ON SOME ANALYTICAL STUDIES OF UNSTEADY AND NON-LINEAR LONGITUDINAL STABILITY

L.M.B.C.Campos*, J.R.C.Azinheira** & A.A. Fonseca***
*Professor Catedrático, **Professor Auxiliar, ***Engineer
Secção de Mecânica Aeroespacial, ISR,
Instituto Superior Técnico - Av. Rovisco Pais
1096 Lisboa Codex, Portugal

Summary

The longitudinal stability of a symmetric airplane is usually studied by linearization about a steady flight condition, leading to the period and damping of the phugoid and short-period modes. In the present communication we introduce two new methods of study of airplane longitudinal stability, which are more general, in that they relieve some of the restrictions made in the usual textbook analysis of the subject. In the first method, which we may designate unsteady longitudinal stability, the three equations of motion are solved by a cyclic recurrence method, involving, for example, a short-period mode with variable airspeed. In the second method, which we may designate non-linear longitudinal stability, the equations of motion are eliminated directly to obtain a single equation of order higher than the second, which is solved by a perturbation technique. One motivation for these extensions of the study of longitudinal stability of airplanes, into the unsteady and non-linear regimes, is to look, within the framework of flight dynamics, for explanations of phenomena like the PIO (Pilot Induced Oscillations), with which have not been completely solved by control theory alone.

§1 - Introduction

The longitudinal stability of an airplane [1-4] is specified by three equations: the balance of forces (lift, drag, thrust and weight) in the rectifying plane of the trajectory, and the balance of pitching moment in the normal direction. Even for flight at low Mach number over a small altitude range, these equation exhibit non-linearities, e.g. in the acceleration terms and in the aerodynamic forces. These are neglected in the usual [5-8] method of linearization, about a mean state of straight and level flight at constant airspeed, leading to the phugoid and short-period modes, and specifying their frequency and damping. In the present communication we present two extensions of the standard textbook analysis, relieving one of the two main restrictions: (I) For "unsteady longitudinal stability" we do not start from a steady mean state and use a recursive method of solution; (II) for "non-linear longitudinal stability" the equations are eliminated without linearization, and solved by a perturbation technique. We start by recalling the equations of longitudinal motion of an airplane (§2), and then present the unsteady (part I) and non-linear (part II) theories, each of which has four sub-cases (Ia to Id, and IIa to IId), and conclude with a discussion of possible applications.

§2- Equations of Motion in Terms of Time or Distance

We consider the longitudinal motion a symmetric aircraft (*Figure 1*), under lift L, drag D, thrust T and weight W balancing the inertia force, and also the pitching moment M equation:

$$m\dot{V} = T - D - W\sin\gamma, \tag{1a}$$

$$mV\dot{\gamma} = L - W\cos\gamma, \tag{1b}$$

$$I(\ddot{\alpha} + \ddot{\gamma}) = M, \tag{1c}$$

where m is the mass, I the transverse moment of inertia, and the variables are the airspeed V, flight path angle γ and angle-of-attack α, with differentiation with regard to time t denoted by a dot (2a)

$$\dot{f} \equiv \partial f / \partial t, \qquad \dot{f} = (\partial f / \partial s)ds / dt \equiv V f', \tag{2a,b}$$

whereas differentiation with regard to the arc length s, or distance along the flight path, is represented by a prime (2b). Using the latter twice:

$$\ddot{f} = (V d / ds)(V d / ds)f = V^2 f'' + V' V f', \tag{2c}$$

the equations of motion can be written:

$$m V' V = T - D - W\sin\gamma, \tag{3a}$$

$$m V^2 \gamma' = L - W\cos\gamma, \tag{3b}$$

$$V^2(\alpha'' + \gamma'') + V V'(\alpha' + \gamma') = M / I, \tag{3c}$$

using derivatives with regard to distance in (3a,b,c) instead of time in (1a,b,c). In these equations it was assumed that thrust lies along the flight path, but a small deviation causes a small error.

§3- Specification of the Aerodynamic Forces and Moments

To study longitudinal motion we need two aerodynamic forces, viz. lift and drag, and one aerodynamic moment, viz. in pitch:

$$L = \frac{1}{2}\rho\ S\ V^2 C_L(\alpha), \tag{4a}$$

$$D = \frac{1}{2}\rho\ S\ V^2 C_D(\alpha), \tag{4b}$$

$$M = \frac{1}{2}\rho\ S\ V^2\ c\ C_M(\alpha), \tag{4c}$$

where ρ is the mass density, S the wing or reference area, c the mean aerodynamic chord, and for flight at low Mach number, neglecting (not only compressibility but) also turbulence effects, the lift C_L, drag C_D and pitching moment C_M coefficients depend only on angle-of-attack α. For flight away from the stall, the lift coefficient is a linear function of angle of attack:

$$C_L(\alpha) = C_{L0} + \alpha\ C_{L\alpha} = C_{L\alpha}(\alpha - \alpha_0), \qquad (5a)$$

$$C_{L\alpha} \equiv \partial C_L / \partial\alpha, \quad C_{L0} \equiv C_L(0), \qquad (5b,c)$$

$$\alpha_0 = -C_{L0} / C_{L\alpha}, \quad C_L(\alpha_0) = 0, \qquad (5d,e)$$

where C_{L0} is the lift coefficient at zero angle of attack, $C_{L\alpha}$ is the lift slope, and α_0 the angle of zero lift. A similar set of relations applies to the pitching moment coefficient:

$$C_{M(\alpha)} = C_{M0} + \alpha\ C_{M\alpha} = C_{M\alpha}(\alpha - \alpha_1), \qquad (6a)$$

$$C_{M\alpha} \equiv dC_M / d\alpha, \quad C_{M0} \equiv C_M(0), \qquad (6b,c)$$

$$\alpha_1 = -C_{M0} / C_{M\alpha}, \quad C_M(\alpha_1) = 0, \qquad (6d,e)$$

where the angle of zero pitching moment α_1, is generally distinct from the angle-of zero lift. The drag is taken to consist of :

$$C_D(\alpha) = C_{Df} + jC_L(\alpha) + k\{C_L(\alpha)\}^2, \qquad (7)$$

friction drag C_{Df}, lift-induced drag with coefficient k, plus a non-parabolic drag term with coefficient j.

Part I - Unsteady Stability and Recursive Method

The method I of solution of the equations of longitudinal motion of an airplane proceeds along the following steps:

(i) assume constant flight path angle $\dot{\gamma} = 0 = \gamma'$, e.g. flight along a constant glide slope in the transverse force balance (1b or 3b), and express angle-of-attack as a function of airspeed $\alpha(V)$;

(ii) substitute this into the longitudinal force balance (1a or 3a), and obtain a non-linear first-order differential equation, to be solved for airspeed: V(t) or V(s);

(iii) substitute V in the pitching moment equation (1c or 3c) with $\ddot{\gamma} = 0$, and solve the second-order differential equation to determine angle-of-attack: $\alpha(t)$ or $\alpha(s)$;

(iv) replace angle-of-attack α form (iii) and airspeed from (ii), back into the transverse force balance (1b or 3b), to specify a first-order correction for flight path angle: $\gamma(t)$ or $\gamma(s)$;

(v) repeat the iteration recursively, until the flight variables α, V, γ show a small change, i.e. differ from one iteration to the next, by less than the required accuracy.

