WELL-CONDITIONED PROCEDURES FOR OPTIMAL ROBUST DESIGN OF FLIGHT CONTROL SYSTEMS

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Abstract

The aim of the paper is to present alternative well-conditioned procedures arising in robust design of flight control systems with respect to unstructured modeling uncertainties and in other applications in which the attenuation of exogenous disturbances is required.These techniques are based on transforming the design problems considered in one and two-block Nehari problems which optimal solutions allow to improve the robustness and disturbance attenuation performances.Unlike the suboptimal H^∞ methods which are frequently used in such applications,the optimal solutions given in this paper based on explicit formulae deduced via singular perturbations theory,are well conditioned near the optimum.Study cases referring to weighted robustness performances with respect to additive,multiplicative and stable-factor uncertainties and the design of a stability augmentation system(SAS) for the longitudinal dynamics of an aircraft by solving a model-matching problem illustrate the described optimal design procedures.

1 Introduction

There are many applications in which the design of a flight control system leads to H^∞ problems; we shall recall here some typical cases like the robust design with respect to different classes of unstructured modeling uncertainties[6],[11],[13],the reduction of the sensitivity together with its complement (the mixed-sensitivity problem)[17],[18] and model-matching problems in which increased maneuverability of the aircraft is required[21].In most applications such problems are solved using the H^∞-techniques based on the results of Doyle and *al.*[1] and the subsequent developments[8],[20].Given a standard system T with two inputs u_1,u_2 and two outputs y_1,y_2,and $\gamma > 0$,these methods give necessary and sufficient conditions as well as explicit formulae to determine a controller K (or a family of parameterized controllers) such that the resulting system $T_{y_1u_1}$,obtained by coupling K to T with $u_2 = Ky_2$,is stable and has the H^∞ norm less than γ.

In order to improve the performances of the resulting system one tends to reduce γ until one of the necessary and sufficient conditions fails;but when approaching the minimum of γ,the explicit formulae invoked above often leads to ill-conditioned computations[10],therefore in such applications only a suboptimal level of attenuation may be obtained.

The aim of the present paper is to describe well-conditioned procedures for some of the typical cases enumerated at the beginning of this section.These applications may be solved by transforming them in one and two-block Nehari problems for which we used the explicit formulae to the optimal solutions derived in Ref.[3] and [5] respectively,using the theory of singular perturbations.

The paper is organized as follows:the next section contains the statement of the optimal weighted robustness problem with respect to additive,multiplicative and stable-factor modeling uncertainties and the procedure to transform them in one-block Nehari problems.We also consider in Section 2 a model matching problem for the short period dynamics of an aircraft and we reduce it to a two-block Nehari problem.In Section 3 we present the well-conditioned procedures to compute the optimal solutions to the one and two-block Nehari problems,respectively.

Numerical cases illustrating the proposed optimal procedures are described in Section 4 in which we considered the weighted robustness problem with respect to additive,multiplicative uncertainties and with respect to normalized coprime factorization for the longitudinal dynamics of an aircraft.We also present in Section 4 the numerical results corresponding to the model-matching problem in which we determine a longitudinal stability augmentation system (SAS) in order to improve the maneuverability and the robustness performances together with the control limitation.

2 Optimal flight control applications; reduction to Nehari-type problems

2.1 Preliminaries

We shall consider the case when the aircraft dynamics is described by using a linearized model corresponding to a certain nominal flight condition within the flight envelope.Let denote by $G(s)$ the matrix function of this nominal model and let (A, B, C, D) a minimal state-space realization of it.Then,according to Ref.16,a double coprime factorization $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ with $\tilde{M}V - \tilde{N}U = I; \tilde{V}M - \tilde{U}N = I$; $M, N, U, V, \tilde{M}, \tilde{N}, \tilde{U}, \tilde{V} \in RH^{\infty}$ (RH^{∞} denoting the set of all proper stable transfer matrices) is given by:

$$\begin{aligned} M(s) &:= (A + BF, B, F, I) \\ \tilde{M}(s) &:= (A + HC, H, C, I) \\ N(s) &:= (A + BF, B, C + DF, D) \\ \tilde{N}(s) &:= (A + HC, -(B + HD), -C, D) \\ U(s) &:= (A + BF, -H, F, 0) \\ \tilde{U}(s) &:= (A + HC, H, -F, 0) \\ V(s) &:= (A + BF, -H, C + DF, I) \\ \tilde{V}(s) &:= (A + HC, -(B + HD), F, I) \end{aligned} \tag{1}$$

where F and H are any matrices of appropriate dimensions for which $A + BF$ and $A + HC$ are hurwitz.Moreover,the set of all controllers $K(s)$ stabilizing $G(s)$ has the following parameterization in terms of the coprime factors above[24]:

$$K = (U + ML)(V + NL)^{-1} = (\tilde{V} + L\tilde{N})^{-1}(\tilde{U} + L\tilde{M}) \tag{2}$$

with respect to the stable system parameter L.

The following three subsections are devoted to the optimal robust design of flight controllers with respect to additive,multiplicative and coprime-factorization uncertainties,respectively.

2.2 Weighted robustness with respect to additive uncertainties

The robust design with respect to additive uncertainties of a flight control system is considered in applications in which the aircraft dynamics is approximated by a nominal model G plus an unknown bounded stable uncertainty[13],[23].Therefore it consists in finding a controller K which stabilizes all perturbed systems $G + \Delta$ where $\Delta \in RH^{\infty}$ and $\| \Delta \|_{\infty} \leq \gamma$ for $\gamma > 0$ as large as possible.From the Small Gain Theorem[25] it follows that K is a solution to this problem if it stabilizes G and $\| K(I - GK)^{-1} \|_{\infty} < \gamma^{-1}$,therefore a maximal robustness radius may be obtained by solving the optimal problem $\min_K \| K(I - GK)^{-1} \|_{\infty}$ with K stabilizing.If we are interested in good robustness properties over a specified range of frequencies of the modeling uncertainties Δ then we shall consider the optimal weighted robustness problem $\min_K \| WK(I - GK)^{-1} \|_{\infty}$ where W denotes a square stable weighting matrix,adequately chosen.Although the robustness problem usually appears in flight control applications together with other design specifications,this problem itself is useful in practical situations when one analyses the largest(in norm) admissible unmodelled dynamics for which a stabilizing controller may be still determined.Another important practical significance of this problem consists in minimizing the influence of output disturbances over controls for a given frequency range of disturbances.

