OPTIMIZATION OF RUNWAY CAPACITY UTILIZATION IN THE CASE OF GENERAL PARETO CURVE

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Abstract

This paper studies an essential application of Air Traffic Management investigating the runway capacity allocation with the aim of pre-tactical planning in the time scope of approximately 24 hours. For the prediction and solving of traffic problems for such planning horizon we have to control flows of aggregated flights rather than individual flights. Hence, the length of the queue arising from a capacity allocation decision is the main feature of flow management. Related standard key performance indicators such as throughput, delay, etc. depend on occurrence and behavior of arrival and departure queues. We study and investigate properties of weighted queues sum as a function of capacity allocation in the case of general Pareto curve. It results in straightforward and effective $O(N^2)$ algorithm giving integer solution for the problem of total weighted queues sum minimization over a time period with arbitrary number of considered time intervals. This integer problem has been solved earlier with the linear programming methods. However, the best complexity of such methods is $O(N^{3.5}L)$ and linear programming solvers based on them give a solution of the problem in a reasonable time for the limited number of considered intervals only. Furthermore, the algorithm developed in this paper constructs an "almost optimal" initial solution depending on the weight of corresponding queues. Significant feature of the initial solution is that it can be easily obtained by hand for an arbitrary number of considered intervals. It is also shown which intervals have to undergo reallocation of capacity in order to achieve an optimal flow. Moreover, the minimal finite solution set which contains an optimal flow for each weight value in the mentioned problem can be easily constructed.

The developed algorithm finds optimal solutions in "interval-to-interval" techniques. The obtained solution method gives us the tool for controlling and managing of flow construction and queue outcome depending on strategies we follow. It gives an optimal solution of considered problem for the whole day with interval length from 5 to 15 minutes on standard PC in maximal 0.1 seconds.

1 Introduction

Under optimization of airport runway capacity we understand its best allocation between arrivals and departures, i.e. construction of arrival/departure flow, that optimally satisfies the predicted traffic demand over a period of time under given conditions and strategies. Airport runway capacity reflects operational limits that an airport has under given operational conditions such as weather, runway configuration, time, etc. Estimates of airport operational limits per time interval are provided by statistical methods combined with air traffic managers' and controllers' experience which are applied to historical airport performance data observed over a long period of

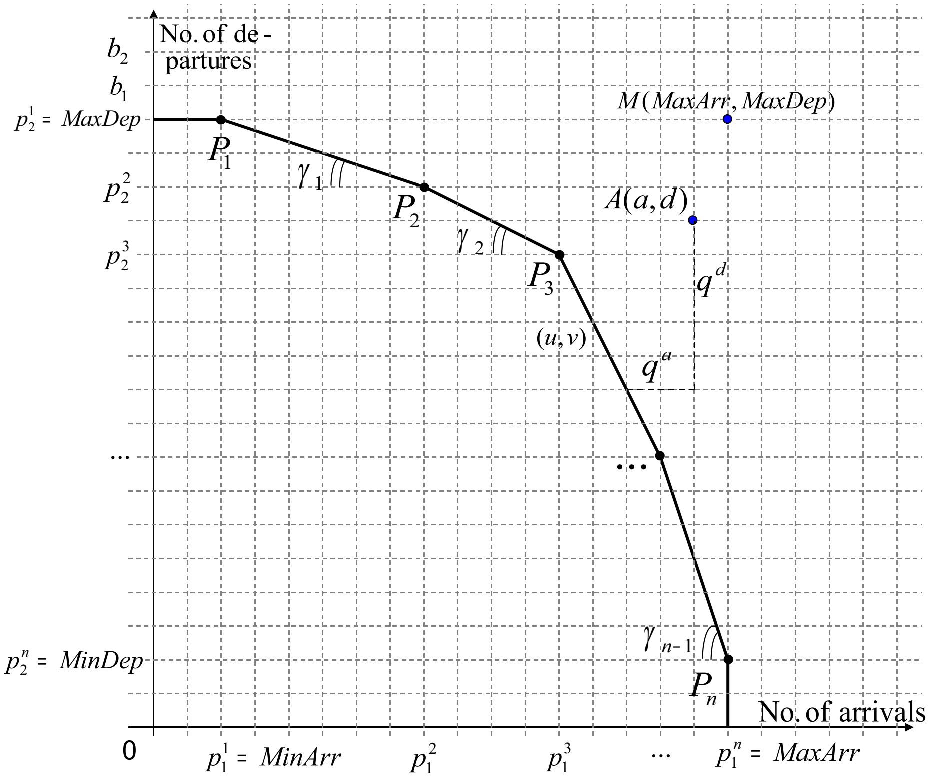

Fig. 1 *Pareto curve PCG*

time [1, 7, 8]. The resulting convex piecewise linear arrival/departure capacity curve with integer coordinates of its knots is often called Pareto curve.

We consider its general form shown in Figure 1 and denoted as PCG. It consists of one horizontal segment

$$[(0, MaxDep), (MinArr, MaxDep)] = [(0, p_2^1), P_1],$$

one vertical segment

$$[(MaxArr, MinDep), (MaxArr, 0)] = [P_n, (p_1^n, 0)],$$

where $MinArr/MinDep$ is the maximal number of arrivals/departures which can be handled in the given time interval by the maximal possible departure/arrival flow $MaxDep/MaxArr$, and $n - 1 \geq 0$ line segments $[P_j, P_{j+1}]$, where each flow point $(u, v) \in [P_j, P_{j+1}]$ satisfies

$$v = -\tan\gamma_j u + b_j \qquad (1)$$

and $P_j = P_j(p_1^j, p_2^j)$, $P_{j+1} = P_{j+1}(p_1^{j+1}, p_2^{j+1})$, $u \in [p_1^j, p_1^{j+1}]$, $j \in \{1, ..., n-1\}$, $\gamma_0 = 0° < \gamma_1 < ... < \gamma_j < ... < \gamma_{n-1} < \gamma_n = 90°$, $0 < b_1 < ... < b_j < ... < b_{n-1} < +\infty$.

Notwithstanding that Pareto curve provides the detailed information about operational limits it is difficult to decide which arrival/departure ratio has to be chosen in each time interval of the planning horizon in order to find a flow, which fulfills the given criteria at the best. The length of the queue arising from a capacity allocation decision is the main feature of flow management. For this reason we consider here one of the main airport runway capacity optimization problems: the problem of minimizing of total weighted queues sum. Based on properties of weighted queues sum an optimal flow for the given weight can be easily constructed or the minimal finite set of flows which contains an optimal solution for all weights $\in [0, 1]$ can be specified in $O(N^2)$ time.

This paper is organized as follows. Section 2 formulates the problem of minimizing of total weighted queues sum. Sections 3 and 4 investigate properties and behavior of weighted queues sums. Here two strategies of flow construction which give priority to arrivals or departures are also studied. It results in straightforward and effective algorithm for finding of an optimal solution in the problem of minimizing of total weighted queues sum with arbitrary number N of considered time intervals. The algorithm is described in Section 5. In this section an "almost optimal" initial solution depending on the weight of corresponding queues is constructed. Mentioned solution can be obtained in a straightforward manner by hand for an arbitrary number of considered intervals on a sheet of squared paper with plotted Pareto curve. It is also shown how and in which intervals capacity in the initial solution has to be reallocated in order to achieve an optimal flow. Effectiveness of the algorithm is illustrated on two numerical examples in Section 6.

The obtained solution method gives us the tool for controlling and managing of queue generation. Moreover, the minimal finite solution set, which contains an optimal flow for each weight value in the mentioned problem, can be easily constructed.

2 Problem Formulation

Let us use the following notations

T – a time period consisting of N time intervals of equal length δ (it is generally assumed that δ = 15 min.)

a_i – arrival demand in the time interval i

d_i – departure demand in the time interval i

q_i^a – arrival queue at the beginning of the time interval $i \in \{1,\dots,N+1\}$

q_i^d – departure queue at the beginning of the time interval $i \in \{1,\dots,N+1\}$

u_i – arrival flow – number of arrivals handled in the time interval i

v_i – departure flow – number of departures handled in the time interval i

to formulate the problem of total weighted queue minimization for the given demand (a_i,d_i), $i \in \{1,\dots,N\}$, where capacity is constrained by Pareto curve PCG (Figure 1):

Problem 1

$$\min_{F^*=(u_1,\dots,u_N,v_1,\dots,v_N)} \alpha \sum_{i=2}^{N+1} q_i^a + (1-\alpha) \sum_{i=2}^{N+1} q_i^d, \tag{2}$$

where $0 \le \alpha \le 1$, *subject to*

$$q_{i+1}^a = \max(0, q_i^a + a_i - u_i) \tag{3}$$

$$q_{i+1}^d = \max(0, q_i^d + d_i - v_i) \tag{4}$$

$$q_1^a \ge 0, q_1^d \ge 0 - \text{initial queues} \tag{5}$$

$$0 \le u_i \le MaxArr \tag{6}$$

$$0 \le v_i \le \min(MaxDep, -\tan\gamma_j u_i + b_j) \tag{7}$$

for

$$i \in \{1,\dots,N\}, \quad j \in \{1,\dots,n-1\}. \tag{8}$$

The introduced problem has been considered and solved, particularly in [1, 2, 3, 4], with linear programming methods. This linear program (LP) has $2N$ variables. It is known [9], that the best complexity for polynomial algorithms solving LP is $O(N^{3.5}L)$, where L is the data length and N is the number of variables in LP. When the time period T is big, for instance the whole day is considered, or/and the length of a time interval $\delta < 15$ minutes the number of considered intervals contributes to the significant increase of the computational time of LP.