This recursive method applies to the study of unsteady stability, because the initial state is not a steady flight condition, and thus the equations of motion are ordinary differential equations with non-constant coefficients, i.e. coefficients which are functions of space or time, as we proceed to illustrate.

§4- Steady Dive Speeds, Minimum Thrust and Minimum Drag

For step (i) of the recursive method I, we assume constant flight path angle in (3b):

$$\gamma = \text{const.}: \quad \cos\gamma = L/W = (\rho S V^2/2W) C_L(\alpha), \qquad (8)$$

where we have used (4a), and substitution into (3a) and use of (4b, 7) leads to:

$$(V^2)' / 2g = -\sin\gamma + T/W - C_{Df}\, \rho S V^2 / 2W - j \cos\gamma (2kW/\rho S V^2). \qquad (9)$$

Assuming that the thrust-to-weight ratio has a dependence on airspeed of the form:

$$T / W = f_0 - f_1 V^2 - f_2 / V^2, \qquad (10)$$

we can simplify (9) to:

$$(V^2)' / 2g = a - bV^2 - d / V^2, \qquad (11)$$

where:

$$a \equiv f_0 - \sin\gamma - j\ \cos\gamma, \qquad (12a)$$

$$b \equiv f_1 + C_{Df}\ \rho S / 2W, \qquad (12b)$$

$$d \equiv f_2 + \cos^2\gamma\ 2kW / \rho\ S, \qquad (12c)$$

are constants over a moderate altitude range, i.e. over which the air density ρ is constant. Re-writing (11) in the form:

$$(V^2)' / 2g = (V^2 - V_+^2)(V^2 - V_-^2) / V^2, \qquad (13)$$

it is clear that $V_\pm$ are the steady dive speeds, for which there is no acceleration along the flight path V' = 0, i.e. longitudinal forces balance. The steady dive speeds are the roots of (11) ≡ (13):

$$(V^2 - V_+^2)(V^2 - V_-^2) = -bV^4 + aV^2 - d, \qquad (14)$$

i.e. are given by:

$$V_\pm^2 = (a / 2b) \left\{1 \pm \sqrt{1 - 4bd / a^2}\right\}. \qquad (15)$$

The condition that $V_\pm$ be real $a^2 > 4bd$, specifies the minimum thrust for a steady dive:

$$(f_0 - \sin\gamma - j\cos\gamma)^2 \geq (2f_1 + C_{Df}\, \rho S/W)(2f_2 + \cos^2\gamma kW/\rho S). \qquad (16)$$

At the minimum thrust, the steady dive speeds coincide:

$$a^2 = 4bd: \qquad V_+ = V_- = V_{md} = \sqrt{a / 2b}, \qquad (17)$$

with the minimum drag speed:

$$V_{md}^2 = (f_0 - \sin\gamma - j\cos\gamma) / (2f_1 + C_{Df}\ \rho S / W). \qquad (18)$$

In the case of level flight γ = 0, at constant thrust $f_1 = 0 = f_2$, and parabolic lift-drag polar j = 0, we obtain:

$$f_0 \geq 2\sqrt{C_{Df}\,k}, \qquad V_{md} = \sqrt{f_0 W / C_{Df}\ \rho S}, \qquad (19a,b)$$

which are well-known particular cases [9-10].

§5- Airspeed as a Function of Distance Along Flight Path

If an aircraft starts a dive or climb at one of the steady airspeeds $V_0 = V_+$ or $V_0 = V_-$, it will remain at that airspeed, because (13) there is no longitudinal acceleration V' = 0. If the initial airspeed $V_0 \neq V_\pm$ is distinct from the airspeeds for steady dive or climb, the solution of (13) specifies [11-14] the airspeed as a function of distance along the flight path V(s), in the inverse form s(V):

$$e^{-2bgs}=\left\{\left[(V/V_+)^2-1\right]/\left[(V_0/V_+)^2-1\right]\right\}^{V_+^2/(V_+^2-V_-^2)}$$

$$\left\{\left[(V_0/V_-)^2-1\right]/\left[(V/V_-)^2-1\right]\right\}^{V_-^2/(V_+^2-V_-^2)}. \quad (20)$$

For a long-distance compared to the aerodynamic scale:

$$s >> \ell \equiv 1/2bg, \quad (21)$$

the airspeed either tends to the upper steady flight speed $V(s) \to V_+$, showing that flight at V_+ is stable, or diverges from the lower steady flight speed ($V(s) - V_-$ increases), showing that flight at V_- is unstable. For small deviations from the initial airspeed:

$$V_0^2 - V^2 << V_0^2 - V_+^2: \quad V(s) = V_1 + V_2\, e^{-s/\ell}, \quad (22)$$

the airspeed varies exponentially:

$$V_1 \equiv V_0 - V_2, \quad (23a)$$

$$2V_2 \equiv V_0\left[1-(V_-/V_0)^2\right]\left[(V_+/V_0)^2-1\right]. \quad (23b)$$

If in addition, the distance along the flight path is small relative to the aerodynamic lengthscale:

$$s^2 << \ell^2: \quad V(s) = V_0 - (V_2/\ell)s, \quad (24)$$

the airspeed variation is linear.

§6- Case Ia: Undamped Shorth-Period with Linear Airspeed

If we denote by $\Phi \equiv \alpha - \alpha_1$ the angle-of-attack relative to the angle of zero pitching moment, and use (4c, 6a) in (3c) we obtain:

$$\Phi'' + (V'/V)\Phi' - (\rho s c\, C_{M\alpha}/2I)\Phi = 0, \quad (25)$$

For flight at constant airspeed:

$$V = \text{const}: \quad \Phi'' + \omega^2\Phi = 0, \quad (26)$$

we obtain an undamped short-period oscillation, with spatial periodicity given by:

$$\omega \equiv \sqrt{-\rho S c\, C_{M\alpha}/2I}, \quad (27)$$

where $C_{M\alpha} < 0$ for static stability, and thus ω is real. In general, for flight at non-constant airspeed, we have instead of (26), an equation (25, 27) with variable coefficients:

$$\Phi'' + (V'/V)\Phi' + \omega^2\Phi = 0. \quad (28)$$

The simplest case is that (24) of airspeed a linear function of distance:

$$V'/V = 1/(s-s_0), \qquad s_0 \equiv \ell V_0/V_2; \quad (29a,b)$$

the differential equation:

$$\left(\Phi'' + \omega^2\Phi\right)(s-s_0) + \Phi' = 0, \quad (30)$$

can be reduced to a Bessel type, and thus the solution is a linear combination of Hankel [15] functions:

$$\Phi(s) = C_1\, H_0^{(1)}\left(\omega(s-s_0)\right) + C_2\, H_0^{(2)}\left(\omega(s-s_0)\right), \quad (31)$$

where the constant C_1, C_2 are determined from the initial angle-of-attack $\Phi(0)$ and its rate $\Phi'(0)$. For long--distance, the asymptotic formula for Hankel functions, shows:

$$\omega(s-s_0) >> 1: \quad \Phi(s) \sim \sqrt{2/\pi\omega(s-s_0)}\left\{C_1\, e^{i\omega(s-s_0)+i\pi/4} + C_2\, e^{-i\omega(s-s_0)-i\pi/4}\right\}, \quad (32)$$

that there is an oscillation with spatial periodicity ω (27), but the amplitude is not constant as in the case of constant airspeed (26), and instead decays like $1/\sqrt{s-s_0}$ for the case (24) of linear airspeed variation. In the case of zero initial rate of angle-of-attack, (31) simplifies to:

$$\dot\Phi(0) = 0: \qquad \Phi(s) = \Phi_0 \quad J_0\left(\omega(s-s_0)\right), \quad (33)$$

in terms of Bessel functions. This is plotted in dimensionless form:

$$\Theta \equiv \Phi(s)/\Phi_0, \qquad X \equiv \omega(s-s_0), \quad (34a,b)$$

in *Figure 2.*

§7- Case Ib: Damped Short-Period with Linear Airspeed

We consider again the equation (28) for angle of attack relative to the angle of zero pitching moment:

$$\Phi(s) = \alpha(s) - \alpha_1, \quad (35)$$

this time including damping λ:

$$\Phi'' + (2\lambda + V'/V)\Phi' + \omega^2\Phi = 0. \quad (36)$$

Considering again a distance along the flight path short compared with the aerodynamic lengthscale (24), and hence a linear airspeed variation (29a), we are led to the differential equation:

$$(s-s_0)\Phi'' + \left[1+2\omega\lambda(s-s_0)\right]\Phi' + \omega^2(s-s_0)\Phi = 0, \quad (37)$$

which can be solved in terms of confluent hypergeometric functions [16]:

$$\Phi(s) = \Phi_0 \exp\left[-\omega(s-s_0)\left(\lambda+\lambda^2-1\right)\right] F\left(1+1/\sqrt{1-1/\lambda^2};1;2\sqrt{\lambda^2-1}\,\omega(s-s_0)\right), \quad (38)$$

where, since the second parameter is unity, the other solution would have a logarithmic singularity at $s = s_0$, and thus is ommitted. To the lowest order, (38) simplifies to:

$$\Phi(s) = \Phi_0\, e^{-\omega\lambda(s-s_0)} \exp\left\{-\omega(s-s_0)\sqrt{\lambda^2-1}\right\}, \quad (39)$$

which is the solution for the short-period mode with constant airspeed:

$$\Phi'' + 2\lambda\,\Phi' + \omega^2\Phi = 0; \quad (40)$$

note that in the case of weak damping (39) simplifies further to:

$$\lambda^2 << 1: \quad \Phi(s) = \Phi_0\, e^{\pm i\omega(s-s_0)}\, e^{-\omega\lambda(s-s_0)}, \quad (41)$$

in agreement with an oscillation with periodicity (27) in the undamped case (26). The effect of the airspeed variation as a linear function of distance appears in (38) in the factor of (39), i.e. the confluent hypergeometric function. The solution (38) is plotted in dimensionless form (34a,b):

$$\Theta_\pm(X) = e^{-\lambda X}\, \mathrm{Re}\left\{e^{iX}\, F\left((1\pm i\lambda)/2;1;2iX\right)\right\}, \quad (42)$$

for $\lambda = 0.1$, where Re {...} denotes the real part of a complex expression. The *Figure 3* for Θ_+ shows a damped oscillation, and *Figure 4* for Θ_- a damped double oscillation, i.e. both factors in curly brackets in (42) are oscillatory. The kind of high-frequency 'beats' seen on Figure 4, could be seen as an stability ressembling PIOs (Pilot Induced Oscillations). The damped and undamped short-period mode, for linear airspeed variation, are compared in *Figure 5*.

§8- Case 1c: Undamped Short-Period with Exponential Airspeed

If the distance along the flight path is not short relative to the aerodynamic lengthscale, then the linear airspeed variation (24), should be replaced by an exponential one (22), provided that airspeed is still close to initial airspeed and far from the steady dive speed. We consider the exponential airspeed variation (28) first for the undamped short-period mode

$$\left(V_1+V_2e^{-s/\ell}\right)\Phi''-\left(V_2/\ell\right)e^{-s/\ell}\Phi'+\omega^2\left(V_1+V_2e^{-s/\ell}\right)\Phi=0. \tag{43}$$

The solution of this differential equation is given in terms of Gaussian hypergeometric functions [17]:

$$\Phi(s)=C_+e^{-i\omega s}F\left(A,B;1+2i\omega\ell;-\left(V_2/V_1\right)e^{-s/\ell}\right)$$
$$+C_-e^{i\omega s}F\left(A-2i\omega\ell,B-2i\omega\ell;1-2i\omega\ell;-\left(V_2/V_1\right)e^{-s/\ell}\right), \tag{44}$$

where the A,B are constants given by:

$$2(A,B)=1+2i\omega\ell\pm\sqrt{1-4\omega^2\ell^2}\,, \tag{45a,b}$$

and the constants of integration $C_\pm$ are specified from the initial angle-of-attack Φ_0 and its rate Φ'_0. To lowest order, for very short distance compared with aerodynamic lengthscale, (44) simplifies to an undamped short period oscillation:

$$s<<\ell:\ \Phi(s)=C_+e^{i\omega s}+C_-e^{-i\omega s}\,, \tag{46}$$

i.e. the solution of (26). The effects of the exponential airspeed variation along the flight path thus appear in the hypergeometric functions in (44). Using the dimensionless form:

$$\Theta=\Phi(s)/\Phi_0=\left(\alpha(s)-\alpha_1\right)/\left(\alpha(0)-\alpha_1\right),\ x\equiv\omega s, \tag{47a,b}$$

$$\Theta(x)=$$

$$=\mathrm{Re}\left[e^{i\Omega x}F\left(A,B;1+2i\Omega;-\mu e^{-x/\Omega}\right)/F\left(A,B;1+2i\Omega;-\mu\right)\right], \tag{48}$$

is plotted for:

$$\omega\ell=1,\quad V_2/V_1=1, \tag{49a,b}$$

in *Figure 6*, showing that the period is close to ω, but the amplitude is not the initial value Φ_0.