Assume that G has no poles on the extended $j\omega$-axis and select W such that it has no transmission zeros on this axis;then we may perform the inner-outer factorizations $M = M_i M_o$ and $WM = P_i P_o$ and the outer-inner factorization $\tilde{M} = \tilde{M}_o \tilde{M}_i$.Using the coprime factorization given in the previous subsection and the representation (2) for all stabilizing controllers,direct calculations give:

$$\| WK(I - GK)^{-1} \|_{\infty} = \| P_i^* WU\tilde{M}_o + \tilde{L}_a \|_{\infty} \tag{3}$$

where $*$ denotes the adjoint operator and $\tilde{L}_a := M_o L \tilde{M}_o \in RH^{\infty}$(for computational details see also Ref.[23]).When performing the decomposition $P_i^* WU\tilde{M}_o = P_+ + P_-$ with P_+ antistable and P_- stable we obtain that $\| WK(I - GK)^{-1} \|_{\infty} = \| P_+ + \hat{L}_a \|_{\infty}$where $\hat{L}_a := \tilde{L}_a + P_- \in RH^{\infty}$;therefore the weighted robust design problem with respect to additive uncertainties leads to the optimal one-block Nehari problem:

$$\min_{\hat{L}_a \in RH^{\infty}} \| P_+ + \hat{L}_a \|_{\infty} \quad ; \quad P_+ \text{ antistable} \tag{4}$$

An alternative solution to this problem has been deduced in Ref.[5] via singular perturbations theory and corresponding explicit formulae are given in Section 3.Then an optimal controller K may be obtained from (2) with $L = M_o^{-1}(\hat{L}_a - P_-)\tilde{M}_o^{-1} \in RH^{\infty}$.

2.3 Weighted robustness with respect to multiplicative uncertainties

An important objective in the design of a flight control system is to ensure good robustness properties with respect to multiplicative uncertainties[13],[17],[18], namely if G denotes the transfer matrix of the aircraft dynamics at certain flight conditions then the flight control system K stabilizes $(I + \Delta)G$ for all stable Δ with $\| \Delta \|_{\infty} \leq \gamma$.According to a well-known result[2], if K is stabilizing for G and $\| GK(I - GK)^{-1} \|_{\infty} < \gamma^{-1}$ then K is a solution to this problem; therefore the greatest robustness radius with respect to multiplicative uncertainties implies to solve the optimal problem $\min_K \| GK(I - GK)^{-1} \|_{\infty}$with K stabilizing G.For a specified range of frequencies of uncertain Δ the problem may be restated as $\min_K \| WGK(I - GK)^{-1} \|_{\infty}$with K stabilizing and W a square stable weighting matrix.Even in the suboptimal case the unweighted robustness problem ($W = I$) with respect to multiplicative perturbations leads to a singular H^{∞} problem(with $D_{12} = 0$)[4],[22].Applications of the suboptimal singular case in flight control systems design may be found in Ref.[4] but there are many practical situations in which this singularity may be avoided. We consider in this paper the case in which the nominal aircraft dynamics $G = NM^{-1}$ satisfies the following conditions: G has no poles on the extended $j\omega$-axis and WN admits an inner-outer factorization Q_iQ_o (this is possible if W is stable and WN is full column rank at all points on the extended $j\omega$ -axis). Under these assumptions which are valid in many aircraft applications, using the parameterization (2) of all stabilizing controllers K and the coprime factorizations from Section 2.1, one obtains:

$$\| WGK(I-GK)^{-1} \|_\infty = \| Q_o\tilde{U}\tilde{M}_i^* + \tilde{L}_m \|_\infty \quad (5)$$

where $\tilde{M}_o, \tilde{M}_i$ are given by the outer-inner factorization $\tilde{M} = \tilde{M}_o\tilde{M}_i$ and $\tilde{L}_m := Q_oL\tilde{M}_o \in RH^\infty$. It follows that when performing the decomposition $Q_o\tilde{U}\tilde{M}_i^* = Q_+ + Q_-$ with Q_+ antistable and Q_- stable, one obtains $\| WGK(I-GK)^{-1} \|_\infty = \| Q_+ + \hat{L}_m \|_\infty$ where $\hat{L}_m = \tilde{L}_m + Q_- \in RH^\infty$. Hence, under the assumptions mentioned above we reduced the weighted robustness problem with respect to multiplicative uncertainties to the one-block Nehari problem:

$$\min_{\hat{L}\in RH^\infty} \| Q_+ + \hat{L}_m \|_\infty \quad ; \quad Q_+ \text{ antistable} \quad (6)$$

The solution $\hat{L}_m$ to this problem allows to determine the controller K, optimally robust with respect to multiplicative perturbations, using (2) with $L = Q_o^{-1}(\hat{L}_m - Q_-)\tilde{M}_o^{-1} \in RH^\infty$.

Remark 1

In the particular case when the nominal system G considered in the previous two subsections is stable then obviously the robustness radius with respect to additive and multiplicative perturbations is infinite since $K = 0$ stabilizes all $G+\Delta$ and $(I+\Delta)G, \Delta \in RH^\infty$. Therefore in the numerical case described in Section 4 we shall consider only unstable dynamics.

2.4 Optimal robustness with respect to normalized left coprime factorization

There are many applications of flight control systems design in which the normalized left coprime factorization is involved[6],[11][13]. This problem is important not only for exclusive robustness reasons but it is also strongly related to the loop-shaping design which is frequently used in the design of flight control systems[6],[13]. Given the normalized left coprime factorization of a nominal dynamics $G = \tilde{M}^{-1}\tilde{N}$; $\tilde{M}, \tilde{N} \in RH^\infty$; $\tilde{M}^*\tilde{M} + \tilde{N}^*\tilde{N} = I$, the robustness problem with respect to this factorization requires to determine a controller K which stabilizes all perturbed systems $(\tilde{M} + \Delta_{\tilde{M}})^{-1}(\tilde{N} + \Delta_{\tilde{N}})$ where $\Delta_{\tilde{M}}, \Delta_{\tilde{N}} \in RH^\infty$ and $\| \Delta_{\tilde{M}} \quad \Delta_{\tilde{N}} \|_\infty \leq \gamma$. If (A, B, C, D) is a minimal realization of G then state-space realizations of the factors $\tilde{M}$ and $\tilde{N}$ are given by $\tilde{M}(s) := (A + HC, H, R^{-\frac{1}{2}}C, R^{-\frac{1}{2}})$ and $\tilde{N}(s) := (A + HC, B + HD, R^{-\frac{1}{2}}C, R^{-\frac{1}{2}}D)$ respectively, where $H := -(YC^T + BD^T)R^{-1}; R := I + DD^T$ and X and Y denote the stabilizing positive definite solutions to the standard Riccati equations[13]:

$$\tilde{A}^TX + X\tilde{A} - XBS^{-1}B^TX + C^TR^{-1}C = 0$$
$$\tilde{A}Y + Y\tilde{A}^T - YC^TR^{-1}CY + BS^{-1}B^T = 0$$

where $\tilde{A} := A - BS^{-1}D^TC$ and $S := I + DD^T$. It is also proved in Ref.[13] that an optimal controller with respect to normalized left coprime factorization is given by $K_o = PQ^{-1}$ with $P, Q \in RH^\infty$ satisfying the minimization condition:

$$\min_{P,Q\in RH^\infty} \left\| \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + \begin{bmatrix} P \\ Q \end{bmatrix} \right\|_\infty \quad (7)$$

and this minimum equals $\sigma_1(1+\sigma_1^2)^{-\frac{1}{2}}$ where σ_1^2 denotes the largest eigenvalue of XY. Since $\tilde{M}^*$ and $\tilde{N}^*$ are antistable, (7) defines a one-block Nehari problem which solution $\begin{bmatrix} P \\ Q \end{bmatrix}$ allows to determine an optimal robust controller $K_o = PQ^{-1}$ with respect to left coprime factorization. Explicit formulae for a realization of K_o are given in Ref.[5]; these formulae are based on balanced realizations in the sense of Jonckheere-Silverman for which X and Y are diagonal and equal.

2.5 Optimal design of a SAS using a model matching approach

Consider the short-period motion of an aircraft described by the second order system:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha & 1 \\ M_\alpha & M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_{\delta_e} \\ M_{\delta_e} \end{bmatrix} \delta_e \quad (8)$$

with the states α-the angle of attack, q -the pitch rate, the control δ_e-elevator deflection, Z_α, M_α, M_q and $Z_{\delta_e}, M_{\delta_e}$ denoting the longitudinal stability and control derivatives, respectively.

The handling qualities of the aircraft require to design a controller (the stability augmentation system-SAS), such that the closed loop system has the poles in a prescribed domain from $\mathbf{C}^-$ (Ref.[15]). We shall choose an "ideal model" which is in fact a second order system satisfying the pole placement requirements:

$$H_m(s) = \frac{\omega_m^2}{s^2 + 2\xi_m\omega_m s + \omega_m^2} \quad (9)$$

The main objectives of our design is to determine a stabilizing controller K such that:

· the difference between the angle of attack in the short period and the output of the ideal model is strongly attenuated over a specified range of frequencies for α_{ref} ;

· the control is limited in order to avoid the actuator saturation;

· the influence of sensor noise over the tracking error and over control is minimized.

Taking into account these objectives, the following model-matching configuration has been considered:

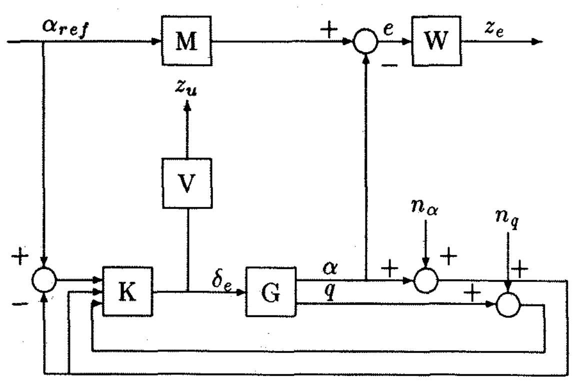

Figure 1: The model-matching configuration

where G denotes the transfer function of the short period, M is the ideal model and V and W are weighting functions penalizing the control and the tracking error, respectively.

In the configuration above a certain redundancy of the measured outputs may be remarked but the numerical results described in Section 4 will show how this redundancy can be removed.

The structure in Figure 1 is equivalent with the γ -attenuation problem illustrated in Figure 2 where $u_1 = \left[\begin{array}{ccc} \alpha_{ref} & n_\alpha & n_q \end{array} \right]^T ; u_2 = \delta_e ; y_1 = \left[\begin{array}{cc} z_e & z_u \end{array} \right]^T ;$ $y_2 = \left[\begin{array}{ccc} \alpha_{ref} - \alpha - n_\alpha & \alpha + n_\alpha & q + n_q \end{array} \right]^T$ for which the standard system will be determined in the next section.

When denoting by $T_{y_1 u_1}(s)$ the transfer matrix from u_1 to y_1 in Figure 2, it follows that the goal of our design is to determine a stabilizing controller K such that $\|T_{y_1 u_1}\|_\infty$ is minimal.

We shall describe in the following a procedure to reduce the model-matching problem to a two-block Nehari one in the usual assumption, available for most of aircrafts, that the short period dynamics is stable.

Let (A_g, B_g, C_g), (A_m, B_m, C_m), (A_w, B_w, C_w, D_w) denote minimal realizations of G, M and W, respectively. To avoid the increase of the order of the controller we chose a constant weight V.

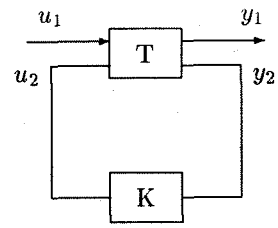

Figure 2: The equivalent γ-attenuation problem

From Figures 1 and 2 we obtain the following realization of the standard system T:

$$\begin{aligned} \dot{x} &= Ax + B_1 u_1 + B_2 u_2 \\ y_1 &= C_1 x + D_{12} u_2 \\ y_2 &= C_2 x + D_{21} u_1 \end{aligned} \qquad (10)$$

where $x := \left[\begin{array}{ccc} x_g^T & x_m^T & x_w^T \end{array} \right]^T$, x_g ,x_m ,x_w denoting the state vectors of G , M and W respectively, and:

$$A = \left[\begin{array}{ccc} A_g & 0 & 0 \\ 0 & A_m & 0 \\ -B_w C_g & B_w C_m & A_w \end{array} \right];$$

$$B_1 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ B_m & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]; B_2 = \left[\begin{array}{c} B_g \\ 0 \\ 0 \end{array} \right];$$

