ON THE DEVELOPMENT OF FLEXIBLE AIRCRAFT EQUATIONS OF MOTION

Kapseong Ro
Korea Aerospace Research Institute
P.O. Box 113, Yusung, Taejon 305-600, Korea

Jewel B. Barlow
Department of Aerospace Engineering
University of Maryland, College Park, MD 20742-3215

Abstract

The governing equations of motion for a flexible aircraft are derived based on a combination of Newtonian and energy approaches. Typically, these equations are developed using the energy approach alone since the complicated motion of an unrestrained elastic body in three dimensional space is more easily captured in the form of kinetic and potential energy. However, it is difficult to capture the details that rationalize the final set of equations of motion which are widely used for analysis and design. On the other hand, combined use of Newtonian and energy approaches provides a clearer exposition of the excellent rationale behind these equations. The complete set of equations of motion, which are kinematically decoupled but kinetically coupled, can be grouped into two sets. These equations are valid only if certain assumptions and constraints are satisfied. The objective of this paper is to provide a clear understanding of these equations - necessary assumptions, constraints, limitations - by deriving them via a combined use of Newtonian and energy approaches in a textbook fashion. For the explicit illustration of the complete modeling process, aerodynamic quasi-steady strip theory and small perturbation theory are employed to obtain a state-space form of linearized longitudinal equations of motion.

1.0 Introduction

Many modern aircraft make use of advanced composite material and active control technology in order to increase the overall aircraft performance and/or cost effectiveness. Those technologies result in weight and stiffness reduction and less inherent stability requirement through the changes in structural material and aircraft geometry, but the airframe flexibility effect in the overall aircraft dynamics must be substantial. Traditional treatment in terms of independent flight dynamic and aeroelastic analysis may cause erroneous results due to the strong interactions between those two regimes. In order to carry out necessary dynamic analysis and successful control law synthesis for such aircraft, a single integrated mathematical model which simultaneously reflects both rigid-body and elastic motions must be sought.

Schmidt and his co-workers used the energy approach throughout the process of deriving the integrated equations of motion for a general elastic airplane[1,2], and applied them to develop numerical models of a forward-swept wing aircraft[2], and an elastic hypersonic aircraft.[3] The energy approach is advantageous because the motion of an unrestrained elastic body in inertial space undergoing large and high ratesf of displacement is rather easily captured in kinetic and potential energy expressions. However, to get down to the equations that are widely used for the analysis and design several assumptions have to be made and a mean axis system is required in advance. Furthermore, references 1 and 2 did not mention the vehicle angular velocity and acceleration terms that contribute to the generalized forces in structural equations. Although they disappear in the final set of linear equations for steady rectilinear flight case, they are important for non-rectilinear flight cases because they introduce inertial coupling effects. These terms are very important in case nonlinear flight simulation is needed to study more general motion such as turning flight.

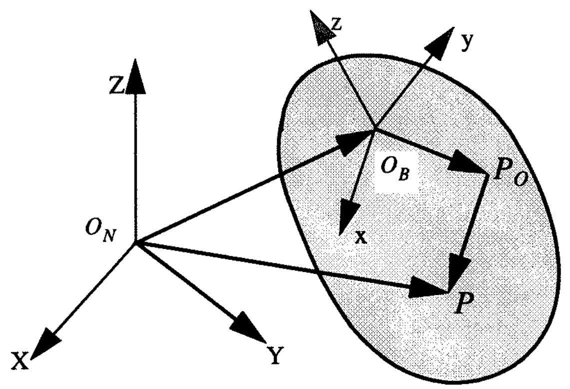
In this paper, the newtonian approach is used in conjunction with the energy approach to develop the integrated equations of motion for a flexible aircraft. Reference 4 very briefly treats this subject. The purpose of present paper is to provide a comprehensive treatment on the dynamic modeling of flexible aircraft, so that one can delineate necessary assumptions, constraints and limitations of resultant mathematical models.

For convenience, the motion is sub-divided into 'Gross Motion' and 'Fine Motion' where the former encompasses the translation of the vehicle mass center and the rotation of aircraft as a whole, and the latter encompasses the motion relative to the moving aircraft such as motion of articulated subsystems and structural components.

2.0 Dynamic Equations for Gross Motion

Consider a general deformable body, and introduce two reference frames, namely an inertial reference frame F_N and a body reference frame F_B which is moving in inertial space (refer to Figure 1). Special care is necessary when introducing a body reference frame because of the structural displacement. In the typical rigid-body dynamic analysis[4], one usually attaches the origin and axes of the body reference frame to a set of material points and introduces the constraint that the coordinates of each mass element are constant in the body reference frame. However, thhis is not possible for a deformable body since the structural distortion results in displacements of material particles relative to any triadic orthonormal based reference frame. In addition, the inertial position of F_B in general may contain not only the mean displacement, but also contributions from the structural displacement. Consequently, the governing dynamic equations can be highly coupled by the rigid and deformation variables. Subsequent analysis and design, based on such equations, may be unnecessarily complicated as significant simplifications are obtainable by imposing certain constraints on the chosen body reference frame and by assuming structural deformations which are sufficiently small. Such a body reference frame is a kind of '*Mean Reference Frame*' and will be discussed shortly. In the meantime, introduce a fictitious body reference frame whose origin, O_B and coordinates are not attached to a material point but fixed at a particular point of interest in the body. Consider an infinitesimal mass element, ρdV located at P_o in the undeformed body configuration as shown in Figure 1.

Figure 1 Flexible Body Motion in 3D Space



As deformation begins, the position of the mass element changes. Letting P be the position of the mass element of the deformed body configuration and denoting by $\vec{R}^{XY}$ a position vector from a point X to a point Y, then $\vec{R}^{P_oP}$ represents the structural displacement vector. It is easy to see

$$\vec{R}^{O_BP} = \vec{R}^{O_BP_o} + \vec{R}^{P_oP} \tag{1.a}$$

$$\frac{{}^Bd}{dt}\vec{R}^{O_BP} = \frac{{}^Bd}{dt}\vec{R}^{O_BP_o} + \frac{{}^Bd}{dt}\vec{R}^{P_oP} \tag{1.b}$$

where ${}^Bd/dt$ represents the time derivative of a vector relative to the body reference frame. It must be noticed that the first term in the right-hand side of equation (1.b) may not be zero due to the relative motion of articulated subsystems, such as engine compressors, fans, rotors, etc. These relative motions appear simply as additional contributions to the angular momentum. Further development disregards the effects of such articulated motion, and only the structural distortion is considered as relative motion. Now, the translational motion of mass element ρdV in inertial space acted upon external force of $d\vec{F}$, is described using Newton's equation,

$$d\vec{F} = \left(\frac{{}^Nd^2}{dt^2}\vec{R}^{O_NP}\right)\rho dV \tag{2}$$

where ${}^Nd/dt$ represents the time derivative of a vector relative to the inertial reference frame. Using the vector relation $\vec{R}^{O_NP} = \vec{R}^{O_NO_B} + \vec{R}^{O_BP}$ and the Coriolis vector differentiation rule,

$$\frac{{}^Nd}{dt}(\vec{R}) = \frac{{}^Bd}{dt}(\vec{R}) + {}^N\vec{\omega}^B \times \vec{R} \tag{3}$$

equation (2) can be expressed as

$$d\vec{F} = \left[\frac{{}^Nd}{dt}{}^N\vec{V}^{O_B} + \frac{{}^Bd}{dt}\left(\frac{{}^Bd}{dt}\vec{R}^{O_BP}\right) + ({}^N\vec{\alpha}^B \times \vec{R}^{O_BP}) + \left(2{}^N\vec{\omega}^B \times \frac{{}^Bd}{dt}\vec{R}^{O_BP}\right) + ({}^N\vec{\omega}^B \times {}^N\vec{\omega}^B \times \vec{R}^{O_BP})\right]\rho dV \tag{4}$$

where ${}^N\vec{V}^{O_B}$, ${}^N\vec{\omega}^B$, ${}^N\vec{\alpha}^B$ represents the linear velocity, angular velocity and angular acceleration of the body reference frame relative to the inertial reference frame. Integrating over the entire volume results in a force equation for the whole body, i.e.

$$\int_V d\vec{F} = \int_V \rho dV\left(\frac{{}^Nd}{dt}{}^N\vec{V}^{O_B}\right) + \int_V \frac{{}^Bd^2}{dt^2}\vec{R}^{O_BP}\rho dV + \left({}^N\vec{\alpha}^B \times \int_V \vec{R}^{O_BP}\rho dV\right) + \left(2{}^N\vec{\omega}^B \times \int_V \frac{{}^Bd}{dt}\vec{R}^{O_BP}\rho dV\right) + \left({}^N\vec{\omega}^B \times {}^N\vec{\omega}^B \times \int_V \vec{R}^{O_BP}\rho dV\right) \tag{5}$$

If the origin of the body reference frame is constrained to coincide with the mass center B^* for every instant, i.e. *the instantaneous center of mass of the deformed body*, (**Constraint 1**) then by definition

$$\int_V \vec{R}^{B^*P}\rho dV = 0 \quad \forall t \tag{6.a}$$

and by consequence

$$\int_V \frac{{}^Bd}{dt}\vec{R}^{B^*P}\rho dV = \int_V \frac{{}^Bd}{dt}\vec{R}^{P_oP}\rho dV = 0 \tag{6.b}$$

Applying (6) to (5) results in

$$\vec{F} = M_T\left(\frac{{}^Nd}{dt}{}^N\vec{V}^{B^*}\right) = M_T{}^N\vec{a}^{B^*} \tag{7}$$

where $\vec{F} = \int_V d\vec{F}$ and $M_T = \int_V \rho dV$.

Equation (7) governs the translational motion of a deformable body moving in inertial space. Notice that the deformation terms vanish by virtue of choosing the instantaneous mass center as the origin of the body reference frame. This result appears in many standard texts and herein simply establishes the notation and context for what follows.

Next, the rotational motion of the body is considered by taking the cross product of (2) with the position vector $\vec{R}^{O_N P}$,

$$\vec{R}^{O_N P} \times d\vec{F} = \vec{R}^{O_N P} \times \left(\frac{{}^N d^2}{dt^2} \vec{R}^{O_N P} \right) \rho dV \tag{8}$$

which yields the moment of the external force about the origin of the inertial reference frame which acts on the infinitesimal mass element ρdV. Integrating (8) over the entire volume yields

$$\vec{M}^{O_N} = \int_V \vec{R}^{O_N P} \times \left(\frac{{}^N d^2}{dt^2} \vec{R}^{O_N P} \right) \rho dV \tag{9}$$

where

$$\vec{M}^{O_N} = \int_V \vec{R}^{O_N P} \times d\vec{F} \tag{10}$$

is the resultant external moment applied to the body about O_N. Next, the angular momentum of infinitesimal mass element about O_N is defined as

$$d({}^N\vec{H}^{O_N}) = \vec{R}^{O_N P} \times \left(\frac{{}^N d}{dt} \vec{R}^{O_N P} \right) \rho dV \tag{11}$$

Similarly, integrating (11) over the entire volume yields the total angular momentum of the body about O_N,

$${}^N\vec{H}^{O_N} = \int_V d({}^N\vec{H}^{O_N}) \rho dV = \int_V \vec{R}^{O_N P} \times \frac{{}^N d}{dt} \vec{R}^{O_N P} \rho dV \tag{12}$$

Now, differentiating (12) relative to the inertial reference frame results in

$$\vec{M}^{O_N} = \frac{{}^N d}{dt} {}^N\vec{H}^{O_N} \tag{13}$$

which implies that the time rate of change of angular momentum of the body about a point fixed in inertial space is equal to the resultant moment applied to the body about the same inertially fixed point. However, it is referenced to a point fixed in inertial space, and the computation of angular momentum is dependent upon the current inertial position of the body. This could be avoided if a body fixed reference point were chosen. If a body fixed point such as O_B is considered, then using the vector equation

$$\vec{R}^{O_N P} = \vec{R}^{O_N O_B} + \vec{R}^{O_B P}$$

the angular momentum of the body about O_N can be expressed as

$$\begin{aligned} {}^N\vec{H}^{O_N} = {} & \left(\vec{R}^{O_N O_B} \times M_T \frac{{}^N d}{dt} \vec{R}^{O_N O_B} \right) + \left(\int_V \vec{R}^{O_B P} \times \frac{{}^N d}{dt} \vec{R}^{O_B P} \rho dV \right) \\ & + \left(\int_V \vec{R}^{O_B P} \rho dV \times \frac{{}^N d}{dt} \vec{R}^{O_N O_B} \right) + \left(\int_V \vec{R}^{O_B P} \times \frac{{}^N d}{dt} \vec{R}^{O_B P} \rho dV \right) \end{aligned} \tag{14}$$

Again, letting O_B be the instantaneous center of the body as in the translational motion, then (14) becomes

$${}^N\vec{H}^{O_N} = \left(\vec{R}^{O_N B^*} \times M_T \frac{{}^N d}{dt} \vec{R}^{O_N B^*} \right) + {}^N\vec{H}^{B^*} \tag{15}$$

where ${}^N\vec{H}^{B^*}$ represents the *'Central Angular Momentum'* which is defined as

$${}^N\vec{H}^{B^*} = \int_V \vec{R}^{B^* P} \times \frac{{}^N d}{dt} \vec{R}^{B^* P} \rho dV \tag{16}$$

It must be noted that (16) only holds for the instantaneous mass center of the body. Differentiating (16) relative to the inertial reference frame leads to

$$\vec{M}^{B^*} = \frac{{}^N d}{dt} {}^N\vec{H}^{B^*} \tag{17}$$

Equation (17) is similar to (13) but the reference point is the instantaneous mass center so that the computation of angular momentum of the body is less demanding. Now, to complete the rotational motion of the body, the central angular momentum must be evaluated. Starting from (17), one can obtain

$${}^N\vec{H}^{B^*} = \underset{\sim}{I}^{B^*} \cdot {}^N\vec{\omega}^B + \int_V \vec{R}^{B^* P_o} \times \frac{{}^B d}{dt} \vec{R}^{P_o P} \rho dV + \int_V \vec{R}^{P_o P} \times \frac{{}^B d}{dt} \vec{R}^{P_o P} \rho dV \tag{18}$$

where $\underset{\sim}{I}^{B^*}$ represents the *'Central Inertia Dyadic'* of the deformed body defined as

$$\underset{\sim}{I}^{B^*} = \int_V \{ (\vec{R}^{B^* P} \cdot \vec{R}^{B^* P}) \underset{\sim}{U} - \vec{R}^{B^* P} \vec{R}^{B^* P} \} \rho dV \tag{19}$$

where U is unit dyadic. It must be noted that the central inertia dyadic is time dependent in the body reference frame due to structural deformation. Unlike the expression for rigid-body angular momentum, additional terms are present which produce substantial couplings between the rigid-body and structural motion variables. However, such coupling effects can be eliminated by imposing an additional constraint on the body reference frame and some assumptions on the characteristics of structural deformation. In order to eliminate the second term from (19) an additional constraint is imposed on the chosen body reference frame (**Constraint 2**) such that

$$\int_V \vec{R}^{B^* P_o} \times \frac{{}^B d}{dt} \vec{R}^{P_o P} \rho dV = 0 \quad \forall t \tag{20}$$

For small structural deformations, it can be assumed that the rate of structural displacement is collinear with its position vector (**Assumption 1**). Thus, the third term in (18) may be dropped. Therefore, the central angular momentum equation becomes

$${}^N\vec{H}^{B^*} = \underset{\sim}{I}^{B^*} \cdot {}^N\vec{\omega}^B \tag{21}$$

and finally the rotational equations of motion may be expressed as

$$\vec{M}^{B^*} = \frac{{}^N d}{dt}{}^N\vec{H}^{B^*} = \frac{{}^N d}{dt}(\underset{\sim}{I}^{B^*} \cdot {}^N\vec{\omega}^B) \tag{22}$$

Equations (7) and (22) along with the body reference frame constraints (6.a) and (20) provide a basis for computing the gross motion of a flexible aircraft. In order to carry out the analysis and numerical study, it is necessary to write these vector equations into their component forms in the body reference frame. Let $\vec{X}_Y$ be the general notation for the component form of a vector $\underset{\sim}{X}$ in a reference frame F_Y, then $\vec{F}_B, \vec{M}_B^{B^*}, {}^N\vec{V}_B^{B^*}, {}^{N}\vec{\omega}_B^{B}$ and $\underset{\sim}{I}_B^{B^*}$ are the component forms of external force, moment, velocity, angular velocity and central inertia dyadic in the body reference frame. Let these component forms be denoted by

$$\vec{F}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \vec{M}_B^{B^*} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}, \quad {}^N\vec{V}_B^{B^*} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad {}^N\vec{\omega}_B^{B} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\underset{\sim}{I}_B^{B^*} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

Moving on, the component form of gross motion can be obtained as

$$\vec{F}_B = M_T \frac{{}^N d}{dt}{}^N\vec{V}_B^{B^*} = M_T\left({}^N\dot{\vec{V}}_B^{B^*} + {}^N\underset{\sim}{\omega}_B^{B}\,{}^N\vec{V}_B^{B^*}\right) \tag{23}$$

$$\begin{aligned} \vec{M}_B^{B^*} &= \frac{{}^N d}{dt}\left(\underset{\sim}{I}_B^{B^*} \cdot {}^N\vec{\omega}^{B}\right) \\ &= \dot{\underset{\sim}{I}}_B^{B^*}\,{}^N\vec{\omega}_B^{B} + \underset{\sim}{I}_B^{B^*}\,{}^N\dot{\vec{\omega}}_B^{B} + {}^N\underset{\sim}{\omega}_B^{B}\left(\underset{\sim}{I}_B^{B^*}\,{}^N\vec{\omega}_B^{B}\right) \end{aligned} \tag{24}$$

where ${}^N\underset{\sim}{\omega}_B^{B}$ represents the skew symmetric form of the angular velocity. However, structural deformation still contributes to the time varying components of the central inertia dyadic. By assuming that the structural deformation is sufficiently small to ignore the time-varying components of the central inertia dyadic (**Assumption 2**), the first term on the right-hand side of (24) disappears and becomes a typical set of euler equations for a rigid-body. Along with (23) and (24), the inertial orientation of the body reference frame is related to the body axis angular velocity components by the kinematic relations,

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & \sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{25}$$

where ψ, θ, ϕ are the Euler angles which are the present choice of variables to define the vehicle's rotational position. These are referred to as yaw or azimuth angle, pitch or elevation angle and roll or bank angle, respectively. It is emphasized that there is no contribution from structural motion to the equations for the gross motion, and the governing equations motion become identical to those of the rigid-body aircraft case. However, this is possible only when the chosen body reference frame satisfies the constraints of (6.a) and (20). From the theory of structures[5], the structural displacements can be expressed by superimposing the product of a spatial function and a time function,

$$\vec{R}^{P_oP} = \sum_{j=1}^{\infty} \vec{\Phi}_j(x, y, z)\xi_j(t) \tag{26}$$

Then, the constraints, expressions (6.a) and (20), can be written as

$$\int_V \vec{R}^{B^*P} \rho dV = \sum_{j=1}^{\infty} \int_V \vec{\Phi}_j(x, y, z)\rho dV \xi_j(t) \tag{27.a}$$

$$\int_V \vec{R}^{B^*P} \times \frac{{}^B d}{dt}\vec{R}^{P_oP} \rho dV = \sum_{j=1}^{\infty} \int_V \vec{R}^{B^*P_o} \times \vec{\Phi}_j(x, y, z)\rho dV \dot{\xi}_j(t) \tag{27.b}$$

2.1 Free Vibration in a Vacuum

The natural modes and frequencies of free vibrations are important physical parameters for a flexible aircraft study. When the natural modes and frequencies of free vibrations are used to represent the structural displacement, the constraint equations (6.a) and (20) are automatically satisfied. By definition, free vibrations mean that no external forces act on the body and its center of mass is not accelerating. The equations of free vibration are

$$\int_V \frac{{}^B d^2}{dt^2}\vec{R}^{P_oP} \rho dV = 0 \tag{28.a}$$

$$\int_V \vec{R}^{B^*P_o} \times \frac{{}^B d^2}{dt^2}\vec{R}^{P_oP} \rho dV = 0 \tag{28.b}$$

Substituting (26) into (28) leads to

$$\int_V \vec{\Phi}_j(x, y, z)\rho dV = 0$$

$$\int_V \vec{R}^{B^*P_o} \times \vec{\Phi}_j(x, y, z)\rho dV = 0$$

where $\bar{\Phi}_j(x, y, z)$ is an eigenfunction in vector form which represents the j^{th} mode shape of free vibration, and in turn they guarantee the constraint conditions from the previous section. In addition to (28), there are the equations for the internal force equilibrium of the free vibrations. These equations may be in the form of vector integral equations derived from stress-strain-displacement relations, or using approximation methods such as Galerkin's method, the Rayleigh-Ritz method or the Finite Element method. The equations may be expressed as

$$[M][\ddot{q}] + [K][q] = [0] \tag{29}$$

where $[M]$, $[K]$ are the generalized mass matrix and the stiffness matrix, and $[q]$ is the generalized displacement vector. After finding such matrices, one can obtain a solution for (29) by solving the generalized eigenvalue problem,

$$\omega^2[M][\phi] = [K][\phi]$$

which gives the free vibration frequencies ω_i^2 and mode shapes $[\phi_i]$. There exist zero frequencies corresponding to rigid-body modes and an infinite number of deformation modes. The natural modes of free vibration are orthogonal to each other and therefore provide great simplicity in deriving the aircraft structural equations of motion.

2.2 Mean Reference Frame

In section 2.0, an arbitrary body reference frame is introduced first and constraints and assumptions are made on the body reference frame and structural deformation to inertially decouple the governing equations between the rigid and structural motion. In other words, a body reference frame can be chosen in such a way that it is very similar to standard reference frames attached to rigid aircraft representations.

Now, it is shown that such a body reference frame is a special kind of mean reference frame. The mean reference frame is defined as a set of body reference frames which satisfy

$$\int_V \frac{{}^B d}{dt}\vec{R}^{O_B P}\rho dV = 0 \;\; \forall t \tag{30.a}$$

$$\int_V \vec{R}^{O_B P} \times \frac{{}^B d}{dt}\vec{R}^{O_B P}\rho dV = 0 \;\; \forall t \tag{30.b}$$

In other words, the mean reference frame is a body reference frame in which the resultant linear and angular momentum of the relative motion vanish at every instant. Again, only structural motion is considered as relative motion. Unlike the chosen body reference frame in the previous section, the origin of the mean reference frame does not necessarily coincide with the instantaneous mass center. If the chosen body reference frame is regarded as a mean reference frame whose origin doesn't coincide with the instantaneous mass center then

$$\vec{R}^{O_B P} = \vec{R}^{O_B B^*} + \vec{R}^{B^* P_o} + \vec{R}^{P_o P} \tag{31}$$

Substituting into (30.a) results in

$$\int_V \frac{{}^B d}{dt}\vec{R}^{O_B P}\rho dV = \int_V \frac{{}^B d}{dt}\vec{R}^{O_B B^*}\rho dV + \int_V \frac{{}^B d}{dt}\vec{R}^{P_o P}\rho dV = 0$$

For a deformable body, $\frac{{}^B d}{dt}\vec{R}^{O_B B^*}$ does not vanish. However, if structural deformation is assumed to be small enough to ignore the time varying components of the inertia dyadic due to structural motion then that is the same as assumption 2 in the previous section and the first term can be dropped. This may be proved using the parallel axis theorem,

$$\underline{I}^{O_B} = \underline{I}^{B^*} + M_T(\vec{R}^{O_B B^*} \cdot \vec{R}^{O_B B^*})\underline{U} \tag{32}$$

and differentiating with respect to time relative to the body reference frame

$$\frac{{}^B d}{dt}\underline{I}^{O_B} = \frac{{}^B d}{dt}\underline{I}^{B^*} + 2M_T\left(\vec{R}^{O_B B^*} \cdot \frac{{}^B d}{dt}\vec{R}^{O_B B^*}\right)\underline{U}$$

By assumption 2, $\frac{{}^B d}{dt}\underline{I}^{O_B}$ and $\frac{{}^B d}{dt}\underline{I}^{B^*}$ are zero and this leads

$$\frac{{}^B d}{dt}\vec{R}^{O_B B^*} = 0$$

which implies that any point which is a fixed distance from the instantaneous mass center qualifies as the origin of the mean reference frame. Therefore, the mean reference frame constraints (30.a) are simplified to

$$\int_V \frac{{}^B d}{dt}\vec{R}^{P_o P}\rho dV = 0 \tag{33.a}$$

Similarly, the mean reference frame condition (30.b) is also simplified to

$$\int_V \vec{R}^{B^* P_o} \times \frac{{}^B d}{dt}\vec{R}^{P_o P}\rho dV = 0 \tag{33.b}$$

Thus far, the origin of the mean reference frame does not coincide with the instantaneous mass center. When the origin does not coincide with the instantaneous mass center, but is a fixed distance from it, the translation equation (5) may not be simplified to (7). Similar arguments also apply to the rotational equation. Therefore, to further simplify the dynamic equations, as in the previous section, the origin of the mean reference frame must be constrained to coincide with the instantaneous mass center of the body. For such a case,

$$\int_V \vec{R}^{B^* P_o}\rho dV = \int_V \vec{R}^{P_o P}\rho dV = 0 \tag{34}$$

which serves as an additional condition for the mean reference frame in addition to equations (30). However, this additional constraint becomes identical to (30.a) when expressed in terms of free vibration modes. Therefore, the conditions for a mean reference frame whose origin coincides with the instantaneous mass center are simplified to

$$\int_V \vec{\Phi}_j(x, y, z)\rho dV = 0 \tag{35.a}$$

$$\int_V \vec{R}^{B^* P_o} \times \vec{\Phi}_j(x, y, z)\rho dV = 0 \tag{35.b}$$

Note that these constraints are identical to (27) in the previous section, so that the body reference frame used in section 2.0 is regarded as a mean reference frame with its origin at the instantaneous mass center.