We have not found any investigations of weighted queues sum as a function of capacity allocation in the case of general Pareto curve. Weighted queues sum was studied by author in [5, 6] for the simple Pareto curve which is presented by three values: the maximal number of landings $MaxArr$ and the maximal number of take-offs $MaxDep$ per time interval by the single-mode and the maximal number of flights $MaxCap$ per time interval by the mixed-mode of capacity utilization at the addressed airport. Having properties of weighted queues sum for the general Pareto curve we can easily construct an optimal flow for given weight α or specify a minimal finite set of flows which contains an optimal solutions for all weights $\alpha \in [0,1]$ in $O(N^2)$ time. Moreover, we get a controlling tool for outcome of arrival/departure queue which allows the construction of an "almost optimal" solution of Problem 1 by hand.

Considering Pareto curve PCG it can be noted that the units of the initial demand (a_i, d_i) in interval $i \in \{1,...,N\}$, which exceed the maximal numbers of arrivals $MaxArr$ or/and departures $MaxDep$, will be moved to the queue for the next interval independently of capacity allocation. These units can not be allocated in interval i and should be accepted as unavoidable queue $(q^{au}_{i+1}, q^{du}_{i+1})$, i.e. the part of the demand for the next interval:

$$\bar{a}_i = \min\{a_i + q^{au}_i, MaxArr\}, \quad q^{au}_1 = q^a_1, \tag{9}$$
$$q^{au}_{i+1} = a_i + q^{au}_i - \bar{a}_i, \quad i \in \{1,...,N\};$$
$$\bar{d}_i = \min\{d_i + q^{du}_i, MaxDep\}, \quad q^{du}_1 = q^d_1, \tag{10}$$
$$q^{du}_{i+1} = d_i + q^{du}_i - \bar{d}_i, \quad i \in \{1,...,N\}.$$

Hence,

$$\begin{aligned} \min\{a_i, MaxArr\} \leq &\ \bar{a}_i \leq MaxArr \\ \min\{d_i, MaxDep\} \leq &\ \bar{d}_i \leq MaxDep \end{aligned}$$

for $i \in \{1,...,N\}$. Thus, a flow for the demand $\{(\bar{a}_i, \bar{d}_i)\}$, $i \in \{1,...,N\}$ which gives no total arrival or/and departure queue always exists.

Any flow $F^* = \{(u_i, v_i)\}$ of the cutted demand $\{(\bar{a}_i, \bar{d}_i)\}$, $i \in \{1,...,N\}$ has the following total weighted queue sum

$$\alpha \sum_{i=2}^{N+1} (q^a_i + q^{au}_i) + (1-\alpha) \sum_{i=2}^{N+1} (q^d_i + q^{du}_i) =$$
$$\left[\alpha \sum_{i=2}^{N+1} q^a_i + (1-\alpha) \sum_{i=2}^{N+1} q^d_i \right] + const. \tag{11}$$

Therefore, a solution flow F^* in Problem 1 can be found with the demand cutted by (9)-(10). However, queue calculated with respect to the demand $\{(\bar{a}_i, \bar{d}_i)\}$, $i \in \{1,...,N\}$ have to be increased on corresponding values of unavoidable queue. For our following investigations we assume without loss of generality that the points of the initial demand (a_i, d_i), $i \in \{1,...,N\}$ in Problem 1 are already dominated by the point $M(MaxArr, MaxDep)$.

3 Weighted Distance Function

In the case of overdemand in an interval the minimal weighted sum of arrival/departure queues is nothing else than the minimal weighted rectilinear distance from the demand point to the points of Pareto curve PCG, which coordinates are dominated by the demand point.

Let us study properties of this function. For this purpose we consider the difference of the weighted rectilinear distances between the point $M(MaxArr, MaxDep)$ and one point of Pareto curve $P_j(p^j_1, p^j_2)$, where $j \in \{1,...,n-1\}$ (see Figure 1)

$$\begin{aligned} d_\alpha(M, P_j) \ = \ & \alpha(MaxArr - p^j_1) + \\ & (1-\alpha)(MaxDep + \tan\gamma_j p^j_1 - b_j), \end{aligned}$$

and the point $M(MaxArr, MaxDep)$ and the next point of Pareto curve $P_{j+1}(p^{j+1}_1, p^{j+1}_2)$

$$\begin{aligned} d_\alpha(M, P_{j+1}) \ = \ & \alpha(MaxArr - p^{j+1}_1) + (1-\alpha) \\ & (MaxDep + \tan\gamma_j p^{j+1}_1 - b_j). \end{aligned}$$

The difference is equal to

$$d_\alpha(M, P_j) - d_\alpha(M, P_{j+1}) = \tag{12}$$
$$(p^{j+1}_1 - p^j_1)(\alpha - (1-\alpha)\tan\gamma_j).$$

Since the term $p^{j+1}_1 - p^j_1$ is always positive, the sign of the difference (12) depends on the second term only. Therefore, the difference (12) is positive for $0 \leq \gamma_j < \arctan\frac{\alpha}{1-\alpha}$, equal to zero for $\gamma_j = \arctan\frac{\alpha}{1-\alpha}$ and negative when $\arctan\frac{\alpha}{1-\alpha} < \gamma_j \leq 90°$.

Because $\gamma_0 = 0° < \gamma_1 < ... < \gamma_{n-1} < \gamma_n = 90°$, weighted rectilinear distance $d_\alpha(M, P_1P_n)$ between the point M and points of the curve P_1P_n is convex function with an unique minimum if there are no γ_j, $j \in \{1,...,n-1\}$ satisfying $\gamma_j = \arctan\frac{\alpha}{1-\alpha}$ and with minimum at any point of the segment P_jP_{j+1} if $\gamma_j = \arctan\frac{\alpha}{1-\alpha}$ for some $j \in \{1,...,n-1\}$. This segment is so

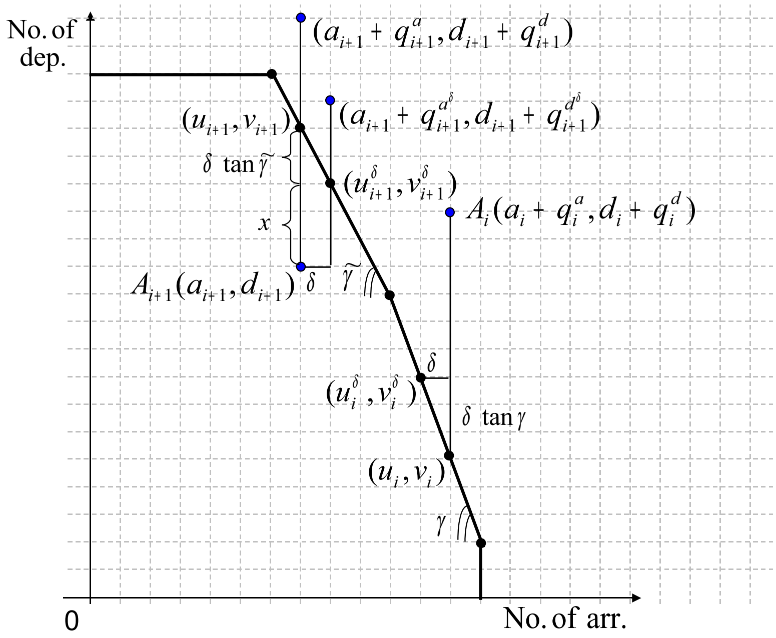

Fig. 2 *Influence of flow choice on weighted queue – Strategy ArrPriority*

called "trade–off" area. Weighted rectilinear distance from any point $A(a,d)$ with overdemand to the part of P_1P_n, which coordinates do not exceed coordinates of the point A, have the same properties as $d_\alpha(M, P_1P_n)$.

4 Two Strategies

Let us investigate two strategies of flow construction and check for which configuration of Pareto curve and for which weight α they contribute to minimization of the total weighted queue.

4.1 Strategy ArrPriority: minimization of arrival queue

Strategy ArrPriority in interval $i \in \{1,\dots,N\}$ with Pareto curve PCG (see Figure 1), demand point $A(a_i,d_i)$ and queue (q^a_i, q^d_i) at the beginning of the time interval leads to the following flow

$$u_i = \min\{a_i + q^a_i, MaxArr\} \quad (13)$$

$$v_i = \min\{d_i + q^d_i, -\tan\gamma_j u_i + b_j, MaxDep\} \quad (14)$$
$$j \in \{1,\dots,n-1\}.$$

To find out the optimal use of this strategy in solution of Problem 1 we assume already calculated in some manner flow F so that in two following one after another intervals i and $i+1$ flow points (u_i,v_i) and (u_{i+1},v_{i+1}) are constructed by (13) and (14), there is overdemand in both intervals and the total arrival demand in both intervals does not exceed $MaxArr$ (Figure 2). Last assumption does not violate the generality and allows to avoid summands which will be cancelled out in the following comparisons.

The part of the total weighted queue which appears with the flow points (u_i,v_i) and (u_{i+1},v_{i+1}), situated on the segments of Pareto curve with the slopes γ and $\tilde{\gamma}$, respectively, is equal to

$$S = (1-\alpha)q^d_{i+1} + (1-\alpha)(q^d_{i+1} - \delta\tan\tilde{\gamma} - x), \quad (15)$$

where $\delta > 0$ and departure queue at the end of interval $i+1$ is nonnegative

$$q^d_{i+1} - \delta\tan\tilde{\gamma} - x \geq 0. \quad (16)$$

We compare (15) with the change of the total weighted queue of the flow F appearing with decreased on value $\delta > 0$ arrival flow $u^\delta_i = u_i - \delta$ in the interval i and with increased at the cost of δ arrival flow $u^\delta_{i+1} = u_{i+1} + \delta$ in the interval $i+1$ (Figure 2). This shift is possible and has sense if and only if

$$u_i - \delta \geq p^1_1 = MinArr \quad (17)$$

$$u_{i+1} + \delta \leq p^n_1 = MaxArr. \quad (18)$$

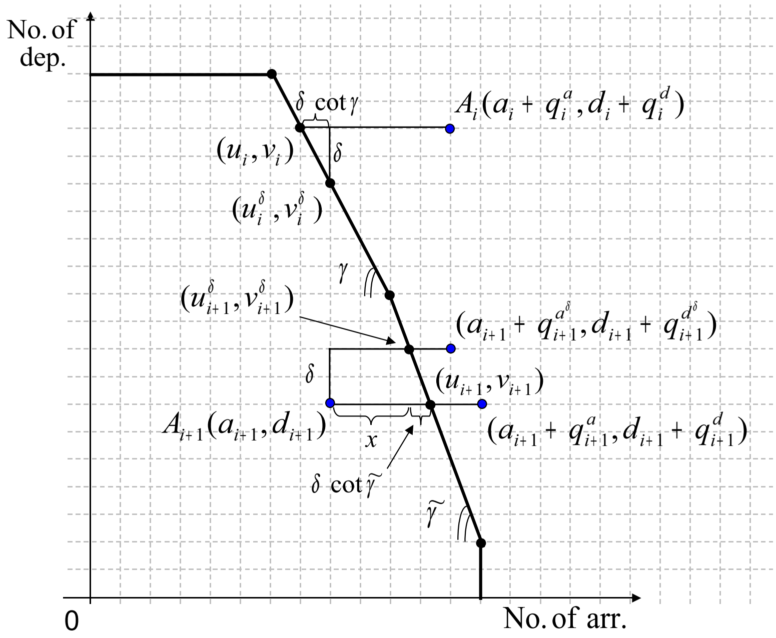

Fig. 3 *Influence of flow choice on weighted queue – Strategy DepPriority*

Let us assume that there are $l \geq 0$ intervals with positive departure queue $q_{i+2+p}^d > 0$ for $p \in \{1, \ldots, l\}$, $i+2+l \leq N+1$ and $q_{i+3+l}^d = 0$. Then the change of the total weighted queue with the shifted flow in intervals i and $i+1$

$$u_i^\delta = u_i - \delta \qquad (19)$$

$$v_i^\delta = v_i + \delta \tan\gamma \qquad (20)$$

$$u_{i+1}^\delta = u_{i+1} + \delta \qquad (21)$$

$$v_{i+1}^\delta = v_{i+1} - \delta \tan\tilde{\gamma}. \qquad (22)$$

is equal to

$$\begin{aligned} S_\delta \;=\; & \alpha\delta + (1-\alpha)(q_{i+1}^d - \delta\tan\gamma) + \\ & (1-\alpha)(q_{i+1}^d - \delta\tan\gamma - x) - \\ & l(1-\alpha)\delta\tan\gamma, \end{aligned} \qquad (23)$$

where departure queue at the end of interval $i+1$

$$q_{i+2}^{dq} = q_{i+1}^d - \delta\tan\gamma - x \qquad (24)$$

is nonnegative, decrease of departure queue $\delta\tan\gamma$ is not greater than the minimum $\min_{p\in\{1,\ldots,l\}} q_{i+2+p}^d$ and $\delta > 0$ is small enough to have flow points (u_i, v_i) and (u_i^δ, v_i^δ), (u_{i+1}, v_{i+1}) and $(u_{i+1}^\delta, v_{i+1}^\delta)$ on the segments of the Pareto curve with the slopes γ and $\tilde{\gamma}$, respectively.

The difference of the total queues of the flow F and of the flow F changed in intervals i and $i+1$ is equal to

$$S - S_\delta = \delta((1-\alpha)((2+l)\tan\gamma - \tan\tilde{\gamma}) - \alpha). \qquad (25)$$

Therefore, S is not greater than S_δ if and only if

$$(2+l)\tan\gamma - \tan\tilde{\gamma} \leq \frac{\alpha}{1-\alpha}. \qquad (26)$$

Let us assume that between two considered intervals are $k \geq 0$ other intervals with in a some manner calculated flow so that departure queue

$$q_{i+p}^d \geq \delta\tan\gamma, \qquad (27)$$

where $p \in \{1, \ldots, k+2+l\}$, $l \geq 0$, $i+k+2+l \leq N+1$ and $q_{i+k+3+l}^d = 0$. Then Strategy ArrPriority in intervals with demand points $A_i(a_i, d_i)$ and $A_{i+k+1}(a_{i+k+1}, d_{i+k+1})$ produces smaller total weighted queue compared to any other type of flow construction when

$$(2+l+k)\tan\gamma - \tan\tilde{\gamma} \leq \frac{(k+1)\alpha}{1-\alpha}. \qquad (28)$$

Use of Strategy ArrPriority in both considered intervals is justified for $\tan\gamma \leq \frac{\alpha}{1-\alpha}$ and $\tan\tilde{\gamma} \leq \frac{\alpha}{1-\alpha}$ accordingly to properties of weighted

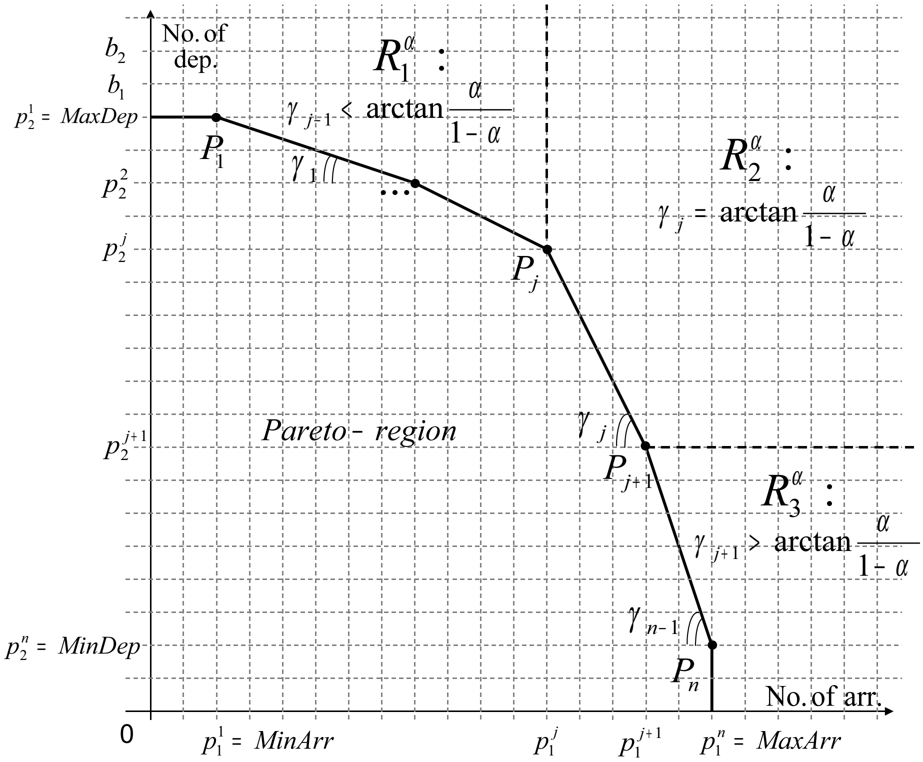

Fig. 4 *Pareto curve PCG and corresponding to* α *regions*

rectilinear distance function. The satisfaction of (28) depends on values of $\tan\gamma$ and $\tan\tilde{\gamma}$. However, it is always satisfied when $\tan\gamma \leq \frac{(k+1)\alpha}{(2+l+k)(1-\alpha)}$.

4.2 Strategy DepPriority: minimization of departure queue

Strategy DepPriority in interval $i \in \{1,\ldots,N\}$ with Pareto curve PCG (see Figure 1), demand point $A(a_i,d_i)$ and queue (q_i^a,q_i^d) at the beginning of the time interval leads to the following flow

$$v_i = \min\{d_i + q_i^d, MaxDep\} \tag{29}$$

$$u_i = \min\{a_i + q_i^a, \cot\gamma_j(b_j - v_i), MaxArr\} \tag{30}$$

$$j \in \{1,\ldots,n-1\}.$$

In the similar way one can show (see Figure 3) that the sum of weighted queues produced by Strategy DepPriority is not greater than the sum appearing by any other strategy when

$$(2+l+k)\cot\gamma - \cot\tilde{\gamma} \leq \frac{(k+1)(1-\alpha)}{\alpha}. \tag{31}$$

Taking into account properties of weighted rectilinear distance function Strategy DepPriority is explained for $\frac{\alpha}{1-\alpha} < \tan\gamma$ and $\frac{\alpha}{1-\alpha} < \tan\tilde{\gamma}$. Then the satisfaction of (31) depends on values of $\cot\gamma$ and $\cot\tilde{\gamma}$. However, (31) is satisfied when $\cot\gamma \leq \frac{(k+1)(1-\alpha)}{(2+l+k)\alpha}$.

5 Algorithm

Results of Sections 3 and 4 lead to the partitioning of the first quadrant into 4 regions illustrated in Figure 4. Here is shown the general case of region R_2^α with "trade-off" area P_jP_{j+1}. However, R_2^α can be based on one knot of Pareto curve only when there does not exist γ_j, $j \in \{1,\ldots,n-1\}$ such that $\gamma_j = \arctan \frac{\alpha}{1-\alpha}$.

It should be noted that the number of the described partitions for Pareto curve PCG is constant and equal to the sum of knots P_1, ..., P_n and segments between them, i.e. $2n-1$ (see Figure 4).

In order to solve Problem 1 for the given weight α an initial solution $F^{start} = \{(u_i,v_i)\}$, $i = \{1,\ldots,N\}$ is constructed. It is found in the following way: if the total demand point $(a_i + q_i^a, d_i + q_i^d)$ in interval $i = \{1, \ldots, N\}$ is

- in Pareto region then $u_i = a_i + q_i^a$, $v_i = d_i + q_i^d$
- in region R_1^α then (u_i, v_i) is calculated by Strategy ArrPriority ((13)-(14))
- in region R_2^α then $u_i = \min\{a_i + q_i^a, p_1^{j+1}\}$, $v_i = -\tan\gamma_j u_i + b_j$
- in region R_3^α then (u_i, v_i) is found by Strategy DepPriority ((29)-(30))

and queue at the end of interval i is equal to $q_{i+1}^a = a_i + q_i^a - u_i$, $q_{i+1}^d = d_i + q_i^d - v_i$.

In general, flow F^{start} is noninteger. Since Problem 1 belong to the pre-tactical planning of capacity, where demand is composed of abstract units only, the problem can be solved in real numbers. However, if we look for an integer solution, it can be constructed easily using the location structure of integer points under Pareto curve which are nearest to it from arrival or departure directions. These points, being consequently connected with lines so that their first coordinate does not decrease and second coordinate does not increase, are always on small line segments with slopes 0°, 45° or 90° (Figure 5). Taking into account properties of weighted rectilinear distance and results of Section 4 we can state that

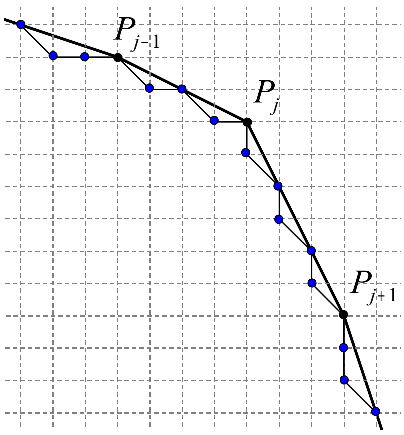

Fig. 5 *Structure of integer points under Pareto curve*

Lemma 1 *Each flow point on each step of calculation of the initial solution F^{start} and following after it reallocation of capacity can be rounded down. However,*

1. *if the flow point is on the segment P_jP_{j+1}, i.e. $\tan\gamma_j = \frac{\alpha}{1-\alpha}$, and there is at least one integer point on P_jP_{j+1} dominated by the demand point in the interval, then the flow point is changed by the nearest to it integer point of P_jP_{j+1} dominated by the demand point;*
2. *if $\alpha = 0.5$, condition (28) (or (31)) is satisfied for two considered intervals and flow point in the first interval is integer through rounding down, then it is necessary to check whether the flow with the departure part rounded up and the arrival part decreased on one unit (the arrival part rounded up and the departure part decreased on one unit) is in Pareto region and gives the better outcome.*

The calculated initial flow F^{start} can be divided into independent parts through intervals without queue. Each resulting part consists of blocks of intervals where tangent of the slope corresponding to the flow is not greater or not smaller than $\frac{\alpha}{1-\alpha}$. To find an optimal solution of Problem 1 we should reallocate the capacity in the blocks of intervals of the flow F^{start} so that the corresponding condition (28) or (31) is not violated or if violated there is no possibility to reallocate.

Let us consider one block of intervals $B = \{i_1, \ldots, i_b\}$, $2 \le b \le N$ where, for instance, the condition (28) has to be checked. We assume that (28) is satisfied for intervals i_1 and i_2

$$(2+l)\tan\gamma_{i_1} - \tan\gamma_{i_2} \le \frac{\alpha}{1-\alpha}, \tag{32}$$

and for intervals i_2 and i_3

$$(2+l-1)\tan\gamma_{i_2} - \tan\gamma_{i_3} \le \frac{\alpha}{1-\alpha}. \tag{33}$$

Add together (32) and (33) we get

$$(2+l)\tan\gamma_{i_1} + l\tan\gamma_{i_2} - \tan\gamma_{i_3} \le \frac{2\alpha}{1-\alpha}. \tag{34}$$

Since $l \tan\gamma_{i_2} > 0$,

$$(2+1+(l-1))\tan\gamma_{i_1} - \tan\gamma_{i_3} \leq \frac{2\alpha}{1-\alpha}, \quad (35)$$

i.e. the condition (28) is satisfied for intervals i_1 and i_3. Analogously, from satisfaction of condition (28) for i_1, i_3 and i_3, i_4 it follows that (28) is fulfilled for i_1, i_4. Therefore, in order to state nonviolation of condition (28) for all pairs of intervals in the block it is enough to have its nonviolation for pairs (i_1, i_2), (i_2, i_3), ..., (i_{b-1}, i_b).

Let us assume that the condition (28) is satisfied in block B for the pairs of intervals (i_1, i_2), ..., (i_{c-1}, i_c) and violated for (i_c, i_{c+1}) where $2 \leq c < b-3$. Necessary check in this case is summarised in Algorithm 1:

Algorithm 1 Check for intervals i_c and i_{c+1}

if (28) is violated **then**
    if $u_{i_c} > MinArr$ and $u_{i_{c+1}} < MaxArr$ **then**
        reallocate capacity;
        if no queue at i_{c+2} **then** $\{i_1, \ldots, i_{c+1}\}$ is checked, go to $\{i_{c+2}, \ldots, i_b\}$;
        else
            if (28) is satisfied **then**
                go to i_c, i_{c+2} and eliminate i_{c+1} from checking procedure;
            else
                go to i_{c-1}, i_{c+1} and eliminate i_c from checking procedure;
            end if
        end if
    else
        go to i_{c+1}, i_{c+2};
    end if
end if

Similar arguments are valid for blocks of intervals where tangent of the slope corresponding to the flow is not smaller than $\frac{\alpha}{1-\alpha}$. Described solution procedure is illustrated on numerical examples in Section 6.

5.1 Minimal finite solution set

Having possible decompositions of the first quadrant into regions by the weighted rectilinear distance $d_\alpha(M(MaxArr, MaxDep), P_1P_n)$ (see Section 3) which has its minimum at the point

$$\begin{cases} P_1, & \text{if } \alpha \in [0, \alpha_1], \\ P_2, & \text{if } \alpha \in [\alpha_1, \alpha_2], \\ \ldots, & \ldots, \\ P_n, & \text{if } \alpha \in [\alpha_{n-1}, 1], \end{cases} \quad (36)$$

where $0 \leq \alpha_1 \leq \ldots \leq \alpha_{n-1} \leq 1$, we can always construct the minimal finite solution set which contains a solution for any weight $\alpha \in [0,1]$.

For this purpose we start the construction of the minimal finite solution set in the interval $[0, \alpha_1]$. First, the initial flow at one endpoint of the interval, for instance $\alpha = 0$, is calculated. Then reallocation of capacity in the initial flow is performed to get an optimal flow at other endpoint $\alpha = \alpha_1$ of the interval. However, sequence of reallocations differs from the solution procedure for the fixed value α where blocks are considered consequently starting from the interval $i = 1$. Now the first two intervals which violate the corresponding condition (28) or (31) and have possibility for reallocation are found in all blocks. Reallocation is performed in a block where the ratio of increase of departure queue to decrease of arrival queue is minimal. Then next two intervals for reallocation in the considered block are found and reallocation is performed in a block with the minimal ratio and so on until an optimal solution for $\alpha = \alpha_1$ is constructed. As a result we get the minimal number of flows with the total queues which are knots of concave piecewise linear curve. This curve forms the lower bound of solutions for $\alpha \in [0, \alpha_1]$. The minimal weighted total queue sum for the fixed $\alpha \in [0, \alpha_1]$ is the minimal rectilinear weighted distance from the origin to the points of this concave curve. Therefore, the knots of the curve generate the minimal finite solution set for $\alpha \in [0, \alpha_1]$.

Alternatively, the initial solution for $\alpha = \alpha_1$ can be used in order to achieve an optimal flow for $\alpha = 0$. But here reallocation is performed in a block where the ratio of increase of arrival queue to decrease of departure queue is minimal.

In order to proceed the interval $[\alpha_1, \alpha_2]$ is considered. The initial flow at one endpoint is calculated and steps described above are performed. After n iterations the minimal finite solution set for $\alpha \in [0,1]$ is composed from obtained flows where possible double solutions with the equal total arrival and departure queue for α_1, ..., α_{n-1} are eliminated. This set consists of flows with the total queues which are knots of concave piecewise linear curve.

Construction of the minimal finite solution set is demonstrated in the next section.

6 Examples

Let us illustrate the developed solution procedure on numerical examples.

6.1 First example

Example 1 *Time period T consist of $N = 4$ time intervals with Pareto curve shown in Figure 6, where $MaxArr = 25$, $MaxDep = 30$. Arrival/departure demand $A_i(a_i, d_i)$, $i = 1, \ldots, 4$ is collected in Table 1.*

Find a flow which solves Problem 1 for $\alpha = 0.7$, $\alpha = 0.6$ *and* $\alpha = 0.5$.

Example 1 is taken from [1] where it is solved for $\alpha = 0.7$ and $\alpha = 0.5$ using LP-methods. The reader can compare solutions and evaluate simplicity and effectiveness of our algorithm.

Table 1 Example 1: arrival/departure demand, unavoidable queue and cutted arrival/departure demand

int.	demand		unav. queue		cutted dem.	
i	a_i	d_i	q^{au}_{i+1}	q^{du}_{i+1}	$\bar{a}_i$	$\bar{d}_i$
1	13	35	0	5	13	30
2	32	2	7	0	25	7
3	24	28	6	0	25	28
4	10	20	0	0	16	20
total	79	85	13	5	79	85

As it was described in Section 2, we start with the calculation of unavoidable queue and cutted demand using formulas (9) and (10) and collect them in Table 1. Both initial $A_i(a_i, d_i)$ and cutted demand $\bar{A}_i(\bar{a}_i, \bar{d}_i)$, $i = 1, \ldots, 4$ is shown in Figure 6. Further cutted demand is used to calculate a flow solving the problem with the initial demand. To find the arising queue for the initial demand we should increase the queue for the cutted demand on the value of unavoidable queue.

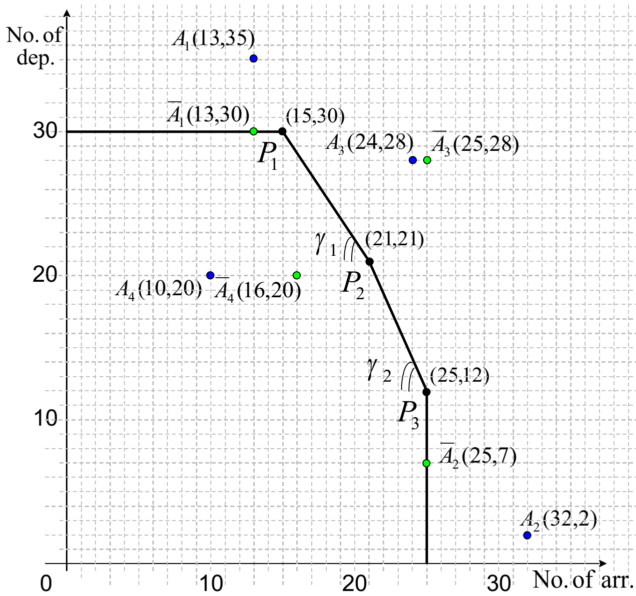

Fig. 6 *Original demand $A_i(a_i, d_i)$ and cutted demand $\bar{A}_i(\bar{a}_i, \bar{d}_i)$, $i = \{1, \ldots, 4\}$*

For defining of regions with respect to the given weight α the points of the Pareto curve at which the weighted rectilinear distance function $d_\alpha(M(25,30), P_1P_3)$ is minimal are found (see Section 3)

$$\begin{cases} P_1 = (15,30), & \text{if } \alpha \in [0, 0.6], \\ P_1P_2, & \text{if } \alpha = 0.6, \\ P_2 = (21,21), & \text{if } \alpha \in [0.6, \frac{9}{13}], \\ P_2P_3, & \text{if } \alpha = \frac{9}{13}, \\ P_3 = (25,12), & \text{if } \alpha \in [\frac{9}{13}, 1]. \end{cases} \tag{37}$$

Therefore, for $\alpha = 0.7$, 0.6 and 0.5 the first quadrant is decomposed into regions shown in Figures 7, 8 and 9, respectively.

6.1.1 $\alpha = 0.7$

Let us calculate the initial flow for $\alpha = 0.7$ and collect it in Table 2. For this value α tangent of the "trade-off" slope is equal to $\frac{\alpha}{1-\alpha} = 2.(3)$. Since demand point $\bar{A}_1$ is on Pareto curve after calculation and shifting of unavoidable queue, flow point in the interval is equal to the demand. There is no queue for the next interval. Therefore, the tangent of the corresponding slope is not saved. The same argumentation is also valid for the second interval with the demand $\bar{A}_2 + (q_2^a, q_2^d)$.

Table 2 Example 1: $\alpha = 0.7$ – initial solution

i	demand		flow		$\tan\gamma_i$	queue	
	$\bar{a}_i$	$\bar{d}_i$	u_i	v_i		q_{i+1}^a	q_{i+1}^d
1	13	30	13	30		0	0
2	25	7	25	7		0	0
3	25	28	25	12	2.25	0	16
4	16	20	16	28	1.5	0	8
	79	85	79	77		0	24

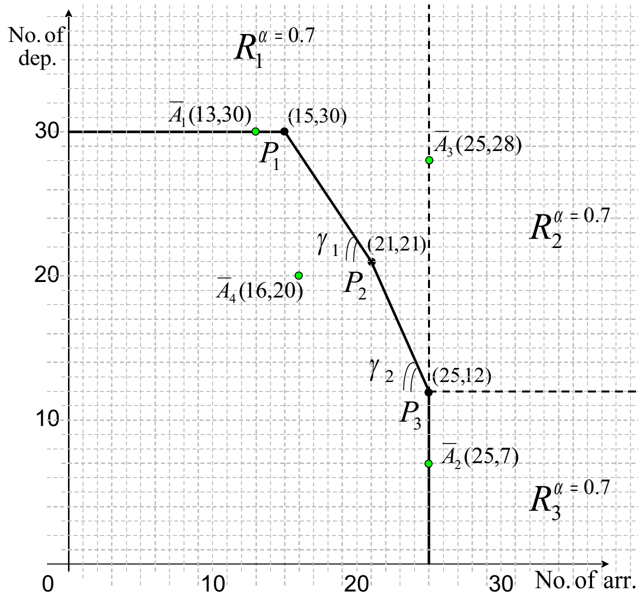

Fig. 7 *Regions for* $\alpha = 0.7$

The demand point $\bar{A}_3$ belongs to $R_1^{0.7}$. Therefore, the flow point in this interval is calculated using strategy ArrPriority with formulas (13) and (14). Since there is departure queue at the end of the third interval, tangent $\tan\gamma = 2.25$ of the segment slope which contains the flow point $(25, 12)$ is saved. The demand point $\bar{A}_4 + (q_4^a, q_4^d) = (16, 36) \in R_1^{0.7}$. Hence, the flow point in the interval $i = 4$ will belong to the segment with tangent of the slope $\tan\gamma = 1.5$. Since the departure part of the flow is not integer, the flow is rounded down.

Table 3 Example 1: $\alpha = 0.7$ - update

i	demand		flow		$\tan\gamma_i$	queue	
	$\bar{a}_i$	$\bar{d}_i$	u_i	v_i		q_{i+1}^a	q_{i+1}^d
1	13	30	13	30		0	0
2	25	7	25	7		0	0
3	25	28	21	21	1.5	4	7
4	16	20	20	22	1.5	0	5
	79	85	79	80		4	12

Now we chose the first block of intervals where tangents are on the one side from the "trade-off" value. The block includes intervals 3 and 4 with tangents that are less than or equal to $\frac{\alpha}{1-\alpha} = 2.(3)$. Therefore, the condition (28) is checked on violation. Reallocation of capacity in these intervals is possible because (17) and (18) are satisfied: $u_3 > 15$ and $u_4 < 25$. With $k = 0$, $l = 0$, $\tan\gamma = 2.25$, $\tan\tilde{\gamma} = 1.5$ the condition (28) is violated. $\delta > 0$ should be chosen so that either one or both flows will be shifted to the nearest to it knot of Pareto curve so far as the minimum of departure queue, which is equal to 8, is not used up. Hence, $\delta = 4$ and after reallocation $\tan\gamma$ will be equal to 1.5. It leads to satisfaction of condition (28). Reallocation is performed with formulas of Section 4.1 and shown in Table 3. Accordingly to Algorithm 1 tangent $\tan\gamma_4$ for the interval $i = 4$ is deleted from the considered block and check is finished. Calculated flow solves the problem with original demand (a_i, d_i), $i \in \{1, .., 4\}$, where queue have to be extended by unavoidable queue from Table 1.

6.1.2 $\alpha = 0.6$

"Trade - off" value for the given weight $\alpha = 0.6$ is equal to $\frac{\alpha}{1-\alpha} = 1.5$. The initial flow calculation in the first two intervals coincide with the described in Section 6.1.1.

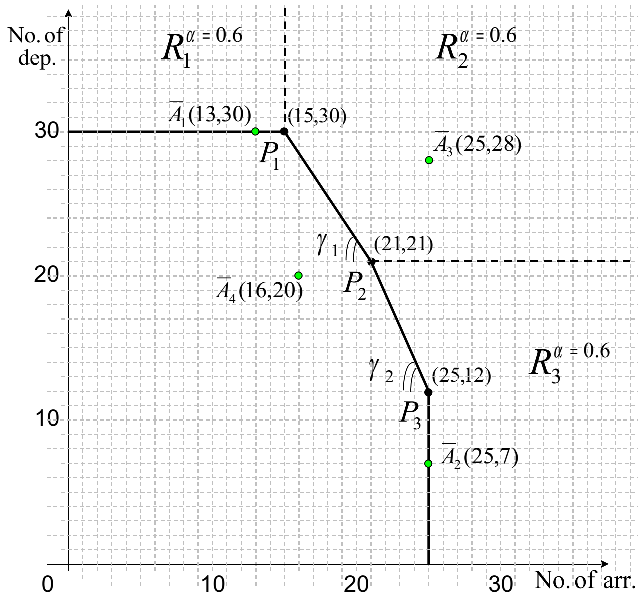

Fig. 8 *Regions for* $\alpha = 0.6$

Table 4 Example 1: $\alpha = 0.6$ - initial solution

i	demand $\bar{a}_i$	$\bar{d}_i$	flow u_i	v_i	$\tan\gamma_i$	queue q^a_{i+1}	q^d_{i+1}
1	13	30	13	30		0	0
2	25	7	25	7		0	0
3	25	28	21	21	1.5	4	7
4	16	20	19	24	1.5	1	3
	79	85	78	82		5	10

The demand point $\bar{A}_3$ belongs to $R_2^{0.6}$. Therefore, $u_3 = 21$ and $v_3 = 21$. Since there is queue at the end of the third interval, tangent $\tan\gamma = 1.5$ of the segment slope which contains the flow point $(21,21)$ is saved (Table 4). The demand point $\bar{A}_4 + (q_4^a, q_4^d) = (20,27) \in R_2^{0.6}$. Hence, the flow point for this interval is on the segment with tangent of the slope $\tan\gamma = 1.5$. Since the departure part of the flow is not integer and conditions of Lemma 1 (conclusion 1) are satisfied, the value $\tan\gamma = 1.5$ is saved and the flow point is changed by the integer point $(19,24) \in P_1P_2$ (see Table 4).

The first block of intervals in the initial flow includes intervals 3 and 4 with tangents that are less than or equal to $\frac{\alpha}{1-\alpha} = 1.5$. The condition (28) is satisfied for the block. Therefore, the initial solution from Table 4 is optimal for $\alpha = 0.6$.

6.1.3 $\alpha = 0.5$

Tangent of the "trade - off" value for $\alpha = 0.5$ is equal to $\frac{\alpha}{1-\alpha} = 1$. The initial flow calculation in the first two intervals is the same as described in Section 6.1.1.

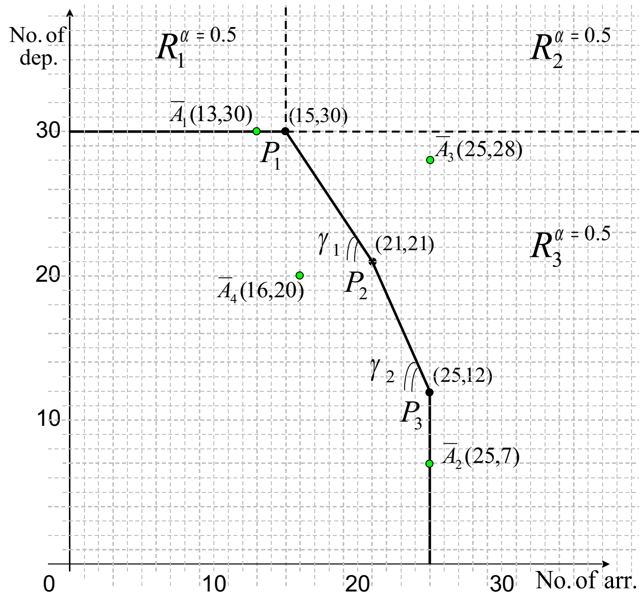

Fig. 9 *Regions for* $\alpha = 0.5$

Since $\bar{A}_3 \in R_3^{0.5}$, the initial flow point in this interval is calculated using strategy DepPriority with formulas (29) and (30). Since there is arrival queue at the end of the third interval, tangent $\tan\gamma = 1.5$ of the segment slope which contains the flow point is saved and the flow is rounded down (Table 5). The demand point $\bar{A}_4 + (q_4^a, q_4^d) = (25,20) \in R_3^{0.5}$. Hence, the initial flow point for this interval will belong to the segment with tangent of the slope $\tan\gamma = 2.25$.

Table 5 Example 1: $\alpha = 0.5$ - initial solution

i	demand $\bar{a}_i$	$\bar{d}_i$	flow u_i	v_i	$\tan\gamma_i$	queue q^a_{i+1}	q^d_{i+1}
1	13	30	13	30		0	0
2	25	7	25	7		0	0
3	25	28	16	28	1.5	9	0
4	16	20	21	20	2.25	4	0
	79	85	75	85		13	0

Since the arrival part of the flow is not integer, the value $\tan\gamma = 2.25$ is saved and the flow is rounded down (Table 5).

Condition (31) is satisfied and $\alpha = 0.5$. Since the flow in the third interval is integer through rounding down, we have to check based on Lemma 1 (conclusion 2) whether the flow $u_3 = 17$, $v_3 = 27$ in the third interval gives the better outcome. As we see in Table 6, the total flow in the forth interval grows. Therefore, the flow calculated in Table 6 solves the problem optimally.

Table 6 Example 1: $\alpha = 0.5$ - update

i	demand		flow		$\tan\gamma_i$	queue	
	$\bar{a}_i$	$\bar{d}_i$	u_i	v_i		q^a_{i+1}	q^d_{i+1}
1	13	30	13	30		0	0
2	25	7	25	7		0	0
3	25	28	17	27	1.5	8	1
4	16	20	21	21	1.5	3	0
	79	85	76	85		11	1

6.1.4 *Minimal finite solution set for Example 1*

Accordingly to Section 5.1 and (37) the minimal finite solution set for any $\alpha \in [0,1]$ is constructed in three steps.

At the first step interval $[0,0.6]$ for the weight α is considered. We take the initial solution for $\alpha = 0$ which coincides with the calculated in Table 5 and check the condition (31) for the unique block $\{3,4\}$. Since $k = 0$, $l = 0$, $\cot\gamma = \frac{2}{3}$, $\cot\tilde{\gamma} = \frac{4}{9}$, $\frac{1-\alpha}{\alpha} = \frac{2}{3}$, the condition (31) is violated. After reallocation we get the flow from Table 6 which satisfies the condition (31). Therefore, the flows from Tables 5 and 6 belong to the minimal finite solution set of Example 1 illustrated in Figure 10 and collected in Table 7.

At the second step we change to $\alpha \in [0.6, \frac{9}{13}]$ and calculate the initial solution for $\alpha = 0.6$ which coincides with the given in Table 4. The condition (28) in intervals 3 and 4 is satisfied for $\alpha = \frac{9}{13}$. Hence, the flow from Table 4 can be included in the minimal finite solution set.

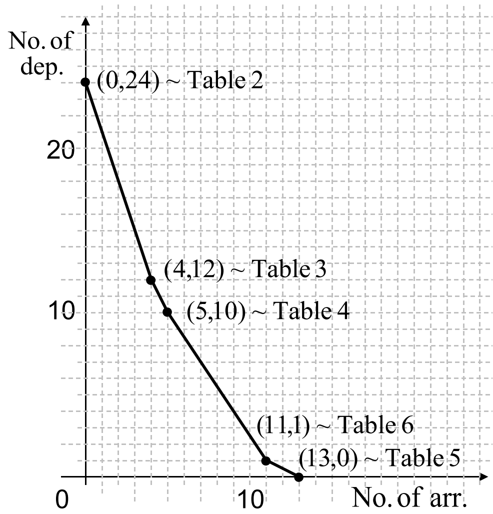

Fig. 10 *Concave solution curve for Example 1*

Finally, $\alpha \in [\frac{9}{13}, 1]$ is considered. The flow from Table 2 is the initial flow for $\alpha = 1$. The condition (28) in intervals 3 and 4 is violated for $\alpha = \frac{9}{13}$. After reallocation the flow from Table 3 is obtained. It satisfies the condition (28). Therefore, the construction of the minimal finite solution set is finished.

Table 7 Minimal finite solution set for Example 1

Table	2	3	4	6	5
α	$[\frac{3}{4},1]$	$[\frac{2}{3},\frac{3}{4}]$	$[\frac{3}{5},\frac{2}{3}]$	$[\frac{1}{3},\frac{3}{5}]$	$[0,\frac{1}{3}]$

The reader can check optimality of solutions for the given weight α by the corresponding condition (28) or (31).

6.2 Second example

Example 2 *Time period T consist of $N = 5$ time intervals, $MaxArr_i = 12$, $MaxDep_i = 12$, $MaxCap_i = 20$ for $i = 1,\dots,5$. Arrival/departure demand is collected in Table 8.*

Find a flow which solves Problem 1 for $\alpha = \frac{1}{3}$, $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

Example 2 is taken from [6] where it is solved with the help of an reallocation method developed for the simple case of Pareto curve. The reader can compare the solution from [6] with

Table 8 Example 2: arrival/departure demand

time int.	arr. dem.	dep. dem.
1	$a_1 = 12$	$d_1 = 12$
2	$a_2 = 12$	$d_2 = 12$
3	$a_3 = 12$	$d_3 = 12$
4	$a_4 = 1$	$d_4 = 10$
5	$a_5 = 2$	$d_5 = 10$
total demand	39	56

the given below constructed by the reallocation method for the general case of Pareto curve.

Since there is no unavoidable queue over the time period T, the cutted demand coincides with the initial demand from Table 8.

In order to identify regions with respect to the fixed weight α the points of the Pareto curve giving minimum of the weighted rectilinear distance function $d_\alpha(M(12,12), P_1P_2)$ are found

$$\begin{cases} P_1 = (8,12), & \text{if } \alpha \in [0, 0.5], \\ P_1P_2, & \text{if } \alpha = 0.5, \\ P_2 = (12,8), & \text{if } \alpha \in [0.5, 1]. \end{cases} \qquad (38)$$

6.2.1 $\alpha = \frac{1}{3}$

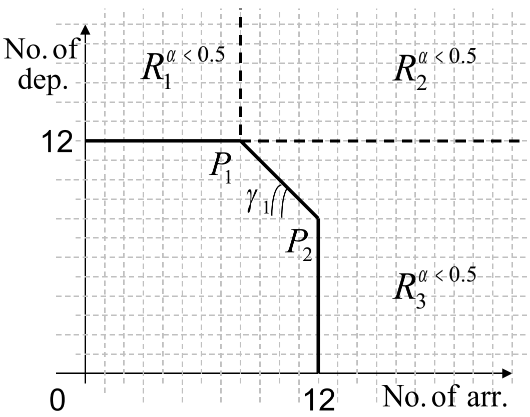

Fig. 11 *Regions for* $\alpha < 0.5$

For $\alpha = \frac{1}{3}$ region $R_2^{\frac{1}{3}}$ is based on the point $P_1 = (8,12)$ (see Figure 11). The initial flow illustrated in Table 9 is calculated using Strategy DepPriority (29)-(30). Since all demand points

Table 9 Example 2: $\alpha = \frac{1}{3}$ – initial solution

i	demand a_i	d_i	flow u_i	v_i	$\tan\gamma_i$	queue q^a_{i+1}	q^d_{i+1}
1	12	12	8	12	1	4	0
2	12	12	8	12	1	8	0
3	12	12	8	12	1	12	0
4	1	10	10	10	1	3	0
5	2	10	5	10	0	0	0
	39	56	39	56		27	0

of the block $\{1,2,3,4\}$ are in the regions $R_2^{\frac{1}{3}}$ and $R_3^{\frac{1}{3}}$, the condition (31) has to be checked:

- $i = 1$ and $i = 2 \Rightarrow k = 0$, $l = 2$ and (31) is violated, but $v_2 = MaxDep$ and can not be shifted up $\Rightarrow$ go to $i = 2$ and $i = 3$
- $i = 2$ and $i = 3 \Rightarrow k = 0$, $l = 1$ and (31) is satisfied $\Rightarrow$ go to $i = 3$ and $i = 4$
- $i = 3$ and $i = 4 \Rightarrow k = 0$, $l = 0$ and (31) is satisfied $\Rightarrow$ the block $\{1,2,3,4\}$ is checked.

Therefore, the flow from Table 9 is an optimal solution of the problem for $\alpha = \frac{1}{3}$.

6.2.2 $\alpha = \frac{1}{2}$

For $\alpha = \frac{1}{2}$ region $R_2^{\frac{1}{2}}$ is based on the segment $P_1P_2 = [(8,12),(12,8)]$ (see Figure 12). The initial flow is given in Table 10.