$$C_1 = \left[\begin{array}{ccc} -D_w C_g & D_w C_m & C_w \\ 0 & 0 & 0 \end{array} \right]; D_{12} = \left[\begin{array}{c} 0 \\ V \end{array} \right];$$

$$C_2 = \left[\begin{array}{cccc} [-1 \quad 0] & 0 & 0 & 0 \\ C_g & 0 & 0 & 0 \end{array} \right]; D_{21} = \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \qquad (11)$$

the zero-matrices having appropriate dimensions.Since A_g was assumed stable and weighting function $W(s)$ is chosen stable one can directly verify that the matrices in (11) satisfy the assumptions to a DF(disturbance feedforward) problem(that is:(i) D_{21} is nonsingular; (ii) $A - B_1 D_{21}^{-1} C_2$ is hurwitz;(iii) $\left[\begin{array}{cc} A - j\omega I & B_2 \\ C_1 & D_{12} \end{array} \right]$ is full column rank for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$);this problem may be transformed in an optimal two-block Nehari problem. Although the method to reduce a DF problem to a two-block Nehari one is known(see for example Ref.[12]) we briefly describe for completeness in Appendix the steps of this procedure. In order to simplify the calculations we considered the case when $D_{21} = I$ and D_{12} is inner; let remark that the more general situation for a DF problem with D_{21} nonsingular, may be transformed in a problem with $D_{21} = I$ by simply scaling the measured output y_2 by D_{21}^{-1}.On the other hand the restriction D_{12} inner may be accomplished even if D_{12} is full column rank by using for example the technique given in Ref.[19] based on the singular value decomposition.We conclude that the design of the SAS using the matching-model approach described above may be transformed in an optimal two-block Nehari problem.

3 Well-conditioned computation of the optimal solutions

We briefly present in this section two alternative procedures to compute optimal solutions to one and two-block Nehari problems which corresponding explicit formulae were deduced via singular perturbations theory in Ref.[3] and [5].

3.1 The one-block Nehari problem case

Consider the one-block Nehari problem for a given antistable system G,consisting in finding a stable system G_o such that $\|G - G_o\|_\infty$ is minimal.

An optimal solution to this problem,derived in Ref.[5] can be obtained by performing the following procedure:

1^{st}*Step* Compute a balanced realization (A, B, C, D) of G with respect to the corresponding Gramians:

$$AP + PA^T - BB^T = 0 \; ; \; QA + A^TQ - C^TC = 0$$

such that $P = Q = diag(\mu_1 I_1, \ldots, \mu_p I_p)$ where $\mu_1 >, \ldots, > \mu_p$ and I_k are $n_k \times n_k$ unit matrices,$k = 1, \ldots, p$.Such a realization can be obtained using the algorithm given by Glover[9];

2^{nd}*Step* Let denote:

$$M_{22} := diag(\mu_2 I_2, \ldots, \mu_p I_p)$$
$$W_{22} := diag\left\{\frac{\mu_i}{\mu_1^2 - \mu_i^2}\right\}_{i=2,\ldots,p}$$

and perform the partitions:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \; ; \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \; ; \; C = [C_1 \; C_2]$$

where A_{11} has the dimensions $n_1 \times n_1$;

3^{rd}*Step* If $B_1 B_1^T$ is nonsingular then an optimal solution to the one-block Nehari problem is given by $G_o(s) := (A_o, B_o, C_o, D_o)$ with:

$$\begin{aligned} A_o &= (A_{12}^T + W_{22}B_2B_1^T)(B_1B_1^T)^{-1}B_1B_2^T - A_{22} - \\ &\quad W_{22}B_2B_2^T \\ B_o &= -(A_{12}^T + W_{22}B_2B_1^T)(B_1B_1^T)^{-1} + W_{22}B_2 \\ C_o &= \mu_1 C_1(B_1B_1^T)^{-1}B_1B_2^T - C_2M_{22} \\ D_o &= -\mu_1 C_1(B_1B_1^T)^{-1}B_1 + D \end{aligned} \tag{12}$$

As it is shown in Ref.[5],in the case when $B_1B_1^T$ is singular,the method can be also applied by performing an orthogonal transformation U such that $UB_1B_1^TU^T = \begin{bmatrix} \hat{B}_1\hat{B}_1^T & 0 \\ 0 & 0 \end{bmatrix}$ with $\hat{B}_1\hat{B}_1^T$ nonsingular and similar formulae to (12) will be obtained;if $B_1 = 0$ the optimal solution to the one-block Nehari problem may be determined using the formulae:

$$\begin{aligned} A_o &= -\begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T + W_{22}B_2B_2^T \end{bmatrix} \\ B_o &= \begin{bmatrix} 0 \\ W_{22}B_2 \end{bmatrix}; \\ C_o &= -\begin{bmatrix} \mu_1 C_1 & C_2M_{22} \end{bmatrix}; D_o = D \end{aligned}$$

therefore,if A has the dimension $n \times n$ then the optimal solution to the one-block Nehari problem has the dimension $n - rank(B_1B_1^T)$.

3.2 The two-block Nehari problem case

We shall describe in the present section a procedure detailed in Ref.[3],to solve the optimal two-block Nehari problem:

$$\inf_{G \in RH^\infty} \left\| \begin{matrix} G_1(s) - G(s) \\ G_2(s) \end{matrix} \right\|_\infty := \gamma_0; G_1, G_2 \in RH_\perp^\infty$$

Usually in applications the suboptimal case $\gamma > \gamma_0$ is solved by using explicit formulae depending on γ. In Ref.[3] the following solution to the suboptimal Nehari problem corresponding to the realization $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}(s) := \left(A, B, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}\right)$is deduced:

$$\begin{aligned} G(s) := \Big(&-\left[A + W(\gamma)C_1^TC_1\right]^T, -QB + C_2^TD_2, \\ &C_1W(\gamma), D_1\Big) \end{aligned} \tag{13}$$

where $W(\gamma) := R(\gamma)\left[\gamma^2 I - QR(\gamma)\right]^{-1}$, Q and $R(\gamma)$ denoting the positive semi-definite solution to the Lyapunov equation:

$$A^TQ + QA - C_1^TC_1 - C_2^TC_2 = 0 \tag{14}$$

and the positive semidefinite stabilizing solution to the Riccati equation:

$$\begin{aligned} AR + RA^T - \left(RC_2^T - BD_2^T\right)\left(\gamma^2 I - D_2D_2^T\right)^{-1}(C_2R - \\ D_2B^T) - BB^T = 0 \end{aligned} \tag{15}$$

respectively.

