AIRCRAFT EQUIVALENT VULNERABLE AREA CALCULATION METHODS

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Abstract

Based on mathematical expectation theory, a method has been deduced to calculate the equivalent singly vulnerable area of aircraft by one threat hit. By simulating the kill event of multiple vulnerable components to 'Model of Filling Boxes with Balls', the expected number of hits required to kill an aircraft has been given through 'inclusion-exclusion principle' in discrete mathematics. The equivalent singly vulnerable area thus can be attained.

The concept of equivalent vulnerable area solves the problem of considering the effect of vulnerable components on the total aircraft vulnerable area by one threat hit and may provide valuable advices on whether the redundancy design technique is adopted or how to determine the number of redundant components in the aircraft conceptual design.

1 Introduction

Aircraft combat survivability (ACS) [1] is defined as the capability of an aircraft to avoid or withstand a man-made hostile environment. Survivability is composed of two focus areas: 1) Susceptibility 2) Vulnerability. For threats those that must hit the aircraft to kill it, the probability of kill of the aircraft P_K (the aircraft's killability) is the product of the probability of hit (the aircraft's susceptibility) P_H and the conditional probability of kill given a hit (the aircraft's vulnerability) $P_{K/H}$. Thus,

$$P_K=P_HP_{K/H} \tag{1}$$

The vulnerability of the aircraft (for a particular threat aspect) is usually expressed as the probability the aircraft is killed given a random (uniformly distributed) hit anywhere on the presented area of the aircraft $P_{K/H}$, or the single-hit vulnerable area of the aircraft A_V.

Vulnerable areas provide a basis for comparing the contribution of different components to aircraft vulnerability and are therefore useful in aircraft design. Knowledge of the most vulnerable components can be assistance in providing the modification advices as the redundancy design technique for example. If the aircraft is subjected to a single random hit, then the total vulnerable area can be obtained by simply summing the singly component vulnerable areas given by [1,2]

$$A_V=\sum_{i=1}^{m} A_{vi} \tag{2}$$

where:

A_{vi} = vulnerable area of the ith singly vulnerable critical component given a hit on the component

m = total number of singly vulnerable critical components, each capable of producing a specified kill level of aircraft [2]

Care must be exercised to identify all critical components, and whether they are multiply or singly vulnerable. All critical components have some level of vulnerability, level of redundancy, if any.

If it is assumed a single hit can damage, at the most, one component, then the first hit upon a multiply vulnerable component aircraft can not kill the aircraft by defeating one of the multiply vulnerable components since the lost function of the killed component can be compensated by another component of the set of components. Hence, Eq.(2) cannot include the contribution of multiple vulnerable components to the total aircraft vulnerable area. Thus, the first hit is not a reliable criterion as to the vulnerability of the aircraft. It is for this reason that an 'equivalent' vulnerable area concept [2] based on the expected number of hits *E(X)* required to kill an aircraft has been devised for considering the effect multiply vulnerable components on the vulnerability of aircraft. But, to our best knowledge, there is no public literature published to give the derivation of the equivalent formula of reference [2]. Without clearly understanding the mathematical formula derivation and all the assumptions involved, it is very hard to put it into reasonable application.

In this paper, we has deduced another equivalent vulnerable area formula based on "Model of Filling Boxes with Balls", which is comparable to the formula in reference [2] in computation results. The detailed derivation gives a full understanding of the kill event of aircraft with multiply vulnerable components.

In the rest of the paper, we first discuss the equivalent vulnerable area method in reference [2]. Then a detailed derivation of our proposed formula is presented. Following is an example to demonstrate the comparison of the two equivalent vulnerable area formulas. Conclusions and recommendations are given in the final section of this paper.

2 Aircraft Equivalent Singly Vulnerable Area

The concept of equivalent singly vulnerable area is applicable only to impacting rounds, and sequential compound damage is excluded. A large number of hits are assumed and respective locative locations of the various hits on the target are assumed to be taken from a uniform population.