§9 - Case 1d: Damped Short-Period with Exponential Airspeed

The more general case would be to combine the damped short-period (36) with exponential airspeed (22), leading to the differential equation:

$$\left(V_1+V_2e^{-s/\ell}\right)\Phi''+\left[2\omega\lambda V_1+V_2\left(2\omega\lambda-1/\ell\right)e^{-s/\ell}\right]\Phi'+$$
$$+\omega^2\left(V_1+V_2e^{-s/\ell}\right)\Phi=0, \tag{50}$$

the solution of which can again be solved in terms of Gaussian hypergeometric functions:

$$\Phi(s)=C_+e^{-\vartheta s/\ell}F\left(A,B;C;-\left(V_2/V_1\right)e^{-s/\ell}\right)$$
$$+C_-e^{(c-1-\vartheta)s/\ell}F\left(A+1-C,B+1-C;2-C;-\left(V_1/V_2\right)e^{-s/\ell}\right), \tag{51}$$

where ϑ, C,A,B are the constants:

$$\vartheta=\omega\ell\left(\lambda\pm\sqrt{\lambda^2-1}\right), \tag{52a}$$

$$C=1+2\omega\ell\sqrt{\lambda^2-1}\,, \tag{52b}$$

$$A,B=1/2+\omega\ell\sqrt{\lambda^2-1}\pm\left|1/4-\lambda\omega\ell+\omega^2\ell^2\left(\lambda^2-1\right)\right|^{1/2}. \tag{52c}$$

In the absence of damping $\lambda = 0$, then (50) simplifies to (43), and likewise (51) simplifies to (44), because $\vartheta = i\omega\ell$ in (52a), $C = 1+2i\omega\ell$ in (52b), and (52c,d) simplifies to (45a,b). The solution (51) is plotted in the dimensionless (47a,b) form:

$$\Theta(x)=e^{-\lambda x}\,\mathrm{Re}\left\{e^{ix}F(x)/F(0)\right\}, \tag{53a}$$

$$F(x)\equiv F\left(1-i\omega\ell,i\omega\ell;1+2i\omega\ell;-\mu e^{-x/\omega\ell}\right), \tag{53b}$$

for the following choices of parameters:

V_2/V_1	$\omega\ell$	*Figure*
1	1	7
2	1	8
1	π	9
1	π	10

and $\lambda = 0.1$. We compare the short-period mode with constant and exponential airspeed (with $\mu = 1$), in *Figure 11* without damping $\lambda = 0$, and in *Figure 12* with damping $\lambda = 0.1$.

§10 - Correction for Flight Path Angle and Succeding Iterations

In implementing steps (i, ii, iii) the recursive method I to study unsteady stability, we have calculated the airspeed (§4,5), then the angle-of-attack in four cases Ia-d (§6-9), assuming a constant flight path angle γ, step (iv) is a correction of the flight path angle, using (3b,4a, 6a):

$$\gamma'/g=-V^{-2}\cos\gamma+\left(\rho S/2\omega\right)C_{L\alpha}\left(\alpha-\alpha_0\right). \tag{54}$$

The simplest estimate would be:

$$|\gamma'|\le\left(\rho SgC_{L\alpha}/2\omega\right)\left[\alpha(s)-\alpha_0\right], \tag{55}$$

implying that:

$$|\gamma-\gamma_0|\le\left(\rho SgC_{L\alpha}/2\omega\right)\left|\int_{s_0}^{s}\alpha(s)ds-\alpha_0\left(s-s_0\right)\right|\equiv\gamma_1. \tag{56}$$

If this is less than the required accuracy $\gamma_1<\varepsilon$ no further iterations are necessary. If greater accuracy is needed, we have to estimate $\gamma(s)$ to feed to the next iteration, e.g.:

$$\gamma'(s)-\gamma_1'(s)=-gV^{-2}\cos\gamma. \tag{56}$$

In the case of small flight path angle:

$$\gamma^2<<1:\ \ \gamma'(s)-\gamma_1'(s)=-gV^{-2}, \tag{57}$$

the integration is straightforward, e.g.:

$$\gamma(s) = \gamma_1(s) + (g\ell) / (V_0 - V_2\, s/\ell), \qquad (58)$$

for linear airspeed variation (24). Having shown how to complete an iteration of the recursive method for unsteady stability, we now turn to non-linear stability.

Part II - Non-Linear Stability and Perturbation Method

The approach for non-linear stability is completely different, in that one eliminates among the three equations of motion to obtain a single equation of higher order. It is simpler to use the equations of longitudinal motion of the airplane with time (1a,b,c) rather than spatial (3a,b,c) derivatives. The elimination for one variable, e.g. angle-of-attack, can lead, at most, to a differential equation of order four, i.e. the order of the system. The order of the equation can be less, e.g. three, if one equation decouples, for a restricted case. The restrictions which limit less the application to flight stability problems, would involve the flight path angle, rather than airspeed or angle-of-attack. These restrictions can be usefull in limiting the considerable complexity of the non-linear equation of motion, which results from the elimination in the system (1a,b,c). In order to see how the elimination can be performed, we need to write the equations of motion explicity in terms of the three flight variables: airspeed, flight path angle and angle-of-attack relative to the angle-of-zero pitching moment.

§11 - Explicit Form of the Equations of Motion

From (c, 6a), the pitching moment equation (1c) is expressed in terms of the angle-of-attack relative to the angle of zero pitching moment (35):

$$\ddot{\gamma} + \ddot{\Phi} + \omega^2 V^{-2} \Phi = 0, \qquad (59)$$

where we have introduced the spatial periodicity (27) of the short-period mode. The transverse force balance (1b), with (4a, 5a) gives:

$$\dot{\gamma} + gV^{-1}\cos\gamma = (\rho S C_{L\alpha} / 2m)(\alpha - \alpha_0), \qquad (60)$$

which may be written:

$$\dot{\gamma} + gV^{-1}\cos\gamma = p(\Phi - \Phi_1), \qquad (61)$$

where:

$$p \equiv \rho S C_{L\alpha} / 2m, \qquad \Phi_1 \equiv \alpha_0 - \alpha_1, \qquad (62a,b)$$

are constant. The longitudinal force balance (1a) with (4b, 7, 6a) and $\alpha - \alpha_0 = \Phi - \Phi_1$, leads to:

$$\dot{V} + g\sin\gamma - T/m = D/m = \left(\rho S V^2 / 2m\right) \left[C_{D0} + jC_{L\alpha}(\Phi - \Phi_1) + kC_{L\alpha}^2(\Phi - \Phi_1)^2\right], \qquad (63)$$

which simplifies to:

$$\dot{V} = q - g\sin\gamma - fV^2\left(1 + h\Phi + \varepsilon\Phi^2\right), \qquad (64)$$

where,

$$q \equiv T/m, \qquad (65)$$

is a constant at constant thrust, and:

$$f \equiv (\rho S/2m)\left[C_{D0} - jC_{L\alpha}\Phi_1 + k\left(C_{L\alpha}\Phi_1\right)^2\right], \qquad (66a)$$

$$fh \equiv (\rho S/2m)\left(jC_{L\alpha} - 2kC_{L\alpha}^2\Phi_1\right), \qquad (66b)$$

$$f\varepsilon \equiv (\rho S/2m)\, k\, C_{L\alpha}^2, \qquad (66c)$$

are all constant.