On the other hand,a result in Ref.[3] states that the optimal Nehari distance γ_0, equals the unique solution to the transcendental equation:

$$\gamma^2 - \rho(QR(\gamma)) = 0 \tag{16}$$

(with ρ denoting the spectral radius), if the equation has a solution, and otherwise it equals $\|G_2\|_\infty$. It follows from this remark that in applications in which γ_0 equals the solution of (16), when applying the suboptimal formulae above with γ close to γ_0, ill-conditioned computations appear since the matrix $\gamma^2 I - QR(\gamma)$ in expression for $W(\gamma)$ tends to be singular. In order to avoid this ill conditioning in Ref.[3] a solution for the optimal problem has been determined via singular perturbations method. We present in the following the algorithm to compute this solution.

Stage1 Determine by using a γ-procedure the unique solution (if exists) of equation (16). Since the dependence $\gamma \to \rho(QR(\gamma))$ is monotonically decreasing [3], the solution of (16) may be easily determined with an imposed level of tolerance, using a γ-procedure starting with $\gamma = \|G_2\|_\infty$. If equation (16) has no solution then we shall use the suboptimal case formula (13) with $\gamma_0 = \|G_2\|_\infty$ since in this case no ill conditioned computations appear.

Remark 2

The following stages of the algorithm will be performed only in the case when γ_0 equals the solution of (16).

Stage 2 Determine a balanced realization of $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ respect to Q and $R(\gamma_0)$ that is:

$$Q(\gamma_0) = R(\gamma_0) = \begin{bmatrix} r_1(\gamma_0)I_1 & 0 \\ 0 & R_{22}(\gamma_0) \end{bmatrix} \qquad (17)$$

where $R_{22}(\gamma_0) = diag(r_2(\gamma_0)I_2, ..., r_p(\gamma_0)I_p)$; I_k are $n_k \times n_k$ unit matrices , $k = 2, ..., p$ and $r_1(\gamma_0) > ... > r_p(\gamma_0) > 0$. Such a balanced realization may be obtained using a similar procedure with the one given by Glover[9], referring to the balancing with respect to Gramians. Denote by $\left(\widetilde{A}, \widetilde{B}, \begin{bmatrix} \widetilde{C_1} \\ \widetilde{C_2} \end{bmatrix}, \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}\right)$ this balanced realization.

Stage 3 Perform the following partitions conformably with (17):

$$\widetilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \widetilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix};$$
$$\begin{bmatrix} \widetilde{C_1} \\ \widetilde{C_2} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Stage 4 Determine the optimal solution $G_0(s) := (A_0, B_0, C_0, D_0)$ to the two-block Nehari problem, where:

$$\begin{aligned} A_0 &= C_{12}^T C_{11} \left(C_{11}^T C_{11}\right)^{-1} \left[A_{21}^T + C_{11}^T C_{12} W_{22}(\gamma_0)\right] - \\ &\quad A_{22}^T - C_{12}^T C_{12} W_{22}(\gamma_0) \\ B_0 &= C_{12}^T C_{11} \left(C_{11}^T C_{11}\right)^{-1} \left(\gamma_0 B_1 - C_{12}^T D_2\right) - \\ &\quad R_{22}(\gamma_0) B_2 + C_{22}^T D_2 \\ C_0 &= -C_{11} \left(C_{11}^T C_{11}\right)^{-1} \left[A_{21}^T + C_{11}^T C_{12} W_{22}(\gamma_0)\right] + \\ &\quad C_{12} W_{22}(\gamma_0) \\ D_0 &= -C_{11} \left(C_{11}^T C_{11}\right)^{-1} \left(\gamma_0 B_1 - C_{12}^T D_2\right) + D_1 \end{aligned} \qquad (18)$$

with $W_{22}(\gamma_0) := R_{22}(\gamma_0)[\gamma_0^2 I - R_{22}^2(\gamma_0)]^{-1}$.
STOP

Remark 3

In the case when $C_{11}^T C_{11}$ is singular the method above may be also used by performing to $C_{11}^T C_{11}$ an orthogonal transformation which gives in (13) for $\gamma^2 = \gamma_0^2 + \epsilon$ a singularly perturbed system which fast component with the dimension equal to $rank(C_{11}^T C_{11})$ may be reduced according to the theory of singular perturbations[3].

In the singular case when $C_{11} = 0$ no ill conditioning will appear in (13).

We conclude that if n denotes the order of G then the optimal solution to the Nehari problem determined above has the dimension $n - rank(C_{11}^T C_{11})$.

4 Numerical results

We present in this section some numerical results of the problems formulated in Section 2,results obtained using the well-conditioned formulae given in Section 3.All calculations have been performed using the MATLAB package.

4.1 The robustness radii for the longitudinal dynamics of an aircraft

We considered the nominal model G of an four-engined,executive jet aircraft,linearized at the equilibrium conditions $V_0 = 236 m/\sec; h_0 = 12200 m; \alpha_0 = 4.2$ deg,having the state-space realization[14]:

$$A = \begin{bmatrix} -2.11e-6 & -4.30e-6 & 0 & -9.81 \\ -3.50e-2 & -6.65e-1 & 2.36e+2 & 0 \\ -1.40e-2 & -2.43e-2 & -7.42e-1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 7.74e-1 \\ -1.06e+1 \\ -6.77 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 5.94e-2 & -5.00e-1 & 5.02 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3.53e+1 \\ 0 \end{bmatrix}$$

where the state components are respectively the forward velocity [m/sec],the heave velocity[m/sec], the pitch rate[deg/sec] and the attitude[deg];the control input is the elevator deflection[deg] and the measured outputs are the normal acceleration[m/sec^2] and the pitch rate.The open-loop eigenvalues are: $-0.5963 \pm 2.4331i; 0.1262; -0145$ the third eigenvalue corresponding to an unstable phugoid mode.

a.The additive uncertainties case In order to obtain good robustness properties beyond $10rad/\sec$ we considered the weighting function $W(s) = \frac{s+1}{0.1s+1}$.Transforming the robustness problem in a one-block Nehari problem as it is shown in Section 2,the optimal solution (12) gives after reducing the uncontrollable and unobservable parts a fourth-order stabilizing controller which ensures the optimal robustness radius with respect to additive uncertainties $\gamma_o^{-1} = 38.3142$.Denoting by $Mmax$ the largest absolute value of all elements of matrices in the computed realization of this controller we obtained $Mmax$= 43.0200.The diagram of the maximum singular value of $K(I-GK)^{-1}$ is plotted in Figure 3,showing a strong attenuation for $\omega > 10rad/\sec$ conformably to the design objectives.