3.0 Dynamic Equations for Fine Motion

In section 2.0, the dynamic equations which govern the gross motion of a flexible aircraft were derived. In section 2.1, results from free vibration theory was stated and utilized as a basis for choosing coordinates to describe structural deformation. In this section dynamic equations which govern the aircraft structural motion in more general circumstances are derived. Due to the infinite dimensional nature of the structural motion, these additional dynamic equations are conveniently derived using the Lagrangian dynamic formulation,

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\xi}_j}T_{rel}\right)-\left(\frac{\partial}{\partial\xi_j}T_{rel}\right)+\frac{\partial}{\partial\xi_j}U = \Lambda_j \quad \text{for} \quad i = 1, \ldots, \infty \tag{36}$$

where T_{rel} is the kinetic energy of the structural motion relative to the body reference frame,

$$T_{rel} = \frac{1}{2}\int_V\left(\left({}^B\frac{d}{dt}\vec{R}^{B*P}\right)\cdot\left({}^B\frac{d}{dt}\vec{R}^{B*P}\right)\right)\rho dV \tag{37}$$

U is the strain energy and Λ_j is the generalized external force. Substituting (26) into the kinetic energy expression (37) leads to

$$T_{rel} = \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_V(\vec{\Phi}_i(x,y,z)\cdot\vec{\Phi}_j(x,y,z))\rho dV\dot{\xi}_i(t)\dot{\xi}_j(t)$$

If free vibration modes are used to express the structural displacement then by the orthogonality condition of free vibration modes the kinetic energy becomes

$$T_{rel} = \frac{1}{2}\sum_{i=1}^{\infty}M_i\dot{\xi}_i^2 \tag{38}$$

where M_i is the i^{th} generalized mass which is defined as

$$M_i = \int_V\vec{\Phi}_i^2(x,y,z)\rho dV$$

Again from the theory of structures the strain energy is conveniently expressed using free vibration modes,

$$U = \frac{1}{2}\sum_{i=1}^{\infty}M_i\omega_i^2\xi_i^2 \tag{39}$$

where ω_i is the i^{th} undamped natural frequency of structural modes. Substituting the kinetic and strain energy expression into Lagrange's equation results in

$$\ddot{\xi}_i(t)+\omega_i^2\xi_i(t) = \frac{1}{M_i}\Lambda_i \text{ for } i = 1, \ldots, \infty \tag{40}$$

In case the structural damping is included, then

$$\ddot{\xi}_i(t)+2\zeta_i\omega_i\dot{\xi}_i(t)+\omega_i^2\xi_i(t) = \frac{1}{M_i}\Lambda_i \quad \text{for } i = 1, \ldots, \infty \tag{41}$$

where ζ_i is structural modal damping factor which must be determined from an experimental measurement. Equation (41) governs the motion of the aircraft structure relative to the body reference frame.

As equation (41) indicates, there are an infinite number of equations which govern the structural motion. This set of equations appears to be uncoupled, but the generalized force remains to be evaluated in more detail. Some judgement and experience is needed to select a finite number of equations for practical usage such as flight simulation and control system design.

4.0 Forces and Moments

This section discusses the external forces and the generalized forces appearing in the equaitions for fine motion for a flexible aircraft flying in the atmosphere. In general, the net external forces in atmospheric flight are aerodynamic (including propulsive reaction) and gravitational. The aerodynamic force acts on the vehicle surface, and can be categorized into two types. The first, denoted by $\vec{f}^M$, is the vehicle's motion dependent aerodynamic force per unit area which depends not only on the instantaneous motion of the vehicle but also on all previous conditions of motion. The second is the aerodynamic force per unit area, denoted by $\vec{f}^D$, which arises from an atmospheric disturbance or a reactive source. Therefore, the total aerodynamic force on the vehicle may be expressed as

$$\vec{A} = \int_S(\vec{f}^M+\vec{f}^D)dS$$

The total external force must include the gravitational force, and hence

$$\vec{F} = \vec{A}+M_T\vec{g}$$

Since the component form of the gravitational force is expressed easily in a vehicle-carried vertical frame F_V it is convenient to introduce the transformation matrix from F_V to F_B, denoted by L_{BV}. Then the component form of total external force in body reference axes is

$$\vec{F}_B = \vec{A}_B+M_T\vec{g}_B = \vec{A}_B+M_TL_{BV}\vec{g}_V \tag{42}$$

where $\vec{A}_B = \begin{bmatrix}X & Y & Z\end{bmatrix}^T$, $\vec{g}_V = \begin{bmatrix}0 & 0 & g\end{bmatrix}^T$ and

$$L_{BV} = \begin{bmatrix}\cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta\end{bmatrix}$$

The external torque acting about the aircraft center of mass is evaluated by taking the cross product of all sources of external force with the position vector $\vec{R}^{B^*P}$,

$$\vec{M}^{B^*} = \int_S \vec{R}^{B^*P} \times (\vec{f}^M + \vec{f}^D)\rho dS + \int_V \vec{R}^{B^*P} \times \vec{g}\rho dV$$

which can be further simplified as

$$\vec{M}^{B^*} = \int_S \vec{R}^{B^*P} \times (\vec{f}^M + \vec{f}^D)\rho dS \tag{43}$$

It can be expressed in body axis component form as $\vec{M}_B^{B^*} = [L\ M\ N]^T$ where L, M, N are rolling, pitching and yawing moment, respectively.

The aerodynamic forces and moments are strictly functions of the modal amplitudes, rates, and accelerations appearing in the fine motion description as well as the motion variables of the mean axes. The strongest of these dependencies are the principal sources of coupling between the fine and gross motion.

Next, the generalized external force must be determined for the structural equations. This may be evaluated from the principle of virtual work,

$$\Lambda_i = \frac{\partial(\delta W)}{\partial(\delta\xi_i)} \tag{44}$$

where W is work done by all the external forces through a virtual displacement. It must be emphasized that the structural equations are based on the body reference frame which is non-inertial so that the external forces must include the inertial force associated with non-uniform motion of the body reference frame as well as the aerodynamic and gravitational forces. Consequently, the total virtual work may be written as

$$\delta W = \delta W^{Inertia} + \delta W^{Gravity} + \delta W^{Air} \tag{45}$$

where $\delta W^{Inertia}$, $\delta W^{Gravity}$ and δW^{Air} are the virtual work done by inertia force, gravitational force and aerodynamic force, respectively. Let $d(\delta W^{Inertia})$ be the virtual work done for an infinitesimal mass element by the inertia force. Then,

$$d(\delta W^{Inertia}) = d\vec{F}^I \cdot \delta\vec{R}^{B^*P} \tag{46}$$

where $d\vec{F}^I$ is the inertial force acting on the infinitesimal mass element, and $\delta\vec{R}^{B^*P}$ is a virtual displacement. Taking the gross motion of the aircraft into account, the inertia force is given by

$$dF^I = \left(\frac{{}^B d^2}{dt^2}\vec{R}^{B^*P} - \frac{{}^N d^2}{dt^2}\vec{R}^{O_N P}\right)\rho dV \tag{47}$$

After necessary substitution and algebraic manipulations among equations (4), (26), (46) and (47), the virtual work done by the inertia force becomes

$$\delta W^{Inertia} = -{}^{N}\vec{\alpha}^{B} \cdot \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_V \vec{\Phi}_i(x,y,z) \times \vec{\Phi}_j(x,y,z)\rho dV \xi_j \delta\xi_i$$
$$-2\,{}^{N}\vec{\omega}^{B} \cdot \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_V \vec{\Phi}_i(x,y,z) \times \vec{\Phi}_j(x,y,z)\rho dV \dot{\xi}_j \delta\xi_i$$

The virtual work done by gravitational force vanishes since

$$\delta W^{Gravity} = \int_V \vec{g} \cdot \delta\vec{R}^{B^*P}\rho dV = \vec{g} \cdot \sum_{i=1}^{\infty}\int_V \vec{\Phi}_i(x,y,z)\rho dV \delta\xi_i = 0$$

Similarly, the virtual work done by aerodynamic force and moment is

$$\delta W^{Air} = \sum_{i=1}^{\infty}\int_S (\vec{f}^M + \vec{f}^D) \cdot \vec{\Phi}_i(x,y,z)\rho dS \delta\xi_i$$
$$+ \sum_{i=1}^{\infty}\int_S \vec{R}^{B^*P} \times (\vec{f}^M + \vec{f}^D) \cdot \vec{\Phi}_i(x,y,z)\rho dS \delta\xi_i$$

Substituting each virtual work expression into (45) leads to the total virtual work

$$\delta W = {}^{N}\vec{\alpha}^{B} \cdot \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_V \vec{\Phi}_i(x,y,z) \times \vec{\Phi}_j(x,y,z)\rho dV \xi_j \delta\xi_i$$
$$-2\,{}^{N}\vec{\omega}^{B} \cdot \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_V \vec{\Phi}_i(x,y,z) \times \vec{\Phi}_j(x,y,z)\rho dV \dot{\xi}_j \delta\xi_i$$
$$+ \sum_{i=1}^{\infty}\int_S (\vec{f}^M + \vec{f}^D) \cdot \vec{\Phi}_i(x,y,z)\rho dS \delta\xi_i$$
$$+ \sum_{i=1}^{\infty}\int_S \vec{R}^{B^*P} \times (\vec{f}^M + \vec{f}^D) \cdot \vec{\Phi}_i(x,y,z)\rho dS \delta\xi_i \tag{48}$$

and using (48) to evaluate generalized forces with (44) results in

$$\Lambda_i = -{}^{N}\vec{\alpha}^{B} \cdot \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_V \vec{\Phi}_i(x,y,z) \times \vec{\Phi}_j(x,y,z)\rho dV \xi_i$$
$$-2\,{}^{N}\vec{\omega}^{B} \cdot \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_V \vec{\Phi}_i(x,y,z) \times \vec{\Phi}_j(x,y,z)\rho dV \dot{\xi}_i$$
$$+ \int_S (\vec{f}^M + \vec{f}^D) \cdot \vec{\Phi}_i(x,y,z)\rho dS$$
$$+ \int_S \vec{R}^{B^*P} \times (\vec{f}^M + \vec{f}^D) \cdot \vec{\Phi}_i(x,y,z)\rho dS \tag{49}$$

As indicated by the first two terms of equations (49), the aircraft structural motion couples inertially with the angular motion of the body reference frame. In general, as the angular velocity increases the aircraft structural modes become more stiff[6]. The last two terms of equations (49) are the aerodynamic contributions to the generalized forces. These terms indicate aerodynamic coupling between the gross and fine motions.

5.0 Small Perturbation Theory

In the preceding sections, the governing equations were derived for the general motion of a flexible aircraft. Carrying out the matrix-vector multiplication of (23), (24) and (25) with structural equations (41) results in six scalar equations for gross motion, three equations for attitude of the aircraft and an infinite number of structural equations of motion. These equations contain the products and transcendental functions of motion variables and are therefore generally nonlinear equations. However, they can be greatly simplified by considering the motion of aircraft as the composite of a steady motion at an equilibrium and a dynamic motion which accounts for small perturbation about the equilibrium. Denoting the equilibrium values by a subscript e and changes from them by the prefix Δ. For example, $u = U_e + \Delta u$, $\phi = \Phi_e + \Delta\phi$, $p = P_e + \Delta p$, $X = X + \Delta X$, $\xi_i = \Xi_i + \Delta\xi_i$ etc. If a specific equilibrium condition of interest is *steady, straight symmetric with wings-level flight,* then the governing dynamic equations for gross motion about the equilibrium point, provided a plane of symmetry C_{xz} exists, are simplified and grouped into longitudinal and lateral/directional equations of motion. These equations are summarized as following.

Longitudinal Equations of Motion

$$\Delta X = M_T\{\Delta\dot{u} + W_e\Delta q + (g\cos\Theta_e)\Delta\theta\} \tag{50.a}$$

$$\Delta Z = M_T\{\Delta\dot{w} - U_e\Delta q + (g\cos\Theta_e)\Delta\theta\} \tag{50.b}$$

$$\Delta M = I_{yy}\Delta\dot{q} \tag{50.c}$$

$$\Delta q = \Delta\dot{\theta} \tag{50.d}$$

Lateral Equations of Motion

$$\Delta Y = M_T\{\Delta\dot{v} + U_e\Delta r - (g\cos\Theta_e)\Delta\phi\} \tag{51.a}$$

$$\Delta L = I_{xx}\Delta\dot{p} - I_{xz}\Delta\dot{r} \tag{51.b}$$

$$\Delta N = I_{xx}\Delta\dot{r} - I_{xz}\Delta\dot{p} \tag{51.c}$$

$$\Delta p = \Delta\dot{\phi} - \sin\Theta_e\Delta\dot{\psi} \tag{51.d}$$

$$\Delta r = \cos\Theta_e\Delta\dot{\psi} \tag{51.e}$$

Next, the structural equations of motion are considered for the same flight condition. Since the aircraft is in a steady flight condition, the angular acceleration, $\dot{\alpha}_{N,B}$, and the angular velocity, $P_e = Q_e = R_e = 0$, is zero. The small perturbational angular velocity terms are multiplied by structural modal amplitudes which are also assumed to be small so that the products of these two quantities are to b e neglected. Consequently the contribution of the angular motion of the aircraft (body reference frame) to structural motion vanishes, and only aerodynamic forces remain as generalized forces acting on the structure. Therefore, (49) becomes

$$\Lambda_i = \int_S (\vec{f}^M + \vec{f}^D)\cdot\vec{\Phi}_i(x,y,z)\rho dS + \int_S \vec{R}^{B*P}\times(\vec{f}^M + \vec{f}^D)\cdot\vec{\Phi}_i(x,y,z)\rho dS \tag{52}$$

For symmetric flight, (52) can be written as

$$\Lambda_i = \int_S \{(\vec{f}_x^M + \vec{f}_x^D)\varphi_x^i(x,y,z) + (\vec{f}_z^M + \vec{f}_z^D)\varphi_z^i(x,y,z) + m_y\varphi_y^i(x,y,z)\}dS \tag{53}$$

by substituting

$$(\vec{f}^M + \vec{f}^D) = (\vec{f}_x^M + \vec{f}_x^D)\hat{\imath} + (\vec{f}_z^M + \vec{f}_z^D)\hat{k}$$

$$\vec{R}^{B*P}\times(\vec{f}^M + \vec{f}^D) = m_y\hat{\jmath}$$

$$\vec{\Phi}_i(x,y,z) = \varphi_x^i(x,y,z)\hat{\imath} + \varphi_z^i(x,y,z)\hat{\jmath} + \varphi_y^i(x,y,z)\hat{k}$$

Dividing the aerodynamic force into steady and perturbational terms leads to the perturbational structural equations of motion for symmetric flight,

$$\Delta\ddot{\xi}_i + 2\zeta_i\omega_i\Delta\dot{\xi}_i + \omega_i^2\Delta\xi_i = \Delta\Lambda_i \tag{54}$$

where

$$\Delta\Lambda_i = \int_S(\Delta\vec{f}_x^M + \Delta\vec{f}_x^D)\varphi_x^i dS + \int_S(\Delta\vec{f}_z^M + \Delta\vec{f}_z^D)\varphi_z^i dS + \int_S \Delta m_y\varphi_y^i dS$$

in which Δm_y is a perturbational moment per unit area on the plane of symmetry. It is reemphasized that the aerodynamic forces and moments are dependent upon the general motion of the aircraft which includes not only the gross motion, but also the structural motion. Therefore, these equations are strongly coupled through the aerodynamic forces even though they are inertially uncoupled. Furthermore, motion dependent aerodynamic forces, as will be seen in the next section, introduce damping effects on the structural equations as well as coupling effects.

<u>6.0 Evaluation of Forces and Moments</u>

Typically, the aerodynamic forces on the aircraft are expressed in the wind axes as drag D, side force C and lift L. Using the transformation matrix F_W to F_B,

$$L_{BW} = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix}$$

where α, β are the angle of attack and sideslip angle which are defined as

$$\alpha = \operatorname{atan}\frac{w}{u},\ \beta = \operatorname{asin}\frac{v}{\sqrt{u^2+v^2+w^2}},\ -\pi \le \alpha \le \pi,\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$$

Then, the aerodynamic forces in body axis components can be expressed as

$$\vec{A}_B = L_{BW}\vec{A}_W + \vec{T}_B$$

where $\vec{A}_W = [-D\ -C\ -L]^T$ and $\vec{T}_B = [T_x\ T_y\ T_z]^T$. For a truly symmetric condition, the sideslip angle and side force are zero, so that the longitudinal forces in the body reference frame become

$$X = L\sin\alpha - D\cos\alpha + T_x$$

$$Z = -L\cos\alpha - D\sin\alpha + T_z$$

The perturbational forces are obtained in a similar manner as in the previous section. Assuming no thrust perturbation, they are

$$\Delta X = (L_e \cos\alpha_e + D_e \sin\alpha_e)\Delta\alpha + (\Delta L \sin\alpha_e - \Delta D \cos\alpha_e) \quad (55.a)$$

$$\Delta Z = (L_e \sin\alpha_e - D_e \cos\alpha_e)\Delta\alpha + (\Delta L \cos\alpha_e + \Delta D \sin\alpha_e) \quad (55.b)$$

where

$$\Delta\alpha = \Delta w / U_e \quad (55.c)$$

The perturbational thrust force is assumed to be zero. Using nondimensional coefficients,

$$L = \bar{q} S_{ref} C_L, \quad D = \bar{q} S_{ref} C_D, \quad M = \bar{q} S_{ref} \bar{c} C_M$$

where is $\bar{q}$ dynamic pressure, S_{ref} is the aerodynamic reference area and C_L, C_D, C_M are the lift, drag and moment coefficients. Again, the perturbational quantities can be expressed as

$$\Delta\bar{q} = \rho_\infty U_e (\Delta u + W_e \Delta\alpha) \quad (56.a)$$

$$\Delta L = (\bar{q}_e S_{ref} \Delta C_L + \Delta\bar{q} S_{ref} C_{L_e}) \quad (56.b)$$

$$\Delta D = (\bar{q}_e S_{ref} \Delta C_D + \Delta\bar{q} S_{ref} C_{D_e}) \quad (56.c)$$

where ρ_∞ is the free stream air density. Now, the nondimensional force and moment coefficients in body axes can be defined as

$$C_{X_e} = C_{L_e} \sin\alpha_e - C_{D_e} \cos\alpha_e \quad (57.a)$$

$$C_{Z_e} = -(C_{L_e} \cos\alpha_e + C_{D_e} \sin\alpha_e) \quad (57.b)$$

and the perturbational quantities are expressed as

$$\Delta C_X = \Delta C_L \sin\alpha_e - \Delta C_D \cos\alpha_e \quad (58.a)$$

$$\Delta C_Z = -(\Delta C_L \cos\alpha_e + \Delta C_D \sin\alpha_e) \quad (58.b)$$

Then the perturbation forces and moments in body axes become

$$\Delta X = \bar{q}_e S_{ref} (\Delta C_X - C_{Z_e} \Delta\alpha) + \rho_\infty U_e (\Delta u + W_e \Delta\alpha) S_{ref} C_{X_e} \quad (59.a)$$

$$\Delta Z = \bar{q}_e S_{ref} (\Delta C_Z + C_{X_e} \Delta\alpha) + \rho_\infty U_e (\Delta u + W_e \Delta\alpha) S_{ref} C_{Z_e} \quad (59.b)$$

$$\Delta M = \bar{q}_e S_{ref} \bar{c} \Delta C_M \quad (59.c)$$

where $\bar{c}$ is the wing mean aerodynamic chord. The generalized forces in structural equations can be also expressed as

$$\Lambda_i = -F_i \cos\alpha - G_i \sin\alpha + H_i \quad (60)$$

where $-F_i$, $-G_i$, $-H_i$ are defined as generalized lift, generalized drag and generalized moment, respectively. Again, using nondimensional coefficients,

$$F_i = \bar{q} S_{ref} C_F^i, \quad G_i = \bar{q} S_{ref} C_G^i, \quad H_i = \bar{q} S_{ref} \bar{c} C_H^i$$

where C_F^i, C_G^i, C_H^i are defined as generalized lift, generalized drag, generalized moment coefficients, respectively. Expressing motion variables as the sum of the steady value and perturbational dynamic values, then the steady generalized forces for a symmetric steady flight condition are

$$\Lambda_i^e = \bar{q}_e S_{ref} (-C_{F_e}^i \cos\alpha_e - C_{G_e}^i \sin\alpha_e) + \bar{q}_e S_{ref} \bar{c} C_{H_e}^i \quad (61)$$

and the perturbational generalized forces are expressed as

$$\Delta\Lambda_i = \bar{q}_e S_{ref} (\Delta C_S^i + C_{R_e}^i \Delta\alpha + \bar{c} \Delta C_H^i) + \rho_\infty U_e (\Delta u + W_e \Delta\alpha) S_{ref} C_{S_e}^i \quad (62)$$

where

$$C_{R_e}^i = (-C_{F_e}^i \sin\alpha_e - C_{G_e}^i \cos\alpha_e) \bar{q}_e S_{ref}$$

$$C_{S_e}^i = (-C_{F_e}^i \cos\alpha_e - C_{G_e}^i \sin\alpha_e) \bar{q}_e S_{ref}$$

$$\Delta C_S^i = (-\Delta C_F^i \cos\alpha_e - \Delta C_G^i \sin\alpha_e) \bar{q}_e S_{ref}$$

Evaluation of aerodynamic force and moment on a flexible aircraft is complicated due to their dependence on the structural motion as well as the rigid-body motion. Computing generalized forces is similarly demanding. In this paper, quasi-steady strip theory[1,2,7], which is a relatively simple and crude approximation method, is adopted to illustrate the complete modeling process. Although it is a rough appoximation, the strip theory is very informative because analytical expressions are avaiable for the aerodynamic coefficients. It should also be mentioned that the airflow over an elastic wing is almost always unsteady in nature so that more sophisticated treatment is necessary, especially when aircraft flutter is of interest. Treating this subject is beyond the scope of this paper, and interested readers are directed to further references[8], and a paper in preparation by the present authors.

6.1 Quasi-Steady Strip Aerodynamic Theory

For a lifting surface such as an aircraft wing or tail, it is possible to obtain analytical approximations for the aerodynamic lift and moment as well as the generalized force by the strip theory approximation. For such a case, the total lift from the lifting surface is the sum of the lift produced by each section of the lifting surface which can be computed from two dimensional airfoil theory. The angle of attack at a particular spanwise location is treated as independent of that at other locations. Figure 2 shows a typical strip with a unit spanwise length. Assuming that the lifting surface is modeled as an engineering beam which is flexible in torsion with no spanwise displacement, then the free vibration modes are expressed as

$$\varphi_x^i(x,y,z) = 0\ ,\quad \varphi_z^i(x,y,z) = \varphi_z^i(y)\ ,\quad \varphi_y^i(x,y,z) = \varphi_y^i(y)$$

Now, the angle of attack of a strip can be expressed as

$$\alpha_s = \frac{w}{u} + \frac{w_e}{u} - \frac{q(l_s+e)}{u} - \frac{q_e}{u} + \theta_e + \theta_e^F + i_s + \frac{w_g}{u} \qquad (63)$$

where

- w_e : structural displacement rate in vertical motion
- q_e : structural displacement rate in angular motion
- θ_e : angular structural displacement
- i_s : section incidence angle
- θ_e^F : angular structural displacement of base at the junction point (jp)
- w_g : atmospheric gust

Using free vibration modes and the frequencies equation, structural displacements and rates are expressed as

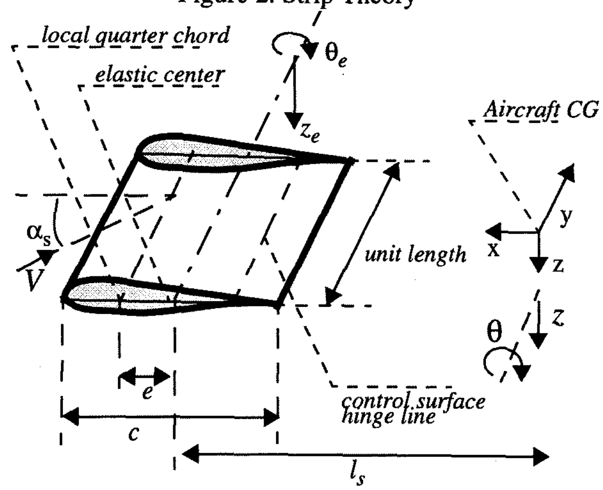
$$z_e = \sum_{i=1}^{\infty} \varphi_z^i(y)\xi_i\ ,\quad w_e = \sum_{i=1}^{\infty} \varphi_z^i(y)\dot{\xi}_i$$

$$\theta_e = \sum_{i=1}^{\infty} \varphi_y^i(y)\xi_i\ ,\quad q_e = \sum_{i=1}^{\infty} \varphi_y^i(y)\dot{\xi}_i$$

Then the equation (63), i.e. the section angle of attack, can be written as

$$\alpha_s = \alpha - \frac{q(l_s+e)}{u} + i_s + \frac{w_g}{u} + \sum_{i=1}^{\infty}\left(\varphi_y^i(y)\xi_i + \frac{\varphi_z^i(y) - \varphi_y^i(y)e}{u}\dot{\xi} + \frac{d\varphi_z^i(jp)}{dx}\xi_i\right) \qquad (64)$$

Figure 2. Strip Theory



6.2 Lift, Moment and Drag

The lift of a section is given by

$$l = \frac{1}{2}\rho_\infty V^2 c_l = \frac{1}{2}\rho_\infty V^2 (c_{l_o} + c_{l_\alpha}\alpha_s + c_{l_\delta}\delta)$$

where c_l is the section lift coefficient, c_{l_o} is the section lift at zero angle of attack, c_{l_α} is the section lift curve slope, c_{l_δ} is the lift coefficients due to trailing edge device deflection δ. Substituting section angle of attack expression (64), integrating over the span, b, of the lifting surface, and adding the contribution from non-lifting surfaces such as fuselage and nacelle result in total aerodynamic lift

$$L = \frac{1}{2}\rho_\infty V^2 S_{ref} C_L$$

where S_{ref} is the aerodynamic reference area, the total lift coefficient C_L is defined as

$$C_L = C_{L_o} + C_{L_\alpha}\alpha + C_{L_{\dot\alpha}}\dot\alpha + C_{L_q}q + C_{L_{wg}}w_g + \sum_{i=1}^{\infty}\left(C_{L_{\xi_i}}\xi_i + C_{L_{\dot\xi_i}}\dot\xi_i\right) + \sum_{r=1}^{m} C_{L_{\delta_r}}\delta_r \qquad (65)$$

where m is the number of control surfaces available for aircraft motion control and δ_r are the deflections of corresponding control surfaces. The coefficient definitions are listed in the Appendix. In order to account for some of the three dimensional finite wing effects, the actual lift coefficients are modified to

$$C_{L_\alpha} = \frac{(C_{L_\alpha})_{2D}}{1 + (1+\tau)\frac{(C_{L_\alpha})_{2D}}{\pi AR}}$$

where τ and AR represent the Glauert correction factor and aspect ratio, respectively. The section moment about the aircraft c.g. is also given by

$$m = \frac{1}{2}\rho_\infty V^2 c_m$$

where c_m is the section moment coefficient given by

$$c_m = \left(c_{m_o}^{c/4} + c_{l_o}\frac{l_s+e}{c}\right) + \left(c_{m_\alpha}^{c/4} + c_{l_\alpha}\frac{l_s+e}{c}\right)\alpha_s + \left(c_{m_\delta}^{c/4} + c_{l_\delta}\frac{l_s+e}{c}\right)\delta$$

Substituting the section angle of attack expression, integrating over the span and adding the contributions from fuselage and nacelle result in the total aerodynamic moment about the aircraft c.g.,

$$M = \frac{1}{2}\rho_\infty V^2 \bar{c} S_{ref} C_M$$

where $\bar{c}$ is the wing mean aerodynamic chord (MAC) and the total lift coefficient C_M is defined as

$$C_M = C_{M_o} + C_{M_\alpha}\alpha + C_{M_{\dot\alpha}}\dot\alpha + C_{M_q}q + C_{M_{wg}}w_g + \sum_{i=1}^{\infty}\left(C_{M_{\xi_i}}\xi_i + C_{M_{\dot\xi_i}}\dot\xi_i\right) + \sum_{r=1}^{m} C_{M_{\delta_r}}\delta_r \qquad (66)$$

The coefficient definitions are listed in the Apendix. The aerodynamic drag on the aircraft is modeled by a drag polar which is given by

$$C_D = C_{D_o} + \frac{C_L^2}{\pi e_o AR}$$

where C_{D_o} is the parasite drag coefficient, e_o is the Oswald efficiency factor and AR is the wing aspect ratio. The drag polar model is a quadratic function of C_L, and can be approximated using the Taylor series expansion about the equilibrium condition. Denote the drag coefficients at equilibrium flight as C_{D_e} then the drag coefficient C_D can be written as

$$C_D = C_{D_e} + C_{D_\alpha}\alpha + C_{D_{\dot\alpha}}\dot\alpha + C_{D_q}q + C_{D_{wg}}w_g + \sum_{i=1}^{\infty}\left(C_{D_{\xi_i}}\xi_i + C_{D_{\dot\xi_i}}\dot\xi_i\right) + \sum_{r=1}^{m} C_{D_{\delta_r}}\delta_r \qquad (67)$$

where coefficients, listed in the Apendix, are defined using the first order terms in the Taylor series expansion of drag polar model. Finally, the total aircraft drag can be written as

$$D = \frac{1}{2}\rho_\infty V^2 S_{ref} C_D$$

6.3 Generalized Forces

The generalized force can be obtained from (52) by applying strip theory. Substituting free vibration modes leads to

$$\Lambda_i = \int_{-0.5b}^{0.5b} \{(\Delta\vec{f}_z^M + \Delta\vec{f}_z^D)\varphi_z^i(y) + m_y\varphi_y^i(y)\}dy \qquad (68)$$

Now, it is possible to express $\Delta\vec{f}_z^M + \Delta\vec{f}_z^D$ in terms of section lift and drag, i.e.

$$f_z^D + f_z^M = -l\cos\alpha - d\sin\alpha \qquad (69)$$

Replacing (69) into (68) and adding contributions from fuselage and nacelle gives the total generalized force as in the form of (60) and F_i, G_i and H_i are now defined as

$$F_i = \int_{-0.5b_w}^{0.5b_w} l^w\varphi_i^{wb}dy + \int_{-0.5b_t}^{0.5b_t} l^t\varphi_i^{tb}dy + (L^{bn})\varphi_i^{bb}\Big|_{wbjp}$$

$$G_i = \int_{-0.5b_w}^{0.5b_w} d^w\varphi_i^{wb}dy + \int_{-0.5b_t}^{0.5b_t} d^t\varphi_i^{tb}dy + (D^{bn})\varphi_i^{bb}\Big|_{wbjp}$$

$$H_i = \int_{-0.5b_w}^{0.5b_W} m_{ec}^w\varphi_i^{wt}dy + 2\int_{-0.5b_t}^{0.5b_t} m_{ec}^t\varphi_i^{tt}dy + (M_{wbjp}^{bn})\frac{d\varphi_i^{bb}}{dx}\Big|_{wbjp}$$

where

l, d	: section lift, drag with superscript for corresponding lifting surface
L^{bn}, D^{bn}	: Lift and Drag contribution from fuselage and Nacelle
m_{ec}	: section moment about elastic center with superscript for corresponding lifting surfaces
M_{wbjp}^{bn}	: Moment contribution from fuselage and nacelle about wing/body junction
φ_i^{wb}, φ_i^{tb}	: bending mode shpae of wing and tail
φ_i^{wt}, φ_i^{tt}	: torsion mode shpace of wing and tail
$\varphi_i^{bb}\Big\|_{wbjp}$	: fuselage bending at wing/body junction
$\frac{d\varphi_i^{bb}}{dx}\Big\|_{wbjp}$	: fuselage bending slope at the wing/body junction

The generalized moment is about the elastic axis of the lifting surface, and hence the moment coefficient term must be modified such that

$$c_m^{ea} = \left(c_{m_o}^{c/4} + c_{l_o}\frac{e}{c}\right) + \left(c_{m_\alpha}^{c/4} + c_{l_\alpha}\frac{e}{c}\right)\alpha_s + \left(c_{m_\delta}^{c/4} + c_{l_\delta}\frac{e}{c}\right)\delta$$

and

$$m_{ec} = \frac{1}{2}\rho_\infty V^2 c_m^{ea}$$

Now, generalized lift and moment can be evaluated by substituting section lift and moment coefficents, and expressed in the form of

$$C_F^i = C_{F_o}^i + C_{F_\alpha}^i\alpha + C_{F_{\dot\alpha}}^i\dot\alpha + C_{F_q}^i q + C_{F_{wg}}^i w_g + \sum_{j=1}^{\infty}\left(C_{F_{\xi_j}}^i\xi_j + C_{F_{\dot\xi_j}}^i\dot\xi_j\right) + \sum_{r=1}^{m} C_{F_{\delta_r}}^i\delta_r \qquad (70.a)$$

$$C_H^i = C_{H_o}^i + C_{H_\alpha}^i\alpha + C_{H_{\dot\alpha}}^i\dot\alpha + C_{H_q}^i q + C_{H_{wg}}^i w_g + \sum_{j=1}^{\infty}\left(C_{H_{\xi_j}}^i\xi_j + C_{H_{\dot\xi_j}}^i\dot\xi_j\right) + \sum_{r=1}^{m} C_{H_{\delta_r}}^i\delta_r \qquad (70.b)$$

Generalized drag depends on the aerodynamic drag contribution of the aircraft which is a quadratic function of aerodynamic lift, so that generalized drag can be assumed as a quadratic function of generalized lift. For such a case, the generalized drag can be obtained using the Taylor series appoximation and can be written as

$$C_G^i = C_{G_e}^i + C_{G_\alpha}^i\alpha + C_{G_{\dot\alpha}}^i\dot\alpha + C_{G_q}^i q + C_{G_{wg}}^i w_g + \sum_{j=1}^{\infty}\left(C_{G_{\xi_j}}^i\xi_j + C_{G_{\dot\xi_j}}^i\dot\xi_j\right) + \sum_{r=1}^{m} C_{G_{\delta_r}}^i\delta_r \qquad (70.c)$$

where the coefficients are the first order terms in the series. Definitions of coefficients in equation set (70) are listed in the Apendix.

7.0 State-Space Form of Dynamic Equations

Equations (50), (54), (59), (62) constitute a complete set of longitudinal equations of motion for a flexible aircraft in steady rectilinear flight. In this section, these equations are expressed in the first order state variable form

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) + \Gamma\underline{v}(t) \qquad (71)$$

where $\underline{x}$, $\underline{u}$, $\underline{v}$, A, B, and Γ are called state vector, input vector, disturbance vector, system matrix, input matrix and disturbance matrix, respectively. The dimension of $\underline{x}$, A, B, and Γ are determined by how many structural equations are included in the model. If N structural equations are included, then

$$\underline{x} \in R^{4+2N}, \underline{u} \in R^m, \underline{v} \in R^2$$
$$A \in R^{(4+2N)\times(4+2N)}, B \in R^{(4+2N)\times m}, \Gamma \in R^{(4+2N)\times 2}$$

where

$$\underline{x} = \left[\Delta u \ \Delta\alpha \ \Delta q \ \Delta\theta \ \Delta\xi_1 \ \Delta\dot{\xi}_1 \ \ldots \ \Delta\xi_N \ \Delta\dot{\xi}_N\right]^T$$
$$\underline{u} = \left[\delta_1 \ \ldots \ \delta_m\right]^T$$
$$\underline{v} = \left[u_g \ w_g\right]^T$$

To compute the matrices of (71) the aerodynamic and generalized force coefficients must be computed first, which can be represented as the sum of the values at the equilibrium flight condition (steady, symmetric wings level flight) and perturbational terms from the equilibrium state. These can be evaluated from the relations given by

$$C_{L_e} = C_{L_o} + C_{L_\alpha}\alpha_e + \sum_{r=1}^{m} C_{L_{\delta_r}}\delta_{r_e} \tag{72.a}$$

$$C_{D_e} = C_{D_o} + \frac{C_{L_e}^2}{\pi e_o AR} \tag{72.b}$$

$$C_{M_e} = 0 = C_{M_o} + C_{M_\alpha}\alpha_e + \sum_{r=1}^{m} C_{M_{\delta_r}}\delta_{r_e} \tag{72.c}$$

$$C^i_{F_e} = C^i_{F_o} + C^i_{F_\alpha}\alpha_e + \sum_{r=1}^{m} C^i_{F_{\delta_r}}\delta_{r_e} \tag{72.d}$$

$$C^i_{G_e} = \frac{2C_{L_e}}{\pi e_o AR}C^i_{F_o} + C^i_{G_\alpha}\alpha_e + \sum_{r=1}^{m} C^i_{G_{\delta_r}}\delta_{r_e} \tag{72.e}$$

$$C^i_{H_e} = C^i_{H_o} + C^i_{H_\alpha}\alpha_e + \sum_{r=1}^{m} C^i_{H_{\delta_r}}\delta_{r_e} \tag{72.f}$$

Note that the static aeroelastic contribution terms

$$\sum_{j=1}^{\infty} C_{L_{\xi_j}}\Xi_j, \ \sum_{j=1}^{\infty} C_{M_{\xi_j}}\Xi_j$$

are taken as zero by assuming that the static incidence angle includes these effects. The perturbational coefficients are computed by

$$\Delta C_L = \Delta C_{L_\alpha}\Delta\alpha + C_{L_{\dot{\alpha}}}\Delta\dot{\alpha} + C_{L_q}\Delta q + C_{L_{wg}}\Delta w_g + \sum_{i=1}^{\infty}\left(C_{L_{\xi_i}}\Delta\xi_i + C_{L_{\dot{\xi}_i}}\Delta\dot{\xi}_i\right) + \sum_{r=1}^{m} C_{L_{\delta_r}}\Delta\delta_r \tag{73.a}$$

$$\Delta C_M = C_{M_\alpha}\Delta\alpha + C_{M_{\dot{\alpha}}}\Delta\dot{\alpha} + C_{M_q}\Delta q + C_{M_{wg}}\Delta w_g + \sum_{i=1}^{\infty}\left(C_{M_{\xi_i}}\Delta\xi_i + C_{M_{\dot{\xi}_i}}\Delta\dot{\xi}_i\right) + \sum_{r=1}^{m} C_{M_{\delta_r}}\Delta\delta_r \tag{73.b}$$

$$\Delta C_D = C_{D_\alpha}\Delta\alpha + C_{L_{\dot{\alpha}}}\Delta\dot{\alpha} + C_{D_q}\Delta q + C_{D_{wg}}\Delta w_g + \sum_{i=1}^{\infty}\left(C_{D_{\xi_i}}\Delta\xi_i + C_{D_{\dot{\xi}_i}}\Delta\dot{\xi}_i\right) + \sum_{r=1}^{m} C_{D_{\delta_r}}\Delta\delta_r \tag{73.c}$$

$$\Delta C^i_F = C^i_{F_\alpha}\Delta\alpha + C^i_{F_{\dot{\alpha}}}\Delta\dot{\alpha} + C^i_{F_q}\Delta q + C^i_{F_{wg}}\Delta w_g + \sum_{j=1}^{\infty}\left(C^i_{F_{\xi_i}}\Delta\xi_j + C^i_{F_{\dot{\xi}_i}}\Delta\dot{\xi}_j\right) + \sum_{r=1}^{m} C^i_{F_{\delta_r}}\Delta\delta_r \tag{73.d}$$

$$\Delta C^i_G = C^i_{G_\alpha}\Delta\alpha + C^i_{G_{\dot{\alpha}}}\Delta\dot{\alpha} + C^i_{G_q}\Delta q + C^i_{G_{wg}}\Delta w_g + \sum_{j=1}^{\infty}\left(C^i_{G_{\xi_i}}\Delta\xi_j + C^i_{G_{\dot{\xi}_i}}\Delta\dot{\xi}_j\right) + \sum_{r=1}^{m} C^i_{G_{\delta_r}}\Delta\delta_r \tag{73.e}$$

$$\Delta C^i_H = C^i_{H_\alpha}\Delta\alpha + C^i_{H_{\dot{\alpha}}}\Delta\dot{\alpha} + C^i_{H_q}\Delta q + C^i_{H_{wg}}\Delta w_g + \sum_{j=1}^{\infty}\left(C^i_{H_{\xi_i}}\Delta\xi_j + C^i_{H_{\dot{\xi}_i}}\Delta\dot{\xi}_j\right) + \sum_{r=1}^{m} C^i_{H_{\delta_r}}\Delta\delta_r \tag{73.f}$$

Using equation set (72) and (73), the aircraft equations of motion (50), (54), (59) and (62) are written as

$$\Delta\dot{u} + W_e\Delta q = X_u\Delta u + X_\alpha\Delta\alpha + X_{\dot{\alpha}}\Delta\dot{\alpha} + X_q\Delta q + X_\theta\Delta\theta + \sum_{j=1}^{N}\left(X_{\xi_j}\Delta\xi_j + X_{\dot{\xi}_j}\Delta\dot{\xi}_j\right) + \sum_{r=1}^{m} X_{\delta_r}\Delta\delta_r + X_{wg}\Delta w_g + X_{ug}\Delta u_g$$

$$\Delta\dot{\alpha} - U_e\Delta q = Z_u\Delta u + Z_\alpha\Delta\alpha + Z_{\dot{\alpha}}\Delta\dot{\alpha} + Z_q\Delta q + Z_\theta\Delta\theta + \sum_{j=1}^{N}\left(Z_{\xi_j}\Delta\xi_j + Z_{\dot{\xi}_j}\Delta\dot{\xi}_j\right) + \sum_{r=1}^{m} Z_{\delta_r}\Delta\delta_r + Z_{wg}\Delta w_g + Z_{ug}\Delta u_g$$

$$\Delta\dot{q} = M_u\Delta u + M_\alpha\Delta\alpha + M_{\dot{\alpha}}\Delta\dot{\alpha} + M_q\Delta q + M_\theta\Delta\theta + \sum_{j=1}^{N}\left(M_{\xi_j}\Delta\xi_j + M_{\dot{\xi}_j}\Delta\dot{\xi}_j\right) + \sum_{r=1}^{m} M_{\delta_r}\Delta\delta_r + M_{wg}\Delta w_g + M_{ug}\Delta u_g$$

$$\Delta\dot{\theta} = \Delta q$$

$$M_i\Delta\ddot{\xi}_i + 2M_i\zeta_i\omega_i\Delta\dot{\xi}_i + M_i\omega_i^2\Delta\xi_i = E^i_u\Delta u + E^i_\alpha\Delta\alpha + E^i_{\dot{\alpha}}\Delta\dot{\alpha} + E^i_q\Delta q + E^i_\theta\Delta\theta + \sum_{j=1}^{N}\left(E^i_{\xi_j}\Delta\xi_j + E^i_{\dot{\xi}_j}\Delta\dot{\xi}_j\right) + \sum_{r=1}^{m} E^i_{\delta_r}\Delta\delta_r + E^i_{wg}\Delta w_g + E^i_{ug}\Delta u_g$$

where X_u, X_α, X_q, etc. are defined as body axis stability derivatives, and they are listed in the Apendix. Finally, expressing the above equations in state-space form gives (71) where the system matrix A, input matrix B, disturbance matrix Γ are computed from,

$$A = \Sigma^{-1}A', \ B = \Sigma^{-1}B', \ \Gamma = \Sigma^{-1}\Gamma'$$

and A', B', Γ, Σ are defined as

$A' =$

$$
\begin{bmatrix}
X_u & X_\alpha & X_q & X_\theta & X_{\xi_1} & X_{\dot{\xi}_1} & \cdots & X_{\xi_N} & X_{\dot{\xi}_N} \\
Z_u & Z_\alpha & Z_q & Z_\theta & Z_{\xi_1} & Z_{\dot{\xi}_1} & \cdots & Z_{\xi_N} & Z_{\dot{\xi}_N} \\
M_u & M_\alpha & M_q & M_\theta & M_{\xi_1} & M_{\dot{\xi}_1} & \cdots & M_{\xi_N} & M_{\dot{\xi}_N} \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
E^1_u & E^1_\alpha & E^1_q & E^1_\theta & \begin{pmatrix} E^1_{\xi_1} \\ -K_1 \end{pmatrix} & \begin{pmatrix} E^1_{\dot{\xi}_1} - \\ 2M_1\zeta_1\omega_1 \end{pmatrix} & \cdots & E^1_{\xi_N} & E^1_{\dot{\xi}_N} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
E^N_u & E^N_\alpha & E^N_q & E^N_\theta & E^N_{\xi_1} & E^N_{\dot{\xi}_1} & \cdots & \begin{pmatrix} E^N_{\xi_N} \\ -K_N \end{pmatrix} & \begin{pmatrix} E^N_{\dot{\xi}_N} - \\ 2M_N\zeta_N\omega_N \end{pmatrix}
\end{bmatrix}
$$

$$
B' = \begin{bmatrix}
X_{\delta_1} & Z_{\delta_1} & M_{\delta_1} & 0 & 0 & E^1_{\delta_1} & \cdots & 0 & 0 & E^N_{\delta_m} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
X_{\delta_m} & Z_{\delta_m} & M_{\delta_m} & 0 & 0 & E^1_{\delta_m} & \cdots & 0 & 0 & E^N_{\delta_m}
\end{bmatrix}^T
$$

$$
\Gamma' = \begin{bmatrix}
X_{ug} & Z_{ug} & M_{ug} & 0 & 0 & E^1_{ug} & \cdots & \cdots & 0 & E^N_{ug} \\
X_{wg} & Z_{wg} & M_{wg} & 0 & 0 & E^1_{wg} & \cdots & \cdots & 0 & E^N_{wg}
\end{bmatrix}^T
$$

$$
\Sigma = \begin{bmatrix}
1 & X_{\dot{\alpha}} & 0 & W_e & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 - Z_{\dot{\alpha}} & 0 & -1.0 & 0 & 0 & \cdots & 0 & 0 \\
0 & M_{\dot{\alpha}} & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & E^1_{\dot{\alpha}} & 0 & 0 & 0 & M_1 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & E^N_{\dot{\alpha}} & 0 & 0 & 0 & 0 & \cdots & 0 & M_N
\end{bmatrix}
$$

8.0 Concluding Remarks

The formulation of a mathematical model for flexible aircraft dynamics is presented. Newton's equation is used to describe the aircraft gross motion, while Lagrange's equation with the virtual work principle is used to describe the fine motion. These two sets of equations are kinematically decoupled although inertial and aerodynamic coupling remain. Assumptions and constraints to achieve this degree of decoupling are explicitly stated. In an effort to provide an extensive tutorial on the subject, a state-space form of the longitudinal equations is obtained using quasi-steady strip aerodynamic theory and small perturbation theory.

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Appendix

Lift Coefficients

- Motion Dependent

$$
C_{L_o} = \frac{2}{S_W}\int_0^{0.5b_W} (c^w_{l_o} + c^w_{l_\alpha} i_w) c^w_s dy + \eta_t \frac{2}{S_T}\int_0^{0.5b_T} (c^t_{l_o} - c^t_{l_\alpha}\varepsilon_o) c^t_s dy + C^{BN}_{L_o}
$$

$$
C_{L_\alpha} = \frac{2}{S_W}\int_0^{0.5b_W} c^w_{l_\alpha} c^w_s dy + \eta_t \frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha}\left(1 - \frac{de}{d\alpha}\right) c^t_s dy + C^{BN}_{L_\alpha}
$$

$$
C_{L_{\dot{\alpha}}} = \eta_t \frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha} \frac{de}{d\alpha}\frac{l^t_s}{U} c^t_s dy
$$

$$
C_{L_q} = \frac{2}{S_W}\int_0^{0.5b_W} -c^w_{l_\alpha}\frac{1}{U}(l_s + e) c^w_s dy + \eta_t \frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha}\frac{1}{U} l_s c^t_s dy + C^{BN}_{L_q}
$$

$$
C_{L_{\xi_i}} = \frac{2}{S_W}\int_0^{0.5b_W} c^w_{l_\alpha}\left(\varphi_i^{wt} + \left.\frac{d\varphi_i^{fb}}{dx}\right|_{wbjp}\right) c^w_s dy +
$$

$$\eta_t \frac{2}{S_T} \int_0^{0.5b_T} c^t_{l_\alpha}\left(\varphi_i^{tt} + \left.\frac{d\varphi_i^{fb}}{dx}\right|_{tbjp}\right) c^t_s dy + \frac{1}{U} C^{BN}_{L_\alpha} \left.\varphi_i^{fb}\right|_{wbjp}$$

$$C_{L_{\dot{\xi}_i}} = \frac{2}{S_W} \int_0^{0.5b_W} c^w_{l_\alpha}\left(\frac{\varphi_i^{wb} - \varphi_i^{wt} e}{U}\right) c^w_s dy$$

$$\eta_t \frac{2}{S_T} \int_0^{0.5b_T} c^t_{l_\alpha}\left(\frac{\varphi_i^{tb} - \varphi_i^{tt} e^t}{U}\right) c^t_s dy + C^{BN}_{L_\alpha} \left.\frac{d\varphi_i^{fb}}{dx}\right|_{tbjp}$$

- Control Dependent (Wing and Tail Devices)

$$C_{L_{\delta A}} = \frac{2}{S_W} \int_0^{0.5b_W} c_{l_{\delta A}} c^w_s dy \ , \ C_{L_{\delta E}} = \eta_t \frac{2}{S_T} \int_0^{0.5b_T} c_{l_{\delta E}} c^t_s dy$$

- Disturbance Dependent

$$C_{L_{w_g}} = \frac{2}{S_W} \int_0^{0.5b_W} c_{l_{\delta F}} \frac{1}{U} c^w_s dy + \eta_t \frac{2}{S_T} \int_0^{0.5b_T} c^t_{l_\alpha} \frac{1}{U} c^t_s dy$$