The equivalent singly vulnerable area A_{VE} for an aircraft consisting of one or more singly vulnerable components and one set of identical multiply vulnerable components is given by [2]

$$A_{VE}=A_V/E(X) \tag{3}$$

where:

A_{VE} = equivalent singly vulnerable area

$A_V = A_{V0}+nA_{VI}$

A_{V0} = summed singly vulnerable area of the aircraft given by Eq.(2)

n = number of identical components constituting the set of multiply vulnerable components

A_{VI} = vulnerable area of each multiply vulnerable component obtained as though the item were a singly vulnerable component

$E(X)$ = expected number of hits on A_V required to kill the aircraft for the number of hits x

$$E(X)=\begin{array}{l} 1+\dfrac{n}{(\frac{n}{\eta}-1)}+\dfrac{n(n-1)}{(\frac{n}{\eta}-1)(\frac{n}{\eta}-2)}+\ldots \\ +\dfrac{n(n-1)\ldots(n-k+2)}{(\frac{n}{\eta}-1)(\frac{n}{\eta}-2)\ldots(\frac{n}{\eta}-k+1)} \end{array} \tag{4}$$

where:

k = number of items in the multiply vulnerable set which must be defeated to result in the specified level of aircraft kill

η = fraction of the summed vulnerable area represented by the set of multiply vulnerable components

$$\eta=(n\,A_{VI})/A_V \tag{5}$$

The quantity η can also be interpreted as the fraction of the lethal hits on the summed vulnerable area A_V that comprises lethal hits on the set of multiply vulnerable components.

3 Derivation of Another Equivalent Vulnerable Area Calculation Method

Based on mathematical expectation theory, by simulating the kill event of identical multiply vulnerable components to 'Model of Filling Boxes with Balls' as is shown in Fig.1, the expected number of hits required to kill an aircraft has been deduced through 'inclusion-exclusion principle' in discrete mathematics. The derivation is on the assumptions that:

(i) Any one threat hit is taken from a uniform distribution.

(ii) The component when hit has only two states, namely kill or no kill.

(iii) The redundancy aircraft has only one set of multiple vulnerable components and the redundancy is achieved through the use of similar components in which each performs identical functions and each has the same 'vulnerable area'.

(iv) Equal or more than k (k>=2) boxes having ball (balls) in the n boxes will result in the kill of aircraft.

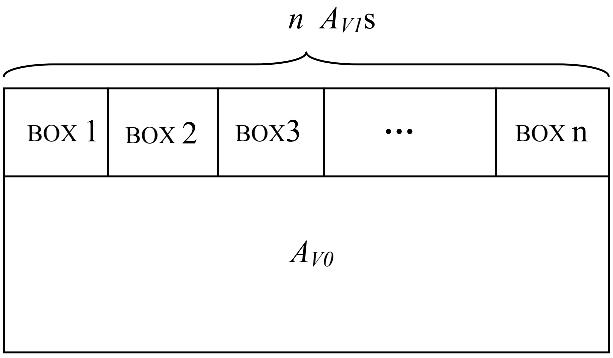

Fig. 1. Model of filling boxes with balls

Let P_i^r be the property that the ith box is empty when r balls are put into the n boxes randomly. Let A_i^r be the subset containing the elements that have property P_i^r in the universal set Ω, written as

$$A_i^r = \{ W \in \Omega : W \text{ has property } P_i^r \} \quad (6)$$

Thus,

$$\left|A_i^r\right| = (n-1)^r, \forall i=1,2,3,\ldots n \quad (7)$$

$$\left|A_{i_1}^r A_{i_2}^r\right| = (n-2)^r, \ \forall\ 1 \le i_1 < i_2 \le n \quad (8)$$

$$\ldots$$

$$\left|A_{i_1}^r A_{i_2}^r \ldots A_{i_j}^r\right| = (n-j)^r,$$

$$\forall\ 1 \le i_1 < i_2 < \ldots < i_j \le n \ \ , j=1,2,\ldots n \quad (9)$$

$$\ldots$$

$$\left|A_{i_1}^r A_{i_2}^r \ldots A_{i_{n-1}}^r\right| = [n-(n-1)]^r = 1,$$

$$\forall\ 1 \le i_1 < \ldots . < i_{n-1} \le n \quad (10)$$

$$\left|A_{i_1}^r A_{i_2}^r \ldots A_{i_n}^r\right| = 0, i_1 = 1, i_2 = 2, \ldots i_n = n \quad (11)$$

In the above equations, $|A|$ denotes the number of combinations of the elements of A, namely, cardinality of A.

Let

$$W_0^r = |\Omega| = n^r \quad (12)$$

$$W_1^r = \sum_{i=1}^{n} |A_i| = C_n^1 (n-1)^r \quad (13)$$

$$W_2^r = \sum_{i \le i_1 < i_2 \le n} \left|A_{i_1}^r A_{i_2}^r\right| = C_n^2 (n-2)^2 \quad (14)$$

$$\ldots$$

$$W_j^r = \sum_{1 \le i_1 < \ldots < i_j \le n} \left|A_{i_1}^r A_{i_2}^r \ldots A_{i_j}^r\right| = C_n^j (n-j)^r, j=1,2,\ldots n \quad (15)$$

$$\ldots$$

$$W_n^r = \left|A_1^r A_2^r \ldots A_n^r\right| = 0 \quad (16)$$

where,

$$C_n^j = \binom{n}{j} = \frac{n!}{(n-j)!\,j!} \quad (17)$$

Let $N_{(m)}^r$ be the number of elements that has at least m properties of the properties $P_1^r, P_2^r, \ldots, P_n^r$ in universal set Ω and $N_{[m]}^r$ be the number of elements that has neither more nor less than m properties of the properties $P_1^r, P_2^r, \ldots, P_n^r$ in universal set Ω. For example,

$$N_{(1)}^r = \left|A_1^r \cup A_2^r \cup \ldots \cup A_n^r\right| \quad (18)$$

$$N_{[0]}^r = \left|\overline{A_1^r}\,\overline{A_2^r} \ldots \overline{A_n^r}\right| \quad (19)$$

Based on "inclusion-exclusion principle" [3,4] in discrete mathematics, we have

$$N_{(1)}^r = \sum_{i=1}^{n} (-1)^{i-1} W_i^r \quad (20)$$

$$N_{[0]}^{r} = \sum_{i=0}^{n} (-1)^{i} W_{i}^{r} \tag{21}$$

$$N_{[m]}^{r} = \sum_{i=0}^{n-m} (-1)^{i} C_{m+i}^{m} W_{m+i}^{r} \tag{22}$$

$$N_{(m)}^{r} = \sum_{i=0}^{n-m} (-1)^{i} C_{m-1+i}^{m-1} W_{m+i}^{r} \tag{23}$$

According to the assumption（iv）and the kill process of aircraft with multiply components that aircraft will be killed when one of the singly-hit components is killed or equal k or more than k components are killed in the set of n multiply components, we have

if $1 \le x \le k-1$, then

$$P(X=x) = \eta^{x-1}(1-\eta) \tag{24}$$

if $x \ge k$, then

$P(X=x) =$

$$\eta^{x-1} \frac{N_{(n-(k-1))}^{x-1}}{n^{x-1}}(1-\eta) + \eta^{x-1} \frac{N_{[n-(k-1)]}^{(x-1)}}{n^{x-1}} \eta \frac{(n-k+1)}{n} \tag{25}$$

where $P(X=x)$ denotes the kill probability for the random variable $X= x$, $x=1,2\ldots,+\infty$.

The expected value $E(X)$, also known as the mean and the expectation, which is a weighted average of the random variable x, and the weights are the probabilities, is given such that

$E(X)=1\times P(x=1)+ 2\times P(x=2)+ 3\times P(x=3)+\ldots$

$$= \sum_{x=1}^{+\infty} x P(X=x) \tag{26}$$

Substituting Eqs.(24) and (25) into Eq.(26) gives

$$E(X) = \sum_{x=1}^{k-1} x P(X=x)|_{1 \le x \le k-1} + \sum_{x=k}^{+\infty} x P(X=x)|_{x \ge k} \tag{27}$$

According to Eqs. (12) through (23), simplifying expression (27) gives

$$E(X) = 1 + \eta + \eta^2 + \ldots + \eta^{k-2} + (1-k)\eta^{k-1} + \sum_{i=0}^{k-2} (-1)^{i} C_{n-k+i}^{n-k} C_{n}^{n-k+1+i} \frac{k\eta_i^{k-1} - (k-1)\eta_i^{k}}{1-\eta_i} \tag{28}$$

where,

$$\eta_i = \frac{k-i-1}{n}\eta \tag{29}$$

As a special case, the ‘equivalent’ vulnerable area reduces to the sum of the component vulnerable areas for an aircraft consisting only of singly vulnerable components. Thus, for the case where n=1 and k=1 referring to singly vulnerable component aircraft,

$$E(X)=1 \tag{30}$$

Summarizing, formulas (28) through (30) are our proposed formulas to calculate the expected number of hits. Substituting them into Eq. (3) can give the equivalent singly vulnerable area of aircraft with one set of multiply redundant components.

4 Example

A sample calculation of A_{VE} is illustrated as follows [2].

Given a single-place twin-engine fighter in which the engines are considered to be the only one set of multiply redundant components and both engines must be killed to result in a kill of the aircraft.

A_{V1}= 10ft^2 =singly vulnerable area of either engine

n= 2 = number of redundant components

k= 2= number of redundant components which must be killed to result in a kill of the aircraft

A_{V0}= 40ft^2 =total vulnerable area for a singly-vulnerable component

Eqs. (4) and (28) all give

$E(X)$=1.4=expected number of hits on A_V required to kill the aircraft

and

A_{VE}=42.85=Equivalent singly vulnerable area

Thus, the contributing of multiply components to the total vulnerable area of aircraft is

[(42.85-40.0)/42.85]*100%=6.65%.

Reference [2] has pointed out that the equivalent singly vulnerable area A_{VE} differs only slightly from the sum of the singly vulnerable component vulnerable area A_{V0} when the multiply vulnerable components are small. Hence, in practice and depending on the objectives of analysis, it is frequently possible to ignore all or all but the most significant multiply vulnerable components in the aircraft vulnerability assessment.

If the vulnerability of the set of multiply vulnerable components is ignored, the expected number of hits required to score a kill on the target is

$$E(X)=1/(1-\eta) \tag{31}$$

Using Eqs. (4) and (28), the quality $1/(1-\eta)$ and *E(X)* are plotted for various combinations of *n* and *k*, as functions of η. We found that the two equations can give the same curves. For convenience, the reciprocals of *E(X)* and $1/(1-\eta)$ is plotted against η in Fig.2. It can be seen that the difference in their values varies. A basis for deciding whether or not to include the contribution of a set of multiply vulnerable components to total aircraft vulnerable area is according to the approaching extent of its 1/*E(X)*curve to the curve ⑥. In those cases where the difference is acceptably small, the contribution of the multiply vulnerable set to total aircraft vulnerability may be ignored and thus the number of the set of redundancy components can also be determined in Fig.2.

5 Conclusions and Recommendations

- Eqs. (28) through (30) are our proposed formulas to calculate the expected number of hits required to kill the aircraft.
- The concept of equivalent singly vulnerable area should be used cautiously, especially when the *n* multiply vulnerable components do not have the same vulnerable area. In this case, the Eqs (4), (28) though (30) are

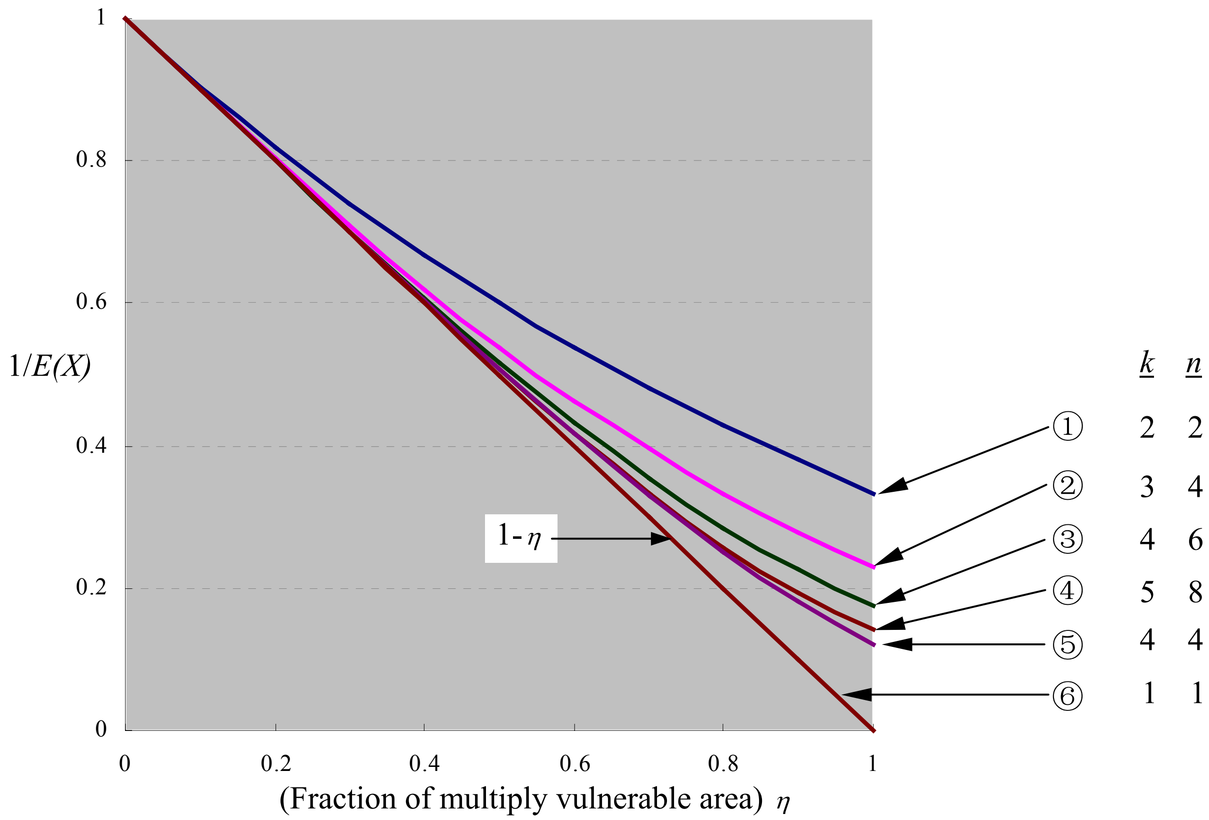

Fig.2. Relationship between *E(X)* and η for various values of *n* and *k*

not rigorously valid.

- When multiply vulnerable components do not have the same vulnerable area, the concept of equivalent singly vulnerable area is also useful, and new equation can be deduced through the abovementioned 'Model of Filling Boxes with Balls' by considering the different areas of each multiply component.
- The concept of an equivalent singly vulnerable (A_{VE}) can be generalized to apply to aircraft having more than one set of multiply vulnerable components [2].
- A number of challenges exist which could allow model including the effect of sequential compound damage in component. Another topic of interest in the equivalent vulnerable area calculation methods is the consideration of the case where overlap area among components exists in a given threat aspect.

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