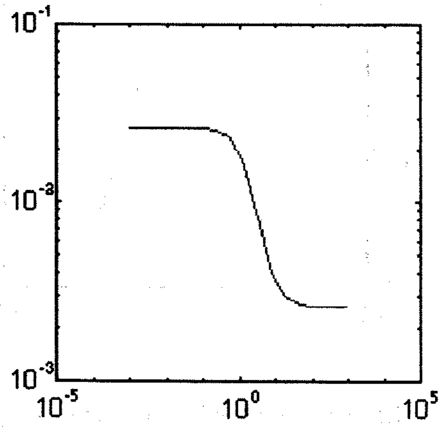

Fig.3 $\overline{\sigma}(K(I-GK)^{-1})$ V's frequency[rad/sec]

b.The multiplicative uncertainties case We considered for this numerical case the weighting function $W(s) = \frac{s+1}{0.001s+1}I_2$ and using the procedure described in Subsection 2.3,we obtained with formulae (12) ,(1) and (2) a stabilizing seventh-order controller which ensure the optimal robustness radius with respect to multiplicative uncertainties $\gamma_o^{-1} = 0.8882$.The parameter $Mmax$ defined above equals in this case 233.4224.In Figure 4 the diagram of the largest singular value of $GK(I-GK)^{-1}$ is given,showing very good robustness properties on the high frequency range.

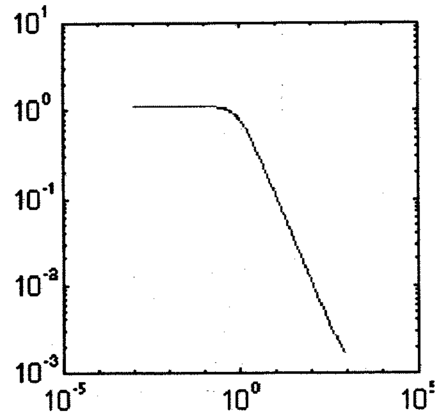

Fig.4 $\overline{\sigma}(GK(I-GK)^{-1})$ V's frequency[rad/sec]

c.The left-coprime factorization case For the nominal model considered we obtained,using formulae (1),(7) and (12) a third-order controller for which the maximal robustness radius with respect to left coprime factorization is attained.This radius equals 0.7371 and $Mmax$= 3.9213.

4.2 Numerical results for the SAS design

For the problem formulated in Section 2.5 we consider the F-8 aircraft at the nominal conditions: flight at 20000 ft,speed 620ft/s and the equilibrium angle of attack $\alpha_0 = 0.078rad$. The short period dynamics is given by (8) where[21]: $Z_\alpha = -0.84\sec^{-1}$, $M_\alpha = -4.8\sec^{-2}$, $M_q = -0.49\sec^{-1}$, $Z_{\delta_e} = -0.11\sec^{-1}$,$M_{\delta_e} = -8.7\sec^{-2}$.According to MIL requirements [15], we choose for the nominal flight conditions above an ideal model (9) with $\omega_m = 3rad/\sec$ and $\xi_m = 0.7$. Since the pilot commands α_{ref} are low frequency references we selected the weighting function for the tracking error $W(s) = \frac{s+300}{s+0.03}$. The weighting function on control is $V = 1$.With these numerical values we transformed first the matching problem in a DF one as it is shown in Appendix and then this problem was reduced to an optimal two-block Nehari problem.By solving the optimal two-block Nehari problem we obtained the stable controller $K(s) := (A_k, B_k, C_k, D_k)$ with $A_k = [A_{k_1} \quad A_{k_2} \quad A_{k_3}]$ where:

$$A_k = \begin{bmatrix} -0.25 & -2.14 & 5.88 \\ 0.01 & -2.82 & -2.24 \\ 0 & 3.26 & -0.51 \\ 0 & 0 & 3.86 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{k_2} = \begin{bmatrix} -11.45 & 0.17 \\ -1.75 & -0.96 \\ 2.54 & 1.58 \\ -5.41 & -0.99 \\ 1.47 & -2.63 \\ 0 & 0.91 \\ 0 & 0 \end{bmatrix}$$

$$A_{k_3} = \begin{bmatrix} -1.04 & 25.47 \\ -0.54 & 2283.01 \\ 2.48 & -1873.50 \\ -2.89 & -1393.50 \\ -5.13 & -894.81 \\ -2.55 & -103.40 \\ 2.95 & -24.59 \end{bmatrix}$$

$$B_k = \begin{bmatrix} 0.45 & 0.45 & 0 \\ -6.63 & -6.63 & 0 \\ 5.70 & 5.70 & 0 \\ 4.06 & 4.06 & 0 \\ 0.84 & 0.84 & 0 \\ -0.05 & -0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 337.15 \end{bmatrix}$$

$$D_k = \begin{bmatrix} -0.94 & -0.94 & 0 \end{bmatrix}$$

The numerical results above show that the control configuration in Figure 1 may be simplified by giving up to the incidence and pitch rate feedback and considering a single input to the controller, namely α_{ref}.It is obvious that in this case the noises n_α and n_q will not affect the controlled outputs z_e and z_u.

The tracking error e for $\alpha_{ref} = 0.2rad$ and the control δ_e responses are illustrated in Figure 5 and 6, respectively(the time axis is scaled in [sec]):

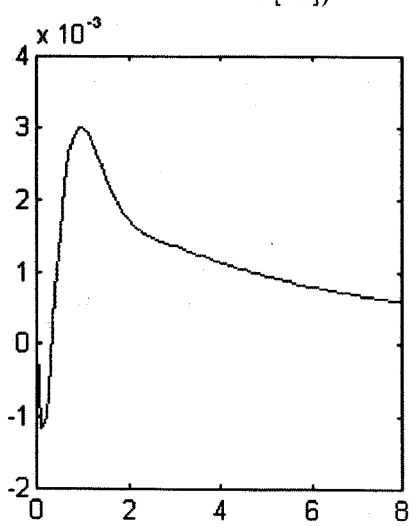

Fig.5 Error response[rad] to $\alpha_{ref} = 0.2rad$

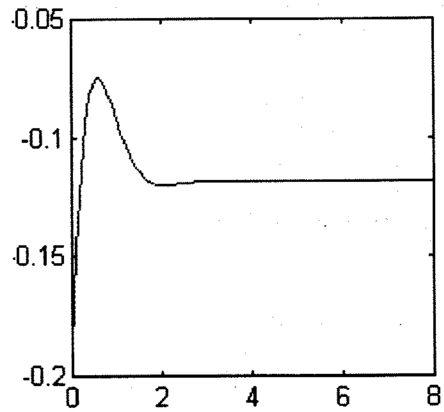

Fig.6 Control response[rad] to $\alpha_{ref} = 0.2rad$

showing that controller ensures a good tracking error (about 1.5% from the reference value) together with an acceptable maximum value of the control.

We also plotted the magnitude diagram for $\frac{e(s)}{\alpha_{ref}(s)}$ (Figure 7) and for $\frac{\delta_e(s)}{\alpha_{ref}(s)}$ (Figure 8)

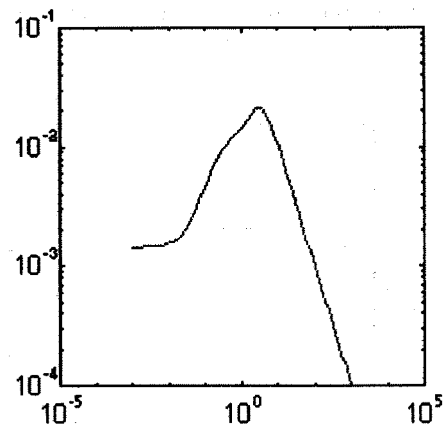

Fig.7 Error magnitude V's frequency[rad/sec]

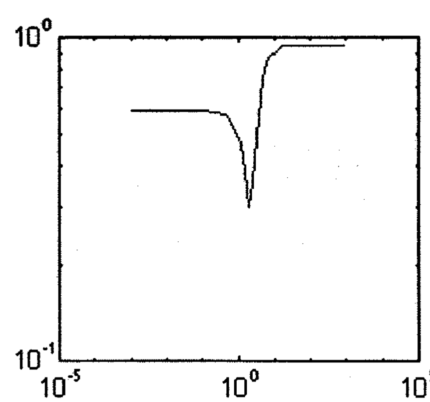

Fig.8 Control magnitude V's frequency[rad/sec]

showing that for low frequency commands (less than $3{\cdot}10^{-2} rad/\sec$) under interest, good tracking error is obtained together with a limited control δ_e.

Note that for the simplified configuration without feedback,the robustness properties with respect to additive and multiplicative uncertainties are very good since both K and G are stable and the resulting system (without the ideal model) reduces to GK.

Let finally remark that during the numerical tests described in this section no ill-conditioned computations appeared,unlike the case when suboptimal procedures are used near the optimum.

Appendix

We shall describe in the following the transformation of a DF problem to a two-block Nehari one in the case when $D_{21} = I$,D_{12} is inner,$D_{11} = 0; D_{22} = 0$(see also Ref.[12]).

Let X, Y be the stabilizing solutions to the standard Riccati equations:

$$A^T X + XA - \left(XB_2 + C_1^T D_{12}\right)\left(B_2^T X + D_{12}^T C_1\right) + C_1^T C_1 = 0$$
$$AY + YA^T - YC_2^T C_2 Y + B_2 B_2^T = 0$$

and perform the double coprime factorization $(A, B_2, C_2) = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$ with:

$$\begin{bmatrix} Y & U \\ -\widetilde{N} & \widetilde{M} \end{bmatrix}\begin{bmatrix} M & -U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

where:

$$\begin{bmatrix} M & -U \\ N & V \end{bmatrix}(s) := \left(A + B_2F, \begin{bmatrix} B_2 & -H \end{bmatrix}, \begin{bmatrix} F \\ C_2 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right)$$

with

$$F := -\left(B_2^T X + D_{12}^T C_1\right)$$

and

$$H := -YC_2^T$$

According to the results in Ref.[24], a parameterized family of stabilizing controllers is then:

$$K = K_1 K_2^{-1}$$

where:

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} M & -U \\ N & V \end{bmatrix}\begin{bmatrix} L \\ I \end{bmatrix}$$

After coupling this family of controllers to the standard system T one obtains the input-output operator:

$$T_{y_1u_1} = T_{11} + T_{12}LT_{21}$$

where:

$$T_{11}(s) := \left(\begin{bmatrix} A + B_2F & -B_2F \\ 0 & A + HC_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_1 + H \end{bmatrix}, [C_1 + D_{12}F \quad -D_{12}F]\right)$$
$$T_{12}(s) := (A + B_2F, B_2, C_1 + D_{12}F, D_{12})$$
$$T_{21}(s) := (A + HC_2, B_1 + H, C_2, I)$$

Since $A - B_1C_2$ is hurwitz (as an assumption for the DF problem) it follows that $T_{21}^{-1} \in RH^\infty$. By denoting $\widetilde{L} = -LT_{21}$ we obtain:

$$T_{y_1u_1} = T_{11} - T_{12}\widetilde{L}$$

Taking into account the choice of F and H we get that T_{12} is inner. Let $T_{12}^\perp$ be a completion such that $\begin{bmatrix} T_{12} & T_{12}^\perp \end{bmatrix}$ is square and inner.A realization of $\begin{bmatrix} T_{12} & T_{12}^\perp \end{bmatrix}$ is then:

$$\left(A + B_2F, \left[B_2 \quad -X^{-1}C_1^T D_{12}^\perp\right], C_1 + D_{12}F, \begin{bmatrix} D_{12} & D_{12}^\perp \end{bmatrix}\right)$$

where $D_{12}^\perp$ is the orthonormal complement of D_{12}.
When writing:

$$T_{y_1u_1} = T_{11} - \begin{bmatrix} T_{12} & T_{12}^\perp \end{bmatrix}\begin{bmatrix} \widetilde{L} \\ 0 \end{bmatrix}$$

from the fact that $\begin{bmatrix} T_{12} & T_{12}^\perp \end{bmatrix}$ is square and inner it follows that:

$$\|T_{y_1u_1}\|_\infty = \left\| \begin{bmatrix} T_{12} & T_{12}^\perp \end{bmatrix}^* T_{11} - \begin{bmatrix} \tilde{L} \\ 0 \end{bmatrix} \right\|_\infty$$