§12 - Simplifications for the Flight Path Angle

The system to be eliminated, viz. (59, 61, 64), e.g. for the angle-of-attack Φ, is quite complicated in the general case (Case IIa). Since the acceleration of the flight path angle is usually small compared with that of angle-of-attack, the next simplification (case IIb) is to replace (59) by:

$$\ddot{\gamma} << \ddot{\Phi}: \quad \ddot{\Phi} + (\omega/V)^2\Phi = 0. \qquad (67)$$

The flight path angle γ is so far unrestricted, but if it is limited to moderate values $\gamma \leq 30°$, such that:

$$\gamma^2 << 1: \quad \sin\gamma = \gamma, \qquad \cos\gamma = 1, \qquad (68a,b)$$

then (61,64) are linearized with regard to flight path angle (case IIc):

$$\dot{\gamma} = -g/V + p(\Phi - \Phi_1), \qquad (69)$$

$$\dot{V} = q - g\gamma - fV^2\left(1 + h\Phi + \varepsilon\Phi^2\right). \qquad (70)$$

If the flight path angle is small $\gamma \leq 10°$, then (case IId) we can neglect it altogether in (70), viz:

$$\dot{V} = q - fV^2\left(1 + h\Phi + \varepsilon\Phi^2\right). \qquad (71)$$

Thus the simplest system (case IId) is (67, 69, 71), and this is sufficient to illustrate the process of elimination and perturbation method of solution.

§13- Elimination for the Angle-of-Attack Relative to Zero Pitching Moment

If we substitute (67), in the form:

$$V = (\pm i/\omega)\Phi^{1/2}\,\ddot{\Phi}^{-1/2}, \qquad (72)$$

into (71), we obtain a non-linear third-order differential equation for the angle-of-attack:

$$\dddot{\Phi} - \Phi^{-1}\dot{\Phi}\ddot{\Phi} \mp 2i(q/\omega)c \mp 2if\omega\left(1 + h\Phi + \varepsilon\Phi^2\right)\Phi^{1/2}\ddot{\Phi}^{1/2} = 0. \qquad (73)$$

Once this equation is solved for the angle-of-attack Φ (t), the airspeed V(t) follows from (72), and the flight path angle γ(t) is obtained by integration of (69).

The equation for angle-of-attack (73) can be rewritten in the form:

$$\ddot{\Phi} + \omega^2(f/q)\Phi\left[1 + h\Phi + \varepsilon\Phi^2\right] = \pm(i\omega/2q)\left(\Phi^{1/2}\ddot{\Phi}^{-1/2}\dddot{\Phi} - \Phi^{-1/2}\dot{\Phi}\ddot{\Phi}^{1/2}\right). \qquad (74)$$

To a linear approximation this simplifies to a short-period mode:

$$\ddot{\varphi} + \Omega_0^2\varphi = 0, \qquad (75)$$

where the short period frequency is given by:

$$\Omega_0^2 = \omega^2(f/q) = \omega^2\left(\rho S\, \overline{C}_D / T\right), \qquad (76a)$$

$$\overline{C}_D = C_{D0} - jC_{L\alpha}\Phi_1 + k\left(C_{L\alpha}\Phi_1\right)^2. \qquad (76b)$$

For flight at constant airspeed the short-period frequency is given (59) by $\Omega = \omega/V$, so that in (76), V^{-2} which is not constant, is replaced by the constant in brackets, which has the same dimensions.

§14 - An Harmonic and other Non-Linearities Involving Derivatives

The equation for angle-of-attack, (74) with (76a) is:

$$\ddot{\Phi}+\Omega_0^2\Phi\left[1+h\Phi+\varepsilon\Phi^2\right]=\pm i\mu\left(\Phi^{1/2}\ddot{\Phi}^{-1/2}\dddot{\Phi}-\Phi^{-1/2}\dot{\Phi}\ddot{\Phi}^{1/2}\right) \tag{77}$$

where:

$$\mu \equiv \omega/2q = (m/2T)\sqrt{-\rho c S C_{n\alpha}/2I}\,. \tag{78}$$

There are two types of non-linearities in (77): (i) those in square brackets ressemble an anharmonic oscillator [18-19]:

$$\ddot{\Phi}+d\psi/d\Phi=0, \tag{79a}$$

with potential ψ having cubic and quartic terms:

$$\psi(\Phi)=\Omega_0^2\left(\frac{1}{2}\Phi^2+\frac{1}{3}\Phi^3+\frac{1}{4}\Phi^4\right), \tag{79b}$$

and are associated with (66a,b,c) the quadratic lift-drag polar (7); (ii) those in curved brackets are associated with the effect on angle-of-attack, of variations of airspeed and flight path angle. The linear short-period mode (75) has sinusoidal solutions with constant amplitude:

$$\varphi(t)=\varphi_0 e^{\pm i\Omega_0 t}, \tag{80}$$

which also satisfy the second set of non-linear terms

$$\varphi^{1/2}\ddot{\varphi}^{-1/2}\dddot{\varphi}=\varphi^{-1/2}\dot{\varphi}\,\ddot{\varphi}^{1/2}, \tag{81}$$

but not the first, so that (80) is not a solution of (77).

§15 - Perturbation Method for Flight Variable and Frequency

We take ε as small parameter in (77), and rewrite it as:

$$\ddot{\Phi}+\Omega_0^2\Phi[1+\varepsilon\Phi(\Phi+\nu)]=\pm i\mu\left(\Phi^{1/2}\ddot{\Phi}^{-1/2}\dddot{\Phi}-\Phi^{-1/2}\dot{\Phi}\ddot{\Phi}^{1/2}\right) \tag{82}$$

where:

$$\nu \equiv h/\varepsilon \tag{83}$$

is a constant. We seek a solution in the form:

$$\Phi(t)=\varphi(t)+\varepsilon\chi(t), \tag{84a}$$

$$\Omega=\Omega_0+\varepsilon\,\Omega_1, \tag{84b}$$

with a perturbation in angle-of-attack $\chi(t)$ and in frequency Ω_1. Substitution of (84a,b) in (82) leads to (75) to zeroth-order (i.e. independent of ε), and the terms linear in ε are:

$$\begin{aligned}&\dddot{\chi}+\left[\pm(i/\mu)\varphi^{-1/2}\ddot{\varphi}^{1/2}-(1/2)\ddot{\varphi}^{-1}\dddot{\varphi}+\varphi^{-1}\dot{\varphi}\right]\ddot{\chi}\\ &-\varphi^{-1}\ddot{\varphi}+\left[\pm(i/\mu)\Omega_0^2\varphi^{-1/2}\ddot{\varphi}^{1/2}+(1/2)\varphi^{-1}\dddot{\varphi}+\varphi^{-2}\dot{\varphi}\ddot{\varphi}\right]\chi\\ &=\mp(i/\mu)\left[2\Omega_0\Omega_1\varphi^{1/2}\ddot{\varphi}^{1/2}+\Omega_0^2\varphi^{3/2}\ddot{\varphi}^{1/2}(\varphi+\nu)\right],\end{aligned} \tag{85}$$

and specify a perturbation equation of third-order, which is linear, with coefficients specified by (80).

§15 - Absence of Modification of the Short-Period Frequency

Substituting (80) in (85) gives:

$$\begin{aligned}&\dddot{\chi}-\Omega_0(\pm 1/\mu-i)\ddot{\chi}+\Omega_0^2\dot{\chi}-\Omega_0^3(\pm 1/\mu-i)\chi\\ &=\mp\left(\omega_0^2/\mu\right)\varphi_0 e^{i\Omega_0 t}\left[2\Omega_1+\Omega_0\varphi_0 e^{i\Omega_0 t}\left(\mu+\varphi_0 e^{i\Omega_0 t}\right)\right].\end{aligned} \tag{86}$$

Thus we seek a forced solution, as a superposition of oscillations at the short-period plus two harmonics:

$$\chi(t)=C_1 e^{i\Omega_0 t}+C_2 e^{i2\Omega_0 t}+C_3 e^{i3\Omega_0 t}. \tag{87}$$

Substitution of (87) into (86) gives:

$$0.C_1=2\Omega_1\Omega_0^2\varphi_0/\mu, \tag{88a}$$

$$\varphi_0^2\nu=(3\mp 7i\mu)C_2, \tag{88b}$$

$$\varphi_0^3\nu=(8\mp 33i\mu)C_3. \tag{88c}$$

From (88a) it follows that the short-period frequency (84b) is unchanged to first-order in the perturbation:

$$\Omega_1=0:\quad \Omega=\Omega_0+0\left(\varepsilon^2\right); \tag{89}$$

substituting (88b,c) in (87) it follows that:

$$\chi(t)=\varphi_0^2 e^{i2\Omega_0 t}\left\{\nu/(3-7i\mu)+\left[\varphi_0/(18-33i\mu)\right]e^{i\Omega_0 t}\right\}, \tag{90}$$

to first order in the perturbation, there are oscillations at the first two harmonics $2\Omega_0$, $3\Omega_0$ of the short-period mode.

§16 - Free Oscillations of the First-Order Perturbation

Besides the forced oscillations (90) of the in homogeneous perturbation equation (86), there are also free oscillations:

$$\chi(t)=\chi_0 e^{i\Omega t}, \tag{91}$$

of the homogeneous equation:

$$\dddot{\chi}-\Omega_0(\pm 1/\mu-i)\ddot{\chi}+\Omega_0^2\dot{\chi}-\Omega_0^3(\pm 1/\mu-i)\chi=0; \tag{92}$$

substitution of (91) into (92), shows that the frequency Ω satisfies a cubic equation:

$$\left(\Omega^2-\Omega_0^2\right)\left[\Omega+(1\pm i/\mu)\Omega_0\right]=0. \tag{93}$$

We should expect $\Omega=\pm\,\Omega_0$ to be roots, because the short-period frequency is unchanged to first order (89). Since (93) is a cubic, the third root follows immediately:

$$\Omega=\pm\Omega_0,\quad -\Omega_0(1\pm i/\mu), \tag{94a,b,c}$$

and it has an imaginary part.

§17 - Existence of Growing and Decaying Free Oscillations

The third root (94c) corresponds (91) to oscillations at the short-period frequency:

$$\chi(t)=\chi_0 e^{-i\Omega_0 t}\exp\left[\pm(\Omega_0/\mu)t\right], \tag{95}$$

with exponentially growing or decaying amplitude. The two possibilities arise from the indetermination of sign in (72). The time scale τ for the growth or decay is given (76a, 78) by:

$$\tau \equiv \mu/\Omega_0 = 1/2\sqrt{fq} \ , \qquad (96)$$

or from (65, 66a, 76b)

$$1/2\tau = \sqrt{(T/m)(\rho S \overline{C}_D/2m)}. \qquad (97)$$

Thus the instability is more noticeable (shorter τ), for higher thrust-to-mass ratios, and high-drag configurations. This would correlate with the PIO, which tends to occur for aircraft with high thrust-to-weight ratios in situations like landing, where drag is high.

§18 - Discussion

We have exhamined analytically the longitudinal stability of airplanes, beyond the usual assumptions underlying the short-period mode and phugoid, namely small perturbation (II) from steady flight (I). Some of the results, like the airspeed law (20) along a glide slope, have been verified in flight test [20], alongside with other predictions of flight dynamics [21-23]. Here we report some aditional results, e.g. that unsteady stability can lead to double oscillations ressembling 'beats' (*Figure 4*), and non-linear stability can lead to oscillations with amplitude growing on a timescale (97). Both of these remarks could be potential explanations for the PIO phenomenon, which has not been completely solved by control methods, and could have an origin in unsteady or non-linear flight stability. The latter is an important subject, which needs to be better understood, due to its implications on high angle-of-attack flight, spins and other safety issues. Analytical methods, alongside with numerical simulations, have a complementary role in understanding these phenomena.

References

[1] Lanchester, F.W., 1945, *Aerodonetics*, Constable.

[2] von Mises, R., 1945, *Theory of Flight*, Dover.

[3] Perkins, C.D. & Hage, R.E., 1950, *Airplane Performance, Stability and Control*, Wiley, N.Y.

[4] Duncan, W.J., 1952, *Principles of Control and Stability of Aircraft*, Cambridge U.P.

[5] Bryan, G.H., 1911, *Stability in Aviation*, MacMillan

[6] Etkin, B., 1974, *Dynamics of flight stability and control*, Wiley, N.Y.

[7] Babister, A.W., 1980, *Aircraft Dynamic Stability and Response*, Pergamon, Oxford.

[8] McRuer, J. & Askhenas, S., 1986, *Aircraft stability and Control*, McGraw-hill, London.

[9] Houghton, E.L. & Brock, R.E., 1960, *Aerodynamics for engineering students*, Edward Arnold.

[10] Lighthill, M.J., 1986, *An informal Introduction to Fluid Mechanics*, Oxford U.P.

[11] Painlevé, P., 1910, Étude sur le regime normal d'un avion, *Tech. Aeron.* **1**, 3-11.

[12] Neumark, S., 1953, Problems of longitudinal stability below the minimum drag speed, and the theory of stability under a constraint, *ARC Report* 2983

[13] Pinsker, W.J.G., 1971, Glide path stability of an aircraft under a speed constraint, *ARC Report* 3705.

