AN INVERSE-DIRECT HYBRID NAVIER-STOKES SOLVER USING PSEUDE-ANALYTIC FUNCTIONS

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An inverse-direct hybrid Navier-Stokes solver using pseudo-analytic function theories, which can deal with numerous parameters and variables appearing in the unsteady three-dimensional compressible viscous solvers, is presented. Thanks to the pseudo-analytic function theory, the integral operators of direct solvers were defined along the physical boundaries besides inlet and outlet boundaries and they can simply integrate with the inverse method. An interchange of necessary informations between direct and inverse methods can be smