VIABLE FEEDBACK SPACE TRAJECTORY CONTROL

Kouamana Bousson

Avionics and Control Laboratory
Department of Aerospace Sciences
University of Beira Interior
6201-001 Covilha, Portugal

Keywords: *viable control, corridor, feedback, orbital control.*

Abstract

The present paper proposes a method for controlling an aerial vehicle inside a corridor without leaving it, the dynamics of the vehicle being subject to bound constraints on the control variables. Such a control method is necessary when the system under control does not need to follow exactly a curvilinear reference trajectory. Therefore, what is in fact required from such a system is to maintain the trajectory close to some nominal reference within a certain admissible tolerance. Based on the known initial and terminal positions of the system, a sequence of waypoints is chosen in such a way that the segment joining two consecutive waypoints remain fully inside the control corridor. Then a predictive control law is designed to control the system from a waypoint to the next until the specified terminal position is reached. An application dealing with orbital control illustrates the method and reveals its potentials for handling the control of complex systems even in case of unknown measurement noise.

1 Introduction

The present paper deals with the problem of driving an aerial vehicle from an initial state x_0 to a final state x_f such that its state trajectory remains within a given corridor K assumed to be compact (that is: closed and bounded), with x_0 and x_f in K as illustrated by Fig. 1. The dynamics of the vehicle is subject to bound-constrained controls.

There are many reasons to consider such control problems for which the references are not single values or line curves, but rather tubes within which the controlled variables should remain. Following reliably a real-valued trajectory, known as a nominal trajectory, is not necessary at all. Indeed, in existing online trajectory control applications, one tries to follow a prescribed trajectory as close as possible, which simply means that the system under control does not need to follow exactly the reference curvilinear trajectory. Therefore, what is in fact required to a control system in an application is mainly to maintain controlled variables close to some nominal reference within acceptable tolerance bounds. This is the case for instance for aircraft control at takeoff or landing, spacecraft orbit control, and aircraft collision avoidance in the context of air traffic control [3]. From the standpoint of actuation, it is interesting to mention that no control alteration is necessary as long as the controlled variables remain within the tolerance bounds. While curvilinear trajectory control has been dealt with in aeronautical and astronautical engineering, work about set-based trajectory control is inexistent and the present paper is an attempt to exploit viability theory [1] to devise methods for coping with the problem.

In section 2 is stated the problem that we intend to solve. Section 3 proposes a solution, and we deal with an application to orbital control in section 4.

2 Problement Statement

Let us consider an aerial vehicle whose dynamics is modeled as:

$$\dot{x}(t) = f\left(x(t), u(t)\right) \tag{1}$$
$$u(t) \in U \tag{2}$$

where $x \in \Re^n$ and $u \in \Re^m$ are respectively the state and the control vectors, f is a nonlinear vector function, U is the control domain assumed to be bounded and convex.

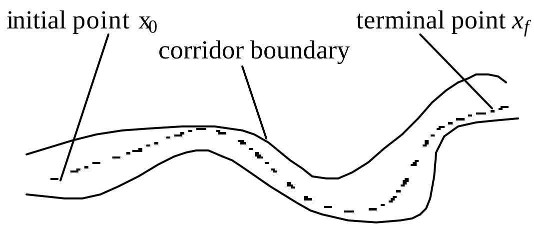

Fig. 1: Navigation corridor

Let $K \subset \Re^n$ be a domain of the state space, and two different states x_0 and x_f laying in K. The problem to be solved is to find a control map v such that the solution of the following vector differential equation:

$$\dot{x}(t) = f\left(x(t), v(t)\right) \tag{3}$$

with initial state condition $x(t_0) = x_0$ and final state condition $x(t_f) = x_f$ remain fully in K (never leave K) from time t_0 to time t_f.

Some considerations are to be taken into account:

- The initial time t_0 is given, but the final time t_f is not (free terminal time).
- The domain K is not necessarily smooth nor convex.
- The control map which is sought should not be too oscillatory.

Domain K may be modelled as a vector inequation of the form:

$$g(x) \leq 0 \tag{4}$$

where g is a vector function defined on the state space

Two main reasons sustain the fact that control map should not be too oscillatory: first, too many oscillations of the control can impart an uncomfortable state dynamics of the system under control; secondly, the oscillations can reduce the actuator performances.

3 Theoretical Development

3.1 General concepts

We can obtain the non-too-oscillatory viable control solutions by setting a bound on the growth of the control. For that purpose, we shall constrain the control system by adjoining a control-dependent constraint as:

$$\dot{u}(t) \in \prod_{i=1}^{m} [a_i, b_i] \tag{5}$$

where a_i and b_i are constant real numbers. Therefore, the model of the vehicle becomes:

$$\dot{x}(t) = f\left(x(t), u(t)\right) \tag{6}$$
$$\dot{u}(t) \in \prod_{i=1}^{m} [a_i, b_i] \tag{7}$$
$$u(t) \in U \tag{8}$$

A solution x of equation (Eq. 1) is said to be *viable* in K from the initial condition $x(t_0) = x_0$ if for each $t \geq t_0$ $x(t) \in K$. A control u which enables the system to have viable solutions in K from time t_0 to time t_f is called a *viable control* [1] in K from t_0 to t_f since it maintains the trajectory of the system under control in domain K between the specified initial and final states.

The model of the control system described by (Eq. 1-2) is equivalent to the following differential inclusion [1]:

$$\dot{x}(t) \in F\left(x(t)\right) = \left\{ f\left(x(t), u(t)\right) : u(t) \in U \right\}, \qquad (9)$$

for $t_0 \leq t \leq t_f$

For a general purpose set K, we introduce the notion of *contingent cone* $T_K(x)$ to K at a point $x \in K$, which is a closed cone of elements y such that:

$$\liminf_{h \to 0+} \frac{d_K(x + hy)}{h} = 0 \qquad (10)$$

where $d_K(z) = \inf_{w \in K} \|z - w\|$ is the distance from z to set K.

It proven in [1] that for the solution of (Eq. 9) to be viable in K, it is necessary and sufficient that the solution meet the following condition:

$$F(x) \cap T_K(x) \neq 0 \qquad (11)$$

However, the case dealt with in the present paper rather uses the equation of set K as given in (Eq. 4). Indeed, the viable control will be sought so that the viable solution x satisfy the following:

For any t such that: $t_0 \leq t \leq t_f$,

$$g(x(t)) \leq 0 \qquad (12)$$

To find a viable control from computational point of view, it is necessary to transform the differential inclusion (Eq. 9) into a difference inclusion. Using a numerical integration scheme one rather obtains a discrete model of differential equation $\dot{x}(t) = f\left(x(t), u(t)\right)$ under the form:

$$x^{(k+1)} = \tilde{f}(x^{(k)}, u^{(k)}), \quad k = 0,1,2,... \qquad (13)$$

where:

$x^{(k)} \equiv x(t_k)$, $u^{(k)} \equiv u(t_k)$, with $t_k = t_0 + k.\Delta t$, and $\Delta t = t_{k+1} - t_k$ is the integration stepsize.

Howe [4] developed real-time simulation algorithms for ordinary differential equations, which is a variation the classical explicit Adams-Moulton predictor-corrector methods. Unlike the Adams methods, the algorithms of Howe best suit online nonlinear control problems as the one we are dealing with here. Howe devised a two-pass and a three-pass real-time predictor-corrector algorithms. For instance, the two-pass algorithm is described as:

$$\begin{aligned} f_k &= f(x^{(k)}, u^{(k)}) \\ \hat{x}^{(k+1/2)} &= x^{(k)} + \frac{\Delta t}{8}(5 f_k - f_{k-1}) \\ \hat{f}_{k+1/2} &= f(\hat{x}^{(k+1/2)}, u^{(k+1/2)}) \\ x^{(k+1)} &= x^{(k)} + \Delta t. \hat{f}_{k+1/2} \end{aligned} \qquad (14)$$

where:

$u^{(k+1/2)} \equiv u(t_{k+1/2})$, with $t_{k+1/2} = t_k + \Delta t/2$.

As may be noticed, the first pass uses an Adams-Bashforth type of predictor algorithm to compute an estimate of the state at time $t_{k+1/2}$ instead of time t_{k+1} as usual. That estimate is then used to compute the derivative at time $t_{k+1/2}$ which, along with derivatives at time

$t_k, t_{k-1}, t_{k-2},...$ is used to compute the state at time t_{k+1}. Taking a linear approximation of the control between times t_k and t_{k+1}, the control in midway (at $t_{k+1/2}$) as it appears in (Eq. 14) may be computed as:

$$\overline{u}^{(k)} \equiv u^{(k+1/2)} = \frac{u^{(k)} + u^{(k+1)}}{2} \tag{15}$$

We have clearly $\overline{u}^{(k)} \in U$ since the control domain U is assumed to be convex (section 2). By the Howe integration algorithm above, (Eq. 13) will rather be written as:

$$x^{(k+1)} = \tilde{f}(x^{(k)}, \overline{u}^{(k)}) \tag{16}$$

The midway control $\overline{u}^{(k)}$, at $t_{k+1/2}$, is used in (Eq. 13) instead of $u^{(k)}$, and function $\tilde{f}$ is defined as:

$$\begin{aligned}&\tilde{f}(x^{(k)}, \overline{u}^{(k)})\\ &= x^{(k)} + \Delta t. f(x^{(k)} + (5f_k - f_{k-1}).\Delta t/8, \overline{u}^{(k)})\end{aligned} \tag{17}$$

The admissible control dynamics as described through (Eq. 7) needs to be discretized so that an estimate of the control domain at each time time interval may be computed. Since the righthand side of (Eq. 7) is constant, the Euler integration scheme yields the exact solution. Indeed, V_k, the admissible control domain at t_k, is obtained as the following set:

$$V_k = u^{(k-1)} + \prod_{i=1}^{m} [\Delta t.a_i, \Delta t.b_i] \tag{18}$$

In the control algorithm which will be described later, $u^{(k-1)}$ will be the actual value of the control at t_{k-1}.

3.2 Waypoint Design

Waypoints are reference points used for trajectory check up in navigation. They allow the navigator to assess drift from nominal track and to estimate delays and distances with respect to a flight plan. To fly from initial point x_0 to final point x_f without leaving domain K, it is suitable to choose waypoints in K.

Let x be a state of the vehicle in K. State x is said to be *between* waypoints $w^{(j)}$ and $w^{(j+1)}$ for any $j = 0,1,..., N-1$, if:

$$\forall i = 1,2,..., n, \quad w_i^{(j)} \le x_i \le w_i^{(j+1)} \tag{19}$$

where x_i denotes the i^{th} coordinate of vector x in the state space, numbers $w_i^{(j)}$ and $w_i^{(j+1)}$ are defined accordingly, and "≤" is considered componentwise.

Assume the state x of the vehicle to be *between* waypoints $w^{(j)}$ and $w^{(j+1)}$. Then, the waypoint $w^{(j)}$ is called the *backward waypoint* and $w^{(j+1)}$ the *forward waypoint* for state x.

Since K may be nonconvex, the waypoints have to be chosen in such a way that the segment joining two consecutive waypoints be fully in K and that the line joining a state in K to its forward waypoint be fully in K as well. Formally, let $w^{(0)}, w^{(1)},..., w^{(N-1)}, w^{(N)}$ be the sequence of waypoints which are chosen for navigating from the initial point x_0 to the final point x_f, with $w^{(0)} \equiv x_0$, and $w^{(N)} \equiv x_f$, then, for any $j = 0,1,..., N-1$, the waypoints $w^{(j)}$ and $w^{(j+1)}$ must meet the following two constraints:

1. $\forall r \in [0,1], \quad \left(r.w^{(j)} + (1-r).w^{(j+1)}\right) \in K$ (20)

2. For any state $y \in K$ which is between $w^{(j)}$ and $w^{(j+1)}$ the following holds:
   $\forall r \in [0,1], \quad \left(r.y + (1-r).w^{(j+1)}\right) \in K$ (21)

3.3 Control Algorithm

Based on the choice of the waypoints as explained above, the idea of the control methodology can now be described:

From the current position of the vehicle, one first needs to determine the forward waypoint w^{next} of the current position x^* as a guide to where to go. From the knowledge of the current control vector u^*, the control domain for the next move has to be computed. The following considerations are made and taken into account in determining the viable control:

- The control for the next move should be such that the remaining distance up to the forward waypoint be minimized.
- The change in the control for the next move needs to be minimized so that it can be without high increase.
- There may be many control vectors which fulfill the two previous requirements, therefore, it is clear that it would be better for the sake of control performance to choose the control vector with minimum norm.

These considerations lead to consider the following criterion:

$$J(u) = \left(x^{(k)} - w^{next}\right)^T Q\left(x^{(k)} - w^{next}\right) + \left(u - u^*\right)^T R\left(u - u^*\right) + \|u\|^2 \tag{22}$$

where $x^{(k)}$ is the predicted state at the next time instant t_k, and Q and R are positive definite matrices. The predicted state is computed using the Howe algorithm described above from the current actual state which corresponds to the state at time t_{k-1}. The full control algorithm is given below:

0. Set the initial control u_0 to an appropriate vector, and the iteration variable t to 1.
1. Determine the current actual position x^* (with navigation sensors) and the current actual control vector u^* of the vehicle. If $\left\|x^* - x_f\right\| \le e$ then stop. /* *e is the required precision.* */
2. Determine the forward waypoint w^{next} of x^*.
3. Compute: $V = u^* + \prod_{i=1}^{m} [\Delta t.a_i, \Delta t.b_i]$.
4. Find $u^{(k)}$ as a solution to the following constrained minimization problem:

$$Min\ J(u) = \left(x - w^{next}\right)^T Q\left(x - w^{next}\right) + \left(u - u^*\right)^T R\left(u - u^*\right) + \|u\|^2$$

subject to: $x = \tilde{f}(x^*, \bar{u})$, $g(x) \le 0$, $u \in U$, and $u \in V$, with $\bar{u} = (u^* + u)/2$.

5. Control the vehicle from time t_{k-1} to t_k using the control vector u defined for $t_{k-1} \le t \le t_k$ as:

$$u(t) = u^* + \frac{u^{(k)} - u^*}{\Delta t}(t - t_{k-1})$$

6. Update $k = k + 1$ and go to 1.

4 Application: Viable Orbital Control

Various forces continually change the characteristics of the satellite orbit with respect to the reference one [2, 5]:

- atmospheric drag, caused by the earth's residual atmosphere at the satellite's altitude effectively slows down the satellite. The atmospheric drag depends on solar activity and varies periodically on eleven-year basis.

- the combined gravitational attraction of the sun and the moon, which reduces inclination of the orbital plane.
- The solar radiation pressure.
- The earth's gravitational potentials.

Due to these disturbances, the satellite may drift from the optimal orbit, giving rise to an actual trajectory that may be above/below or at the left/right side of the reference orbit. Therefore, orbital control aims at maintaining the satellite on the same reference orbit characteristics so that it can fulfill its mission. This simply means the drifts from the nominal orbit have to be monitored and corrected. When the perturbations are well known, then the orbital control law is designed based on the atmospheric drag. Indeed, the satellite trajectory is intentionally drifted by the control, so that the effect of the atmospheric drag cancels out this drift and brings the satellite to the reference orbit.

Considering only the disturbing acceleration due to the earth's gravitational potentials, the equations of motion for a satellite in the earth's spherical gravitational field can be expressed in an inertial spherical reference as:

$$
\begin{aligned}
\ddot{r} &= r\dot{q}^2 \sin^2 f + r\dot{f}^2 - \frac{m}{r^2} \\
&\quad + \frac{3}{2} m J_2 a_e^2 \frac{3\cos^2 f - 1}{r^4} + u_r \\
\ddot{q} &= \frac{-2\dot{r}\dot{q}}{r} - 2\dot{q}\dot{f} \cot f + \frac{u_q}{r \sin f} \\
\ddot{f} &= \frac{-2\dot{r}\dot{f}}{r} + \dot{q}^2 \sin f \cos f \\
&\quad + 3mJ_2 \frac{a_e^2}{r^5} \cos f \sin f + \frac{u_f}{r}
\end{aligned}
\tag{23}
$$

where $J_2 = 0.0010826$, a_e is the earth's equatorial radius, r is the radial distance from the earth center to the satellite, q is the angle counted from the x-axis in the xy-plane to the projection of $\vec{r}$ onto the xy-plane, f is the angle measured from the z-axis to vector $\vec{r}$, and u_r, u_q and u_f are the thrust acceleration components related to the spherical reference.

By setting:

$$
\begin{aligned}
& x_1 = r,\ x_2 = q,\ x_3 = f,\ x_4 = \dot{r},\ x_5 = \dot{q},\ x_6 = \dot{f}, \\
& x = [x_1, x_2, x_3, x_4, x_5, x_6\ ]^T, \\
& u = [u_r, u_q, u_f]^T,
\end{aligned}
$$

then (Eq. 23) may be written as (Eq. 1).

The differential inclusions modeling the control change are:

$$
\begin{aligned}
-65 \le \dot{u}_r \le 65 \\
-27 \le \dot{u}_q \le 27 \\
-52 \le \dot{u}_f \le 52
\end{aligned}
$$

The control domain U is defined through the following inequalities:

$$
\begin{aligned}
0 \le u_r \le 35000 \\
0 \le u_q \le 1500 \\
0 \le u_f \le 27000
\end{aligned}
$$

The problem copes with a satellite on an elliptical orbit with a 55º inclination with respect to the equatorial plane, a semi-minor axis of 5500 nautical miles and a semi-major axis of 6000 nautical miles. The application deals with controlling the satellite on its orbit within a virtual corridor around the nominal orbit, starting from lowest edge of the orbit. The orthogonal section of the orbit corridor is assumed to be circular of radius 5 nautical miles.

For the validation the method presented above, the computed state at each time-step, based on the Howe algorithm, is corrupted with 10% gaussian random noise, which made the signal-to-noise ratio 10-to-1 for each simulated state measurement. This noise injection mimicks the other disturbances (sun and moon gravity, solar radiation pressure, atmospheric drag) not accounted for in the satellite motion model described by (Eq. 23). Therefore, the simulated and corrupted state at each time-step is considered as the actual position referred to in step 1 of the control algorithm in the previous section. The result of the simulation is depicted by fig. 2, where the viable orbit trajectory is represented in the orbital plane coordinate reference system. The inner circle represents earth and the controlled trajectory is inside the corridor represented by the two outer ellipses.

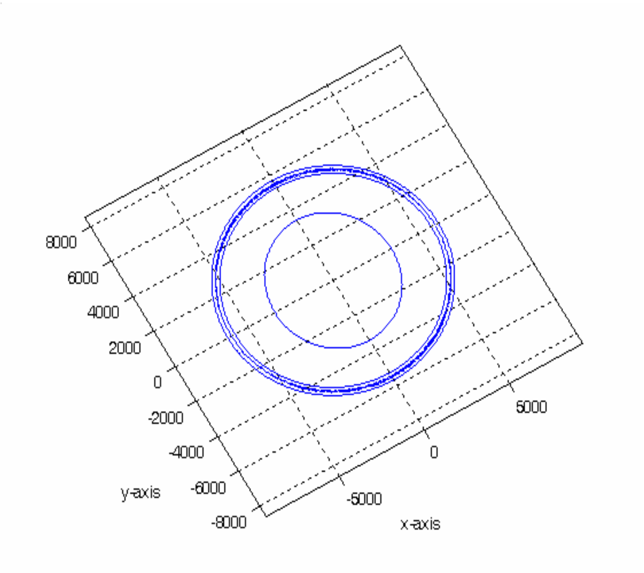

Fig. 2: Viable orbital trajectory

5 Conclusion

A newly developed method for viable trajectory control inside a corridor is presented in the present paper. Such a control method is necessary when the system under control does not necessarily need to follow exactly a curvilinear reference trajectory. Therefore, what is in fact required from such a system is to maintain the trajectory close to some ideal reference within a certain acceptable tolerance. Based on the known initial and terminal positions of the system, a sequence of waypoints is chosen in such a way that the segment joining two consecutive waypoints remain fully inside the control corridor. Then a predictive control law is designed to control the system from a waypoint to the next until the specified terminal position. An application dealing with orbital control illustrates the method and reveals its potentials for handling the control of complex systems even in case of unknown measurement noise.

Acknowledgement

This research was conducted in the Center for Aerospace Sciences and Technology at the University of Beira Interior, Covilhã, Portugal, and supported by the Portuguese Foundation for Sciences and Technology.

References

[1] Aubin J. P. *Viability Theory*. Birkhäuser, Boston, 1991.

[2] Battin R. H. *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Book series, New York, NY, 1987.

[3] Bousson K. Automatic guidance of aircraft for collision avoidance in terminal areas. *Russian-American Journal of Actual Problems of Aviation and Aerospace Systems,* Vol. 6, 2(12), pp. 49-57, 2001.

[4] Howe R. M. A new family of real-time predictor-corrector integration algorithms. *Simulation,* Vol. 57, No. 3, pp. 177-186, 1991.

[5] Lee S., Cochran Jr J. E. Orbital maneuvers via feedback linearization and bang-bang control. *Journal of Guidance, Control, and Dynamics,* Vol. 20, No. 1, pp. 104-110, 1997.