[14] Campos, L.M.B.C. & Aguiar, A.J.M.N., 1989, On the inverse phugoid problem as an instance of non-linear stability in pitch. *Aero. J.* **1559/1**, 241-253.

[15] Watson, G. N., 1944, *Bessel functions*, Cambridge U.P.

[16] Ince, E.L., 1926, *Ordinary differential equations*, Dover.

[17] Forsyth, A.R., 1927, *Treatise of differential equations*, MacMillan.

[18] Landau, L.D. & Lifshitz, E.F., 1953, *Mécanique*, Ed. Mir. Moscow.

[19] Cabannes, H., 1967, *Mécanique*, Dunod, Paris

[20] Campos, L.M.B.C., Fonseca, A.A. & Azinheira, J.R.C., 1995, Some elementary aspects of non--linear airplane speed stability in constrained flight, *Progr. Aerosp. Sci.* **31**, 137-169.

[21] Campos, L.M.B.C., 1984, On the influence of atmospheric disturbances on aircraft aerodynamics, *Aeron. J.* **1085**, 257-264.

[22] Campos, L.M.B.C., 1986, On aircraft flight performance in a perturbed atmosphere, *Aeron. J.* **1305**, 302-312.

[23] Campos, L.M.B.C., 1989, On a pitch control low for constant glide slope through windshears, *Aeron. J.* **1559/2**, 290-300.

Legends for the Figures

Figure 1 - Longitudinal motion of an aeroplane under lift L, drag D, thrust T and weight W, and pitching moment M.

Figure 2 - Short-period oscillation, without damping, for linear airspeed variation as a function of distance along the flight path, specified by a Bessel function.

Figure 3 - As *Figure 2*, but with damping $\lambda = 0.1$, and thus specified by a confluent hypergeometric function.

Figure 4 - As *Figure 3*, for another solution, showing double oscillations or 'beats'.

Figure 5 - Comparison of short-period oscillation, for a linear airspeed variation, without and with damping.

Figure 6 - Short-period oscillation, without damping, for an exponential airspeed as a function of distance along the flight path, specified by a Gaussian hypergeometric function, for the parameters (49a,b).

Figure 7 - As *Figure 6* with damping $\lambda = 0.1$.

Figure 8 - As *Figure 7* with $V_2/V_1 = 2$.

Figure 9 - As *Figure 7* with $\omega\ell = \pi$.

Figure 10 - As *Figure 7* with $V_2/V_1 = 2$ and $\omega\ell = \pi$.

Figure 11 - Comparison of short-period mode, for constant and exponential airspeed, without damping.

Figure 12 - As *Figure 11*, with damping.