Moment Coefficients

- Motion Dependent

$$C_{M_o} = \left[C^{BN}_{M_o} + \left(h_{cg} - \frac{1}{4}\right) C^{BN}_{L_o}\right] +$$

$$\frac{2}{S_W c} \int_0^{0.5b_W} \left[c^{c/4}_{m_o} + c^w_{l_o} \frac{l_s + e}{c^w_s} + \left(c^{c/4}_{m_\alpha} + c^w_{l_\alpha} \frac{l_s + e}{c^w_s}\right) i_W\right] c^w_s c^w_s dy +$$

$$\eta_t \frac{2}{S_W c}\left[\int_0^{0.5b_W} c_{m_{ac}} c^t_s c^t_s dy + \int_0^{0.5b_W} -l^t_s (c^t_{l_o} - c^t_{l_\alpha} \varepsilon_o) c^t_s dy\right]$$

$$C_{M_\alpha} = \frac{2}{S_W c} \int_0^{0.5b_W} \left(c^{c/4}_{m_\alpha} + c^w_{l_\alpha} \frac{l_s + e}{c^w_s}\right) c^w_s c^w_s dy +$$

$$\eta_t \frac{2}{S_W c} \int_0^{0.5b_T} -l^t_s c^t_{l_\alpha}\left(1 - \frac{de}{d\alpha}\right) c^t_s dy + \left[C^{BN}_{M_\alpha} + \left(h_{cg} - \frac{1}{4}\right) C^{BN}_{L_\alpha}\right]$$

$$C_{M_{\dot{\alpha}}} = \eta_t \frac{2}{S_W c} \int_0^{0.5b_T} -l^t_s c^t_{l_\alpha} \frac{de}{d\alpha} \frac{l^t_s}{U} c^t_s dy$$

$$C_{M_q} = \frac{2}{S_W c} \int_0^{0.5b_W} \left(c^{c/4}_{m_\alpha} + c^w_{l_\alpha} \frac{l_s + e}{c^w_s}\right)\left(\frac{l_s + e}{c^w_s}\right) c^w_s c^w_s dy +$$

$$\eta_t \frac{2}{S_W c} \int_0^{0.5b_T} -l^t_s c^t_{l_\alpha} \frac{l_s}{U} c^t_s dy$$

$$C_{M_{\xi_i}} = \frac{2}{S_W c} \int_0^{0.5b_W} \left(c^{c/4}_{m_\alpha} + c^w_{l_\alpha} \frac{l_s + e}{c^w_s}\right)\left(\varphi_i^{wt} + \left.\frac{d\varphi_i^{fb}}{dx}\right|_{wbjp}\right) c^w_s c^w_s dy +$$

$$\eta_t \frac{2}{S_W c} \int_0^{0.5b_T} -l^t_s c^t_{l_\alpha}\left(\varphi_i^{tt} + \left.\frac{d\varphi_i^{fb}}{dx}\right|_{tbjp}\right) c^t_s dy +$$

$$\left(h_{cg} - \frac{1}{4}\right)\left(\frac{1}{U} C^{BN}_{L_\alpha} \left.\varphi_i^{fb}\right|_{wbjp}\right)$$

$$C_{M_{\dot{\xi}_i}} = \frac{2}{S_W c} \int_0^{0.5b_W} \left(c^{c/4}_{m_\alpha} + c^w_{l_\alpha} \frac{l_s + e}{c^w_s}\right)\left(\frac{\varphi_i^{wb} - \varphi_i^{wt} e}{U}\right) c^w_s c^w_s dy +$$

$$\eta_t \frac{2}{S_W c} \int_0^{0.5b_T} -l_s c^t_{l_\alpha}\left(\frac{\varphi_i^{tb} - \varphi_i^{tt} e^t}{U}\right) c^t_s dy + \left(h_{cg} - \frac{1}{4}\right) C^{BN}_{L_\alpha} \left.\frac{d\varphi_i^{fb}}{dx}\right|_{tbjp}$$

- Control Dependent (Wing and Tail Devices)

$$C_{M_{\delta A}} = \frac{2}{S_W c} \int_0^{0.5b_W} \left(c_{m_{\delta A}} + c_{l_{\delta A}} \frac{l_s + e}{c^w_s}\right) c^w_s c^w_s dy$$

$$C_{M_{\delta E}} = \eta_t \frac{2}{S_W c} \int_0^{0.5b_T} -l^t_s c_{l_{\delta E}} c^t_s dy$$

- Disturbance Dependent

$$C_{M_{w_g}} = \frac{2}{S_W c} \int_0^{0.5b_W} \left(c^{0.25c}_{m_\alpha} + c^w_{l_\alpha} \frac{l_s + e}{c^w_s}\right) \frac{1}{U} c^w_s dy +$$

$$\eta_t \frac{2}{S_W c} \int_0^{0.5b_T} -l^t_s c^t_{l_\alpha} \frac{1}{U} c^t_s dy$$

Drag Coefficients

$$C_{D_e} = C_{D_o} + \frac{C^2_{L_e}}{\pi e AR}$$: Drag Polar Model

C_{D_e} : equilibrium drag coefficient

C_{D_o} : parasite and trim drag

C_{L_e} : equilibrium lift coefficient

- Motion Dependent

$$C_{D_\alpha} = \frac{\partial C_D}{\partial \alpha} = 2\frac{C_{L_e}}{\pi e AR} C_{L_\alpha} \ , \ C_{D_{\dot{\alpha}}} = \frac{\partial C_D}{\partial \dot{\alpha}} = 2\frac{C_{L_e}}{\pi e AR} C_{D_{\dot{\alpha}}}$$

$$C_{D_q} = \frac{\partial C_D}{\partial q} = 2\frac{C_{L_e}}{\pi e AR} C_{L_q} \ , \ C_{D_{\xi_i}} = \frac{\partial C_D}{\partial \xi_i} = 2\frac{C_{L_e}}{\pi e AR} C_{L_{\xi_i}}$$

$$C_{D_{\dot{\xi}_i}} = \frac{\partial C_D}{\partial \dot{\xi}_i} = 2\frac{C_{L_e}}{\pi e AR} C_{L_{\dot{\xi}_i}}$$

- Control Dependent (Wing and Tail Devices)

$$C_{D_{\delta A}} = \frac{\partial C_D}{\partial \delta_A} = 2\frac{C_{L_e}}{\pi e AR} C_{L_{\delta A}} \ , \ C_{D_{\delta E}} = \frac{\partial C_D}{\partial \delta_E} = 2\frac{C_{L_e}}{\pi e AR} C_{L_{\delta E}}$$

- Disturbance Dependent

$$C_{D_{w_g}} = \frac{\partial C_D}{\partial w_g} = 2\frac{C_{L_e}}{\pi e AR} C_{L_{w_g}}$$

Generalized Lift Coefficients

- Motion Dependent

$$C^i_{F_o} = \frac{2}{S_W} \int_0^{0.5b_W} (c^w_{l_o} + c^w_{l_\alpha} i_w) c^w_s \varphi_i^{wb} dy +$$

$$\eta_t \frac{2}{S_T} \int_0^{0.5b_T} (c^t_{l_o} - c^t_{l_\alpha} \varepsilon_o) c^t_s \varphi_i^{tb} dy + C^{BN}_{L_o} \varphi_i^{fb}(wbjp)$$

$$C^i_{F_\alpha} = \frac{2}{S_W}\int_0^{0.5b_W} c^w_{l_\alpha} c^w_s \varphi_i^{wb}\,dy + \eta_t\frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha}\left(1-\frac{de}{d\alpha}\right)c^t_s\varphi_i^{tb}\,dy + C^{BN}_{L_\alpha}\varphi_i^{fb}(wbjp)$$

$$C^i_{F_{\dot\alpha}} = \eta_t\frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha}\frac{de}{d\alpha}\frac{l^t_s}{U}c^t_s\varphi_i^{tb}\,dy$$

$$C^i_{F_q} = \frac{2}{S_W}\int_0^{0.5b_W} -c^w_{l_\alpha}\frac{1}{U}(l_s+e)c^w_s\varphi_i^{wb}\,dy + \eta_t\frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha}\frac{1}{U}l_s c^t_s\varphi_i^{tb}\,dy + C^{BN}_{L_q}\varphi_i^{fb}(wbjp)$$

$$C^i_{F_{\xi_j}} = \frac{2}{S_W}\int_0^{0.5b_W} c^w_{l_\alpha}\left(\frac{\varphi_j^{wb}-\varphi_j^{wt}e}{U}\right)c^w_s\varphi_i^{wb}\,dy + \eta_t\frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha}\left(\varphi_j^{tt}+\left.\frac{d\varphi_j^{fb}}{dx}\right|_{tbjp}\right)c^t_s\varphi_i^{tb}\,dy + \frac{1}{U}C^{BN}_{L_\alpha}\varphi_j^{fb}\Big|_{wbjp}\varphi_i^{fb}(wbjp)$$

$$C^i_{F_{\dot\xi_j}} = \frac{2}{S_W}\int_0^{0.5b_W} c^w_{l_\alpha}\left(\frac{\varphi_j^{wb}-\varphi_j^{wt}e}{U}\right)c^w_s\varphi_i^{wb}\,dy + \eta_t\frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha}\left(\frac{\varphi_j^{tb}-\varphi_j^{tt}e^t}{U}\right)c^t_s\varphi_i^{tb}\,dy + C^{BN}_{L_\alpha}\left.\frac{d\varphi_j^{fb}}{dx}\right|_{tbjp}\varphi_i^{fb}(wbjp)$$

- Control Dependent (Wing and Tail Devices)

$$C^i_{F_{\delta A}} = \frac{2}{S_W}\int_0^{0.5b_W} c_{l_{\delta A}}c^w_s\varphi_i^{wb}\,dy\ ,\ C^i_{F_{\delta E}} = \eta_t\frac{2}{S_T}\int_0^{0.5b_T} c_{l_{\delta E}}c^t_s\varphi_i^{tb}\,dy$$

- Disturbance Dependent

$$C^i_{F_{wg}} = \frac{2}{S_W}\int_0^{0.5b_W} c_{l_{\delta F}}\frac{1}{U}c^w_s\varphi_i^{wb}\,dy + \eta_t\frac{2}{S_T}\int_0^{0.5b_T} c^t_{l_\alpha}\frac{1}{U}c^t_s\varphi_i^{tb}\,dy$$

Generalized Moment Coefficients

- Motion Dependent

$$C^i_{H_o} = \left[C^{BN}_{M_o}+\left(h_{wbjp}-\frac{1}{4}\right)C^{BN}_{L_o}\right]\left.\frac{d\varphi_i^{fb}}{dx}\right|_{wbjp} + \frac{2}{S_W c}\int_0^{0.5b_W}\left[c^{c/4}_{m_o}+c^w_{l_o}\frac{e}{c^w_s}+\left(c^{c/4}_{m_\alpha}+c^w_{l_\alpha}\frac{e}{c^w_s}\right)i_W\right]c^w_s c^w_s\varphi_i^{wt}\,dy + \eta_t\frac{2}{S_W c}\left[\int_0^{0.5b_W} c_{m_{ac}}c^t_s c^t_s\varphi_i^{tt}\,dy + \int_0^{0.5b_W}\left(c^t_{l_o}-c^t_{l_\alpha}\frac{e^t}{c^t_s}\varepsilon_o\right)c^t_s c^t_s\varphi_i^{tt}\,dy\right]$$

$$C^i_{H_\alpha} = \left[C^{BN}_{M_\alpha}+\left(h_{wbjp}-\frac{1}{4}\right)C^{BN}_{L_\alpha}\right]\left.\frac{d\varphi_i^{fb}}{dx}\right|_{wbjp} + \frac{2}{S_W c}\int_0^{0.5b_W}\left(c^{c/4}_{m_\alpha}+c^w_{l_\alpha}\frac{e}{c^w_s}\right)c^w_s c^w_s\varphi_i^{wt}\,dy + \eta_t\frac{2}{S_W c}\int_0^{0.5b_T}\left(1-\frac{de}{d\alpha}\right)c^t_{l_\alpha}\frac{e^t}{c^t_s}c^t_s c^t_s\varphi_i^{tt}\,dy$$

$$C^i_{H_{\dot\alpha}} = \eta_t\frac{2}{S_W c}\int_0^{0.5b_T}\left(\frac{-l^t_s-e^t}{c^t_s}\right)\frac{de}{d\alpha}c^t_{l_\alpha}\frac{1}{U}c^t_s c^t_s\varphi_i^{tt}\,dy$$

$$C^i_{H_q} = \frac{2}{S_W c}\int_0^{0.5b_W}\left(c^{0.25c}_{m_\alpha}+c^w_{l_\alpha}\frac{e}{c^w_s}\right)\left(\frac{e}{c^w_s}\right)c^w_s c^w_s\varphi_i^{wt}\,dy + \eta_t\frac{2}{S_W c}\int_0^{0.5b_T}\frac{1}{U}c^t_{l_\alpha}\frac{e^t}{c^t_s}c^t_s c^t_s\varphi_i^{tt}\,dy$$

$$C^i_{H_{\xi_j}} = \frac{2}{S_W c}\int_0^{0.5b_W}\left(c^{c/4}_{m_\alpha}+c^w_{l_\alpha}\frac{e}{c^w_s}\right)\left(\varphi_j^{wt}+\left.\frac{d\varphi_j^{fb}}{dx}\right|_{wbjp}\right)c^w_s c^w_s\varphi_i^{wt}\,dy + \eta_t\frac{2}{S_W c}\int_0^{0.5b_T} c^t_{l_\alpha}\frac{e^t}{c^t_s}\left(\varphi_i^{tt}+\left.\frac{d\varphi_j^{fb}}{dx}\right|_{tbjp}\right)c^t_s c^t_s\varphi_i^{tt}\,dy + \left(h_{cg}-\frac{1}{4}\right)\left(\frac{1}{U}C^{BN}_{L_\alpha}\varphi_j^{fb}\Big|_{wbjp}\right)\left.\frac{d\varphi_i^{fb}}{dx}\right|_{wbjp}$$

$$C^i_{H_{\dot\xi_j}} = \frac{2}{S_W c}\int_0^{0.5b_W}\left(c^{0.25c}_{m_\alpha}+c^w_{l_\alpha}\frac{e}{c^w_s}\right)\left(\frac{\varphi_j^{wb}-\varphi_j^{wt}e}{U}\right)c^w_s c^w_s\varphi_i^{wt}\,dy + \eta_t\frac{2}{S_W c}\int_0^{0.5b_T} c^t_{l_\alpha}\frac{e^t}{c^t_s}\left(\frac{\varphi_j^{tb}-\varphi_j^{tt}e^t}{U}\right)c^t_s c^t_s\varphi_i^{tt}\,dy + \left(h_{cg}-\frac{1}{4}\right)C^{BN}_{L_\alpha}\left.\frac{d\varphi_j^{fb}}{dx}\right|_{tbjp}$$