Table 10 Example 2: $\alpha = \frac{1}{2}$ – initial solution

i	demand a_i	d_i	flow u_i	v_i	$\tan\gamma_i$	queue q^a_{i+1}	q^d_{i+1}
1	12	12	12	8	1	0	4
2	12	12	12	8	1	0	8
3	12	12	12	8	1	0	12
4	1	10	1	12	0	0	10
5	2	10	2	12	0	0	8
	39	56	39	48		0	42

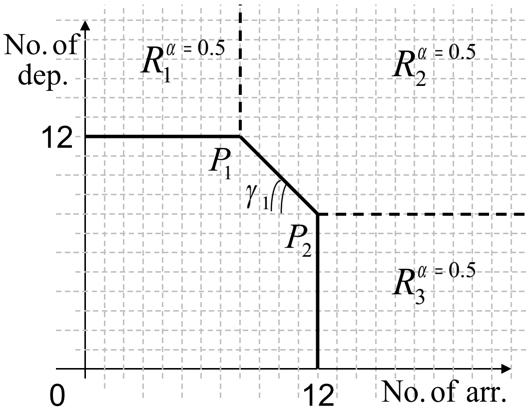

Fig. 12 *Regions for* $\alpha = 0.5$

Since all demand points appear in regions $R_1^{\frac{1}{2}}$ and $R_2^{\frac{1}{2}}$, we have to check condition (28) for the block $\{1,2,3,4,5\}$. It is performed by Algorithm 1:

- $i = 1$ and $i = 2 \Rightarrow k = 0$, $l = 3$ and (28) is violated, but $u_2 = MaxArr$ and can not be shifted up $\Rightarrow$ go to $i = 2$ and $i = 3$
- $i = 2$ and $i = 3 \Rightarrow k = 0$, $l = 2$ and (28) is violated, but $u_3 = MaxArr$ and can not be shifted up $\Rightarrow$ go to $i = 3$ and $i = 4$

Table 11 Example 2: $\alpha = \frac{1}{2}$ – update 1

i	demand a_i	d_i	flow u_i	v_i	$\tan\gamma_i$	queue q^a_{i+1}	q^d_{i+1}
1	12	12	12	8	1	0	4
2	12	12	12	8	1	0	8
3	12	12	8	12	1	4	8
4	1	10	5	12	0	0	6
5	2	10	2	12	0	0	4
	39	56	39	52		4	30

- $i = 3$ and $i = 4 \Rightarrow k = 0$, $l = 1$, (28) is violated and $u_4 < MaxArr$. Hence, reallocation is performed and collected in Table 11, interval $i = 3$ is eliminated from the future consideration (marked grey in Table 11) $\Rightarrow$ go to $i = 2$ and $i = 4$
- $i = 2$ and $i = 4 \Rightarrow k = 1$, $l = 1$ and (28) is violated and $u_4 < MaxArr$. Hence, reallocation is performed and collected in Table 12, interval $i = 4$ is eliminated from the future consideration $\Rightarrow$ go to $i = 2$ and $i = 5$

Table 12 Example 2: $\alpha = \frac{1}{2}$ – update 2

i	demand a_i	d_i	flow u_i	v_i	$\tan\gamma_i$	queue q^a_{i+1}	q^d_{i+1}
1	12	12	12	8	1	0	4
2	12	12	9	11	1	3	5
3	12	12	8	12	1	7	5
4	1	10	8	12	1	0	3
5	2	10	2	12	0	0	1
	39	56	39	55		10	18

- $i = 2$ and $i = 5 \Rightarrow k = 2$, $l = 0$ and (28) is violated and $u_5 < MaxArr$. Hence, reallocation is performed and collected in Table 13. There is no departure queue in the interval $i = 5$. Hence, the block $\{1,2,3,4,5\}$ is checked and the flow in Table 13 is optimal for $\alpha = 0.5$.

Table 13 Example 2: $\alpha = \frac{1}{2}$ – update 3

i	demand a_i	d_i	flow u_i	v_i	$\tan\gamma_i$	queue q^a_{i+1}	q^d_{i+1}
1	12	12	12	8	1	0	4
2	12	12	8	12	1	4	4
3	12	12	8	12	1	8	4
4	1	10	8	12	1	1	2
5	2	10	3	12	0	0	0
	39	56	39	56		13	14

6.2.3 $\alpha = \frac{3}{4}$

For $\alpha \in [\frac{1}{2}, 1]$ region R_2^{α} is based on the point $P_2 = (12, 8)$ (see Figure 13). Therefore, the flow illustrated in Table 10 is the initial flow for any

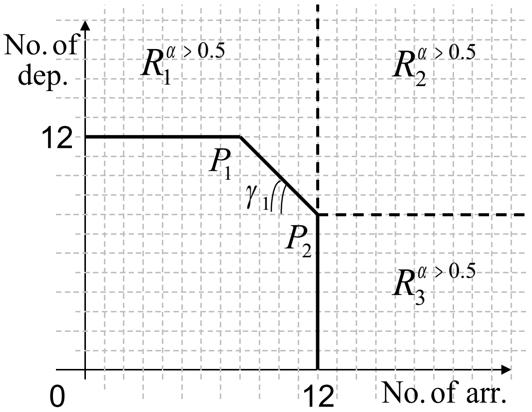

Fig. 13 *Regions for* $\alpha > 0.5$

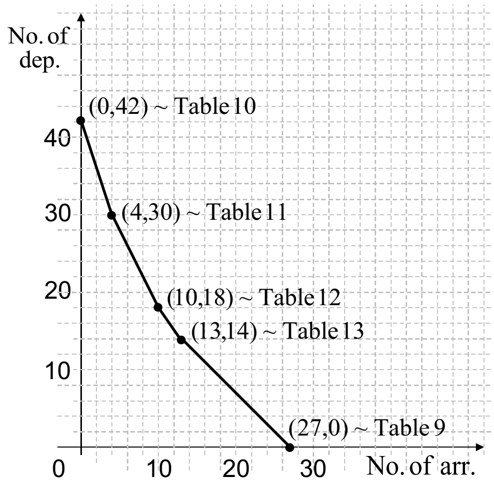

Fig. 14 *Concave solution curve for Example 2*

$\alpha \in [\frac{1}{2}, 1]$. All demand points are in the regions $R_1^{\frac{3}{4}}$ and $R_2^{\frac{3}{4}}$. Hence, the condition (28) for the block $\{1,2,3,4,5\}$ has to be checked:

- $i = 1$ and $i = 2 \Rightarrow k = 0$, $l = 3$ and (28) is violated, but $u_2 = MaxArr$ and can not be shifted up $\Rightarrow$ go to $i = 2$ and $i = 3$
- $i = 2$ and $i = 3 \Rightarrow k = 0$, $l = 2$ and (28) is satisfied $\Rightarrow$ go to $i = 3$ and $i = 4$
- $i = 3$ and $i = 4 \Rightarrow k = 0$, $l = 1$, (28) is satisfied $\Rightarrow$ go to $i = 4$ and $i = 5$
- $i = 4$ and $i = 5 \Rightarrow k = 0$, $l = 0$, (28) is satisfied $\Rightarrow \{1,2,3,4,5\}$ is checked.

Thus, the initial solution from Table 10 is an optimal solution for $\alpha = \frac{3}{4}$.

6.2.4 *Minimal finite solution set for Example 2*

Based on Section 5.1 and (38) construction of the minimal finite set is performed in two steps.

On the first hand, the initial solution for $\alpha = 0$ is calculated and capacity in it is reallocated in order to achieve an optimal flow for $\alpha = \frac{1}{2}$. The solution from Table 9 is the initial solution for $\alpha = 0$ and as it can be easily checked an optimal solution for $\alpha = \frac{1}{2}$. This flow solves Problem 1 for any $\alpha \in [0, \frac{1}{2}]$. Therefore, we put the solution from Table 9 into the minimal finite solution set illustrated in Figure 14 and collected in Table 14.

On the second hand, the initial solution for $\alpha = 1$ is taken to find an optimal flow for $\alpha = \frac{1}{2}$. Since the initial solution for $\alpha = 1$ coincides with the illustrated in Table 10 and there is only one block in it, Tables 10-13 from Section 6.2.2 finalize finding of the minimal finite solution set of Example 2.

Table 14 Minimal finite solution set for Example 2

Table	10	11	12	13	9
α	$[\frac{3}{4}, 1]$	$[\frac{2}{3}, \frac{3}{4}]$	$[\frac{4}{7}, \frac{2}{3}]$	$[\frac{1}{2}, \frac{4}{7}]$	$[0, \frac{1}{2}]$

The reader can check optimality of solutions for the given weight α by condition (28) for flows from Tables 10-13 and by condition (31) for the flow from Table 9.

7 Conclusions

Our investigation results in a straightforward and effective $O(N^2)$ algorithm giving integer solution for the problem of total weighted queues sum minimization over a planning horizon with the corresponding set of general Pareto curves for arbitrary number of considered time intervals. That is also an advantage compared to the linear programming approach. Moreover, the minimal finite solution set, which contains an optimal flow for each weight value in the above mentioned problem, can be calculated easily. Flows from this set produce minimal weighted delay and maximal weighted throughput. Additionally, an "almost optimal" solution can be obtained by hand for an arbitrary number of considered intervals.

With the constructed algorithm we get a controlling tool for the outcome of the arrival/departure queue. The algorithm can be extended, for instance, for design of a flow which allows to achieve the maximal number of punctual flights. In general, the presented algorithm can be used for any problem of allocation of resources when their interdependence is given by convex piecewise linear curve.

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