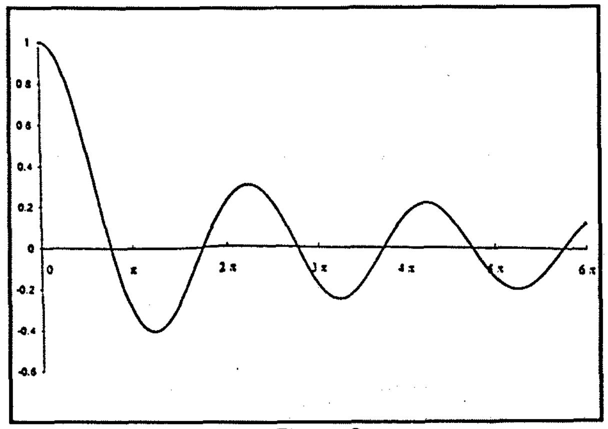

Figure 2

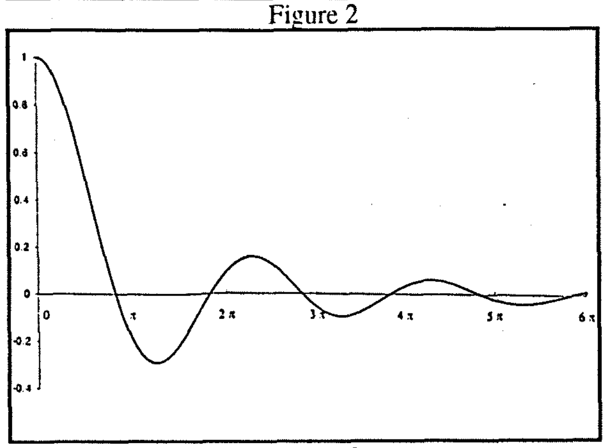

Figure 3

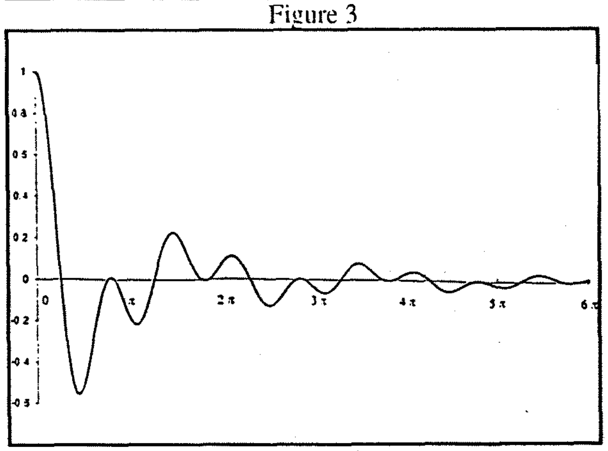

Figure 4

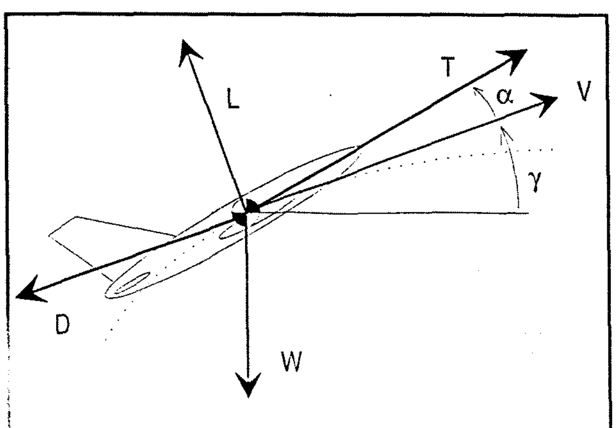

Figure 1

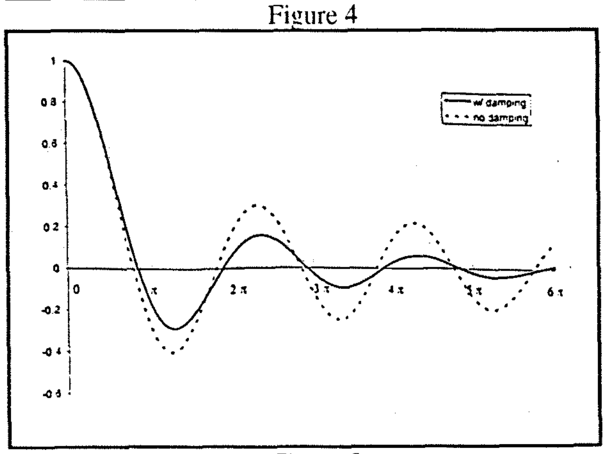

Figure 5

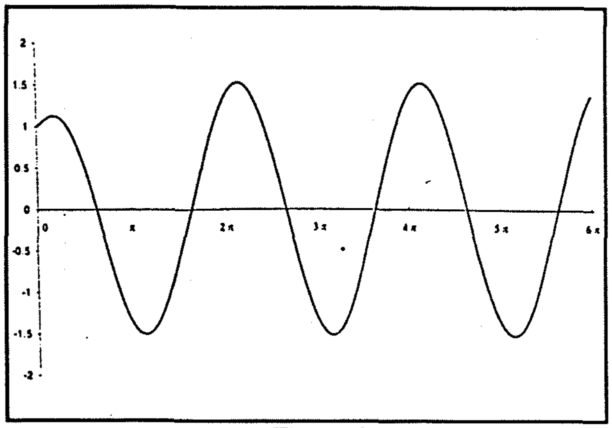

Figure 6

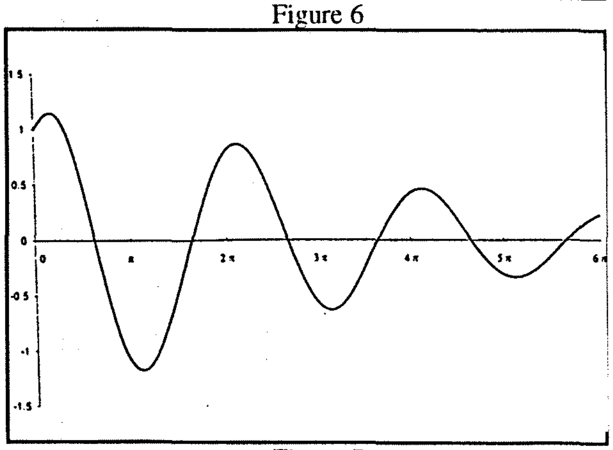

Figure 7

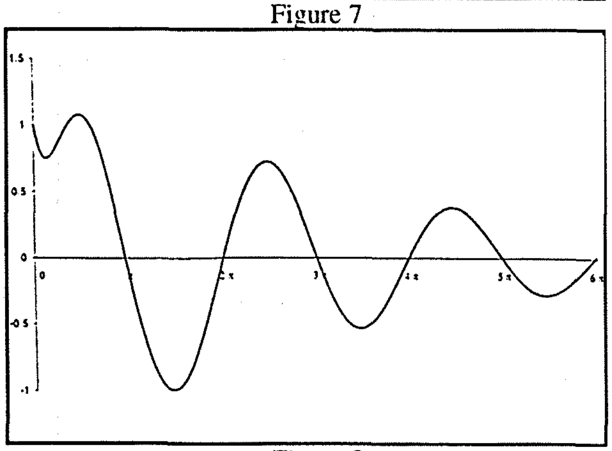

Figure 8

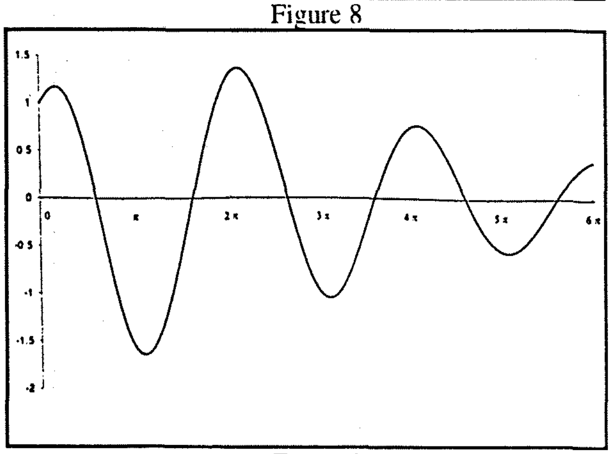

Figure 9

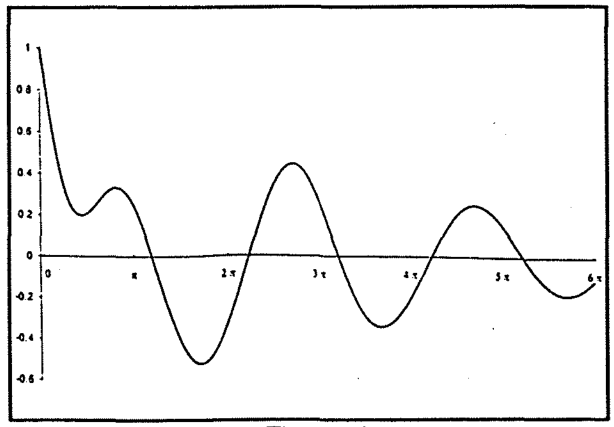

Figure 10

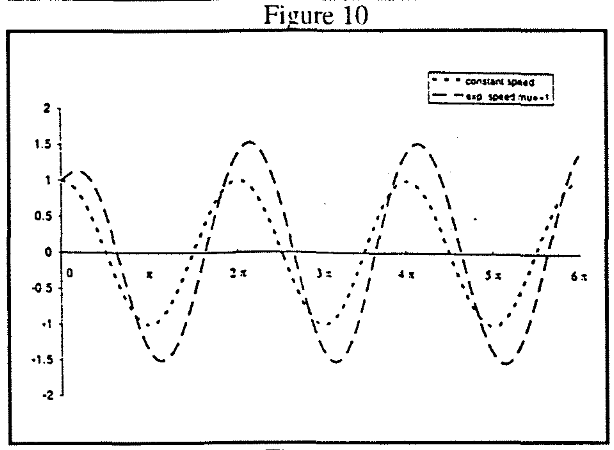

Figure 11

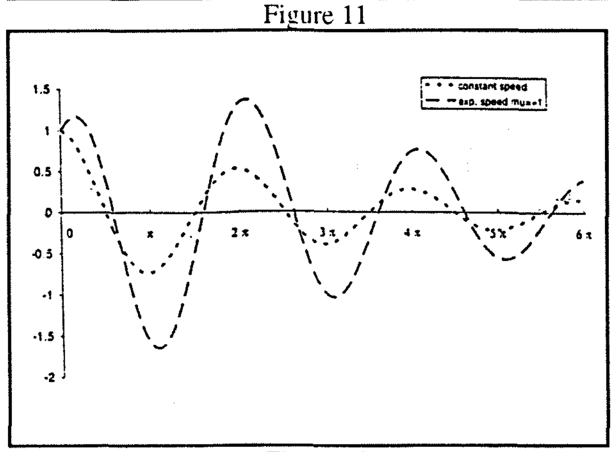

Figure 12