- Control Dependent (Wing and Tail Devices)

$$C^i_{H_{\delta A}} = \frac{2}{S_W c}\int_0^{0.5b_W}\left(c_{m_{\delta A}}+c_{l_{\delta A}}\frac{e}{c^w_s}\right)c^w_s c^w_s\varphi_i^{wt}\,dy$$

$$C^i_{H_{\delta E}} = \eta_t\frac{2}{S_W c}\int_0^{0.5b_T}\frac{e^t}{c^t_s}c_{l_{\delta E}}c^t_s c^t_s\varphi_i^{tt}\,dy$$

- Disturbance Dependent

$$C^i_{H_{wg}} = \frac{2}{S_W c}\int_0^{0.5b_W}\left(c^{0.25c}_{m_\alpha}+c^w_{l_\alpha}\frac{e}{c^w_s}\right)\frac{1}{U}c^w_s c^w_s\varphi_i^{wt}\,dy + \eta_t\frac{2}{S_W c}\int_0^{0.5b_T} c^t_{l_\alpha}\frac{e^t}{c^t_s}\frac{1}{U}c^t_s c^t_s\varphi_i^{tt}\,dy$$

Generalized Drag Coefficients

- Motion Dependent

$$C^i_{G_\alpha} = \frac{\partial C^i_{G_\alpha}}{\partial\alpha} = 2\frac{C_{L_e}}{\pi eAR}C^i_{F_\alpha}\ ,\ C^i_{G_{\dot\alpha}} = \frac{\partial C^i_G}{\partial\dot\alpha} = 2\frac{C_{L_e}}{\pi eAR}C^i_{F_{\dot\alpha}}$$

$$C^i_{G_q} = \frac{\partial C^i_G}{\partial q} = 2\frac{C_{L_e}}{\pi eAR}C^i_{F_q}\ ,\ C^i_{G_{\xi i}} = \frac{\partial C^i_G}{\partial\xi_i} = 2\frac{C_{L_e}}{\pi eAR}C^i_{F_{\xi i}}$$

$$C^i_{G_{\dot\xi_i}} = \frac{\partial C^i_G}{\partial\dot\xi_i} = 2\frac{C_{L_e}}{\pi eAR}C^i_{F_{\dot\xi_i}}$$

- Control Dependent (Wing and Tail Devices)

$$C^i_{G_{\delta A}} = \frac{\partial C^i_G}{\partial\delta_A} = 2\frac{C_{L_e}}{\pi eAR}C^i_{F_{\delta A}}\ ,\ C^i_{G_{\delta E}} = \frac{\partial C^i_G}{\partial\delta_E} = 2\frac{C_{L_e}}{\pi eAR}C^i_{F_{\delta E}}$$

- Disturbance Dependent

$$C^i_{G_{wg}} = \frac{\partial C^i_G}{\partial w_g} = 2\frac{C_{L_e}}{\pi eAR}C^i_{G_{wg}}$$

X - Derivatives

- Motion Dependent

$$X_u = \frac{1}{M_T}\rho_\infty U_e S_{ref} C_{X_e}$$

$$X_\alpha = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}(C_{L_\alpha}\sin\alpha_e - C_{D_\alpha}\cos\alpha_e) - \bar{q}_e S_{ref} C_{Z_e} + \rho_\infty U_e W_e S_{ref} C_{X_e}\right\}$$

$$X_{\dot{\alpha}} = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}\left(C_{L_{\dot{\alpha}}}\sin\alpha_e - C_{D_{\dot{\alpha}}}\cos\alpha_e\right)\right\}$$

$$X_q = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}(C_{L_q}\sin\alpha_e - C_{D_q}\cos\alpha_e)\right\}$$

$$X_\theta = \{-g\cos\Theta_e\}$$

$$X_{\xi_i} = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}\left(C_{L_{\xi_i}}\sin\alpha_e - C_{D_{\xi_i}}\cos\alpha_e\right)\right\}$$

$$X_{\dot{\xi}_i} = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}\left(C_{L_{\dot{\xi}_i}}\sin\alpha_e - C_{D_{\dot{\xi}_i}}\cos\alpha_e\right)\right\}$$

- Control Dependent Derivatives

$$X_{\delta A} = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}(C_{L_{\delta A}}\sin\alpha_e - C_{D_{\delta A}}\cos\alpha_e)\right\}$$

$$X_{\delta E} = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}(C_{L_{\delta E}}\sin\alpha_e - C_{D_{\delta E}}\cos\alpha_e)\right\}$$

- Disturbance (Gust) Dependent Derivatives

$$X_{ug} = \frac{1}{M_T}\rho_\infty U_e S_{ref} C_{X_e}$$

$$X_{wg} = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}(C_{L_{wg}}\sin\alpha_e - C_{D_{wg}}\cos\alpha_e)\right\}$$

Z- Derivatives

- Motion Dependent

$$Z_u = \frac{1}{M_T U_e}\rho_\infty U_e S_{ref} C_{Z_e}$$

$$Z_\alpha = \frac{1}{M_T U_e}\left\{\bar{q}_e S_{ref}\left(C_{X_e} - C_{L_\alpha}\cos\alpha_e - C_{D_\alpha}\sin\alpha_e\right) \rho_\infty U_e W_e S_{ref} C_{Z_e}\right\}$$

$$Z_{\dot{\alpha}} = \frac{1}{M_T U_e}\left\{\bar{q}_e S_{ref}\left(C_{L_{\dot{\alpha}}}\cos\alpha_e - C_{D_{\dot{\alpha}}}\sin\alpha_e\right)\right\}$$

$$Z_q = \frac{1}{M_T U_e}\left\{\bar{q}_e S_{ref}(C_{L_q}\cos\alpha_e - C_{D_q}\sin\alpha_e)\right\}$$

$$Z_\theta = -\frac{g}{U_e}\cos\Theta_e$$

$$Z_{\xi_i} = \frac{1}{M_T U_e}\left\{\bar{q}_e S_{ref}\left(C_{L_{\xi_i}}\cos\alpha_e - C_{D_{\xi_i}}\sin\alpha_e\right)\right\}$$

$$Z_{\dot{\xi}_i} = \frac{1}{M_T}\left\{\bar{q}_e S_{ref}\left(C_{L_{\dot{\xi}_i}}\cos\alpha_e - C_{D_{\dot{\xi}_i}}\sin\alpha_e\right)\right\}$$

- Control Dependent Derivatives

$$Z_{\delta A} = \frac{1}{M_T U_e}\left\{\bar{q}_e S_{ref}(C_{L_{\delta A}}\cos\alpha_e - C_{D_{\delta A}}\sin\alpha_e)\right\}$$

$$Z_{\delta E} = \frac{1}{M_T U_e}\left\{\bar{q}_e S_{ref}(C_{L_{\delta E}}\cos\alpha_e - C_{D_{\delta E}}\sin\alpha_e)\right\}$$

- Disturbance (Gust) Dependent Derivatives

$$Z_{ug} = \frac{1}{M_T}\rho_\infty U_e S_{ref} C_{Z_e}$$

$$Z_{wg} = \frac{1}{M_T U_e}\left\{\bar{q}_e S_{ref}(C_{L_{wg}}\cos\alpha_e - C_{D_{wg}}\sin\alpha_e)\right\}$$

M- Derivatives

- Motion Dependent

$$M_u = 0\ ,\ M_\alpha = \frac{1}{I_y}(\bar{q}_e S_{ref}\bar{c} C_{M_\alpha})\ ,\ M_{\dot{\alpha}} = \frac{1}{I_y}\left(\bar{q}_e S_{ref}\bar{c} C_{M_{\dot{\alpha}}}\right)$$

$$M_q = \frac{1}{I_y}(\bar{q}_e S_{ref}\bar{c} C_{M_q})\ ,\ M_{\xi_i} = \frac{1}{I_y}\left(\bar{q}_e S_{ref}\bar{c} C_{M_{\xi_i}}\right)$$

$$M_{\dot{\xi}_i} = \frac{1}{I_y}\left(\bar{q}_e S_{ref}\bar{c} C_{M_{\dot{\xi}_i}}\right)$$

- Control Dependent Derivatives

$$M_{\delta A} = \frac{1}{I_y}(\bar{q}_e S_{ref}\bar{c} C_{M_{\delta A}})\ ,\ M_{\delta E} = \frac{1}{I_y}(\bar{q}_e S_{ref}\bar{c} C_{M_{\delta E}})$$

- Disturbance (Gust) Dependent Derivatives

$$M_{ug} = 0\ ,\ M_{wg} = 0$$

E^i - Derivatives

- Motion Dependent

$$E^i_u = \rho_\infty U_e S_{ref}(C^i_{S_e} + C^i_{H_e}\bar{c})$$

$$E^i_\alpha = \left\{\bar{q}_e S_{ref}(C^i_{R_e} + C^i_{H_e}\bar{c} - C^i_{F_\alpha}\cos\alpha_e - C^i_{G_\alpha}\sin\alpha_e) \rho_\infty U_e W_e S_{ref}(C^i_{S_e} + C^i_{H_e}\bar{c})\right\}$$

$$E^i_{\dot{\alpha}} = \left\{\bar{q}_e S_{ref}\left(-C^i_{F_{\dot{\alpha}}}\cos\alpha_e - C^i_{G_{\dot{\alpha}}}\sin\alpha_e + C^i_{H_{\dot{\alpha}}}\bar{c}\right)\right\}$$

$$E^i_q = \left\{\bar{q}_e S_{ref}(-C^i_{F_q}\cos\alpha_e - C^i_{G_q}\sin\alpha_e + C^i_{H_q}\bar{c})\right\}$$

$$E^i_\theta = 0$$

$$E^i_{\xi_j} = \left\{\bar{q}_e S_{ref}\left(-C^i_{F_{\xi_j}}\cos\alpha_e - C^i_{G_{\xi_j}}\sin\alpha_e + C^i_{H_{\xi_j}}\bar{c}\right)\right\}$$

$$E^i_{\dot{\xi}_j} = \left\{\bar{q}_e S_{ref}\left(-C^i_{F_{\dot{\xi}_j}}\cos\alpha_e - C^i_{G_{\dot{\xi}_j}}\sin\alpha_e + C^i_{H_{\dot{\xi}_j}}\bar{c}\right)\right\}$$

- Control Dependent Derivatives

$$E^i_{\delta A} = \left\{\bar{q}_e S_{ref}(-C^i_{F_{\delta A}}\cos\alpha_e - C^i_{G_{\delta A}}\sin\alpha_e + C^i_{H_{\delta A}}\bar{c})\right\}$$

$$E^i_{\delta E} = \left\{\bar{q}_e S_{ref}(-C^i_{F_{\delta E}}\cos\alpha_e - C^i_{G_{\delta E}}\sin\alpha_e + C^i_{H_{\delta E}}\bar{c})\right\}$$

- Disturbance (Gust) Dependent Derivatives

$$E^i_{ug} = \rho_\infty U_e S_{ref}(C^i_{S_e} + C^i_{H_e}\bar{c})$$

$$E^i_{wg} = \left\{\bar{q}_e S_{ref}(-C^i_{F_{wg}}\cos\alpha_e - C^i_{G_{wg}}\sin\alpha_e + C^i_{H_{wg}}\bar{c})\right\}$$