

Ideally, $B_{mc}^{-1}B_c$ should be close to I, but a reasonably sufficient condition for that case has not yet been identified.

As for the error dynamics, since we have assumed perfect measurement, then real error, e, is identical to measured error, $\tilde{e}$, so the time derivative of e can be written by taking the derivative of equation 40, giving:

$$\begin{aligned}\dot{e}(t) = &B_{mc}[B_{mc}^{-1}B_c\Lambda(t)\hat{\Lambda}^{-1}(t) - I]\dot{r}(t)\\ &+ B_c\dot{\Lambda}(t)\hat{\Lambda}^{-1}(t)r(t)\\ &- \gamma B_c\Lambda(t)\hat{\Lambda}^{-1}(t)\mathrm{K}(t,\varepsilon)B_{mc}^{-1}e(t) \qquad (42)\end{aligned}$$

To ensure stability of these dynamics the eigenvalues of the matrix $-\gamma B_c\Lambda\hat{\Lambda}^{-1}\mathrm{K}(r,\varepsilon)B_{mc}^{-1}$ should be negative. However, a reasonably sufficient condition for that case has not yet been identified.

5 Conclusion

5.1 Summary of Findings

This paper proposed a method to: 1) invert a MIMO system's design control dynamics to construct a control signal, 2) simultaneously estimate and adapt to differences between real and design actuator effectiveness, and 3) automatically switch control goals to attempt to maintain critical control upon detection of (an) ineffective actuator(s). The estimation and adaptation is based on indirect model reference adaptive control (indirect MRAC).

The proposed method was demonstrated for a simplified simulation of a small v-tail UAV. The method was able to quickly identify and adapt to a partial and subsequent total actuator failure. In the latter case, control of sidelsip was automatically abandoned in order to preserve the higher priority control of heading, altitude and airspeed.

Finally, this method was analyzed for sensitivity to disturbances, measurement noise, and modeling uncertainty. It was found that a fundamental trade-off exists between the speed of convergence and the sensitivity to disturbances, noise, and uncertainty in the state function, especially for small actuator commands. A sufficient condition that guarantees stability in the presence of modeling uncertainty in the input matrix could not yet be determined.

5.2 Future Work

Several areas of investigation remain for this proposed control method, including consideration of control bandwidth limitations and saturation. Furthermore, the analysis presented in this paper suggests that some knowledge of the spectral intensity of the expected disturbances and measurement noise could be used to set the proposed design parameters, γ and ε.

The analysis presented in this paper showed that the proposed method is well suited for systems where the actuators are typically commanded away from zero, since the estimates of actuator effectiveness are less susceptible to error due to disturbances and measurement noise in such cases. Propellers are one such kind of actuator, so it is desired to attempt to adapt this method to a multi-rotor UAV. Unfortunately, quadrotors are not controllable in the event of a single actuator failure, but a hexarotor may be. However, a hexarotor is an example of an over-actuated system (i.e. more actuators than degrees of control freedom), so the input matrix, B_{mc} is not square. Therefore, future work will be to adapt this method to control of an over-actuated system.

Finally, the analysis presented in this paper showed that this method has a fundamental trade-off between the speed of adaptation and sensitivity to disturbances, measurement noise, and error in the modeling of the state function. Future work will be to investigate the possibility of decoupling the adaptation gain from robustness as has been achieved by the $\mathcal{L}_1$ control scheme as discussed in [3].

Acknowledgments

The author was supported through the Global COE Program "Global Center of Excellence for Mechanical Systems Innovation," by the Japanese Ministry of Education, Culture, Sport, Science and Technology.

References

[1] Kumpati S. Narendra and Anuradha M. Annaswamy. *Stable Adaptive Systems*. Prentice Hall, Englewood Cliffs, NJ, 1989.

[2] Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ, third edition.

[3] Evgeny Kharisov, Naira Hovakimyan, and Karl J. m. Comparison of several adaptive controllers according to their robustness metrics. In *AIAA Guidance, Navigation, and Control Conference*, number AIAA 2010-8047, Toronto, Ontario, August 2010. AIAA.

Copyright Statement