Direct calculations show that a minimal realization of $\begin{bmatrix} T_{12} & T_{12}^\perp \end{bmatrix}^* T_{11}$ is:

$$\left(\begin{bmatrix} -(A + B_2F)^T & 0 \\ 0 & A + HC_2 \end{bmatrix}, \begin{bmatrix} -XB_1 \\ B_1 + H \end{bmatrix}, \begin{bmatrix} B_2^T & -F \\ -(D_{12}^\perp)^T C_1 X^{-1} & 0 \end{bmatrix}\right)$$

Consider now the partition:

$$\begin{bmatrix} T_{12} & T_{12}^\perp \end{bmatrix}^* T_{11} = \begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \end{bmatrix}$$

where $\hat{G}_2(s) := \left(-(A + B_2F)^T, -XB_1, -\left(D_{12}^\perp\right)^T C_1 X^{-1}\right)$ is antistable and perform the decomposition $\hat{G}_1 = \hat{G}_{1a} + \hat{G}_{1s}$ in which:

$$\hat{G}_{1s}(s) := (A + HC_2, B_1 + H, -F)$$
$$\hat{G}_{1a}(s) := \left(-(A + B_2F)^T, -XB_1, B_2^T\right)$$

with $\hat{G}_{1s}(s)$ stable and $\hat{G}_{1a}(s)$ antistable. Denoting by $\hat{L} = \tilde{L} - \hat{G}_{1s}$ the DF problem has been transformed in a two-block Nehari problem consisting in determining for the given antistable systems $\hat{G}_{1a}$ and $\hat{G}_2$ a stable system $\hat{L}$ such that $\left\| \begin{matrix} \hat{G}_{1a} - \hat{L} \\ \hat{G}_2 \end{matrix} \right\|_\infty$ is minimized.

References

(1)Doyle, J.C., Glover, K., Khargcnekar, P., Francis, B.,"State Space Solutions to Standard H_2 and H_∞ Control Problems" *IEEE-Trans.Autom.Control*, **34**(1989), pp.18-36.

(2)Doyle,J.C.,Stein,G.,"Multivariable feedback design:concepts for a classical/modern synthesis", *IEEE-Trans.Autom.Control*,**26**(1981),pp.4-16.

(3)Dragan, V., Halanay, A., Stoica, A.,"Two-block Nehari and H^∞ problems ", preprint no. 1/1995, Institute of Mathematics of the Romanian Academy.

(4)Dragan, V., Halanay, A., Stoica, A., "Construction of controllers near optimally robust with respect to multiplicative perturbations",Preprint no.36/1995,Institute of Mathematics of the Romanian Academy.

(5)Dragan, V., Halanay, A., Stoica, A.,"A procedure to compute an optimal robust controller", *Proceedings of Eurocontrol,Rome,1995*,pp.1006-1009.

(6)Duc,G.,Mammar,S.,La Ballois S.,"Practical design of H^∞ control using coprime factors", *Proceedings of Eurocontrol Gronningen,1993*,pp.727-732.

(7)Elliot, J.R.," NASA's advanced control law program for the F-8 digital fly-by-wire aircraft ", *IEEE-Trans.Autom.Control*,**22**(1977), pp.753-757.

(8)Gahinet,P.,"A linear matrix inequality approach to H^∞ control", *Int.J.Robust Nonlinear Control*,**4**(1994), pp.421-448.

(9)Glover, K. ,"All optimal Hankel - norm approximations of linear multivariable systems and their L_∞ -error bounds " *International Journal of Control*, **39**(1984), pp. 1115-1193.

(10)Habets,L.C.G.,*Robust stabilization in the gap-topology*, Springer Verlag,1991.

(11)Hyde,R.A.,"The application of scheduled H_∞ controllers to a VSTOL aircraft", *IEEE-Trans. Autom.Control*, **38**(1993), pp.1021-1039.

(12)Juang,J.C.,Jonckheere,E.A.,"Data reduction in the mixed-sensitivity H^∞ design problem", *IEEE-Trans.Autom.Control*,**34**(1989),pp.861-866.

(13)McFarlane,D.C.,Glover,K.,*Robust controller design using normalized coprime factor plant descriptions*, Springer Verlag,1989.

(14)McLean,A.,*Automatic fligt control systems*, Prentice Hall International,1990.

(15)MIL-F-8785C "Flying qualities of piloted airplanes ", 5 Nov. 1980.

(16)Nett,C.N.,Jacobson,C.A.,Balas,M.J.,"A connection between state-space and doubly coprime fractional representation", *IEEE-Trans.Autom.Control*, **29**(1984),pp.831-832.

(17)Rogers,W.L.,Collins,D.J.,"X-29 H_∞ controller synthesis", *J.Guidance, Control and Dynamics*, **15** (1992), pp.962-967.

(18)Safonov,M.,Chiang,R.Y.,"CASCD using the state-space L^∞ theory-a design example", *IEEE-Trans. Autom.Control*, **33**(1988), pp.477-479.

(19)Safonov,M.G.,Limebeer,D.J.N.,"Simplifying the H^∞-theory via loop shifting",Proc.IEEE Conf. on Decision and Control,Austin,TX,December 7-9,1988.

(20)Scherer,C.W.,*The Riccati inequality and state space H^∞ optimal control*, Ph.D. Thesis, University of Wurzburg,1990.

(21)Sparks, A., Banda, S.S.,"Application of structured singular value synthesis to a fighter aircraft " , *Journal of Guidance*, **16**(1993), pp.940-947.

(22)Stoorvogel,A.A.,"Robust stabilization of systems with multiplicative perturbations", *Proceedings MTNS,Kobe,Japan*,1991.

(23)Vidyasagar,M.,Kimura,H.,"Robust controllers for uncertain linear multivariable systems", *Automatica*, **22**(1986), pp.85-94.

(24)Youla, D.C., Jabr, H.A.,Bongiorno, J.J.," Modern Weiner-Hoph design of optimal controllers, part II: the multivariable case ", *IEEE-Trans.Autom.Control*, **21**(1976), pp.319-338.

(25)Zames,G.,"On input-output stability of time-varying nonlinear feedback systems,part.1", *IEEE-Trans.Autom.Control*,**11**(1966),pp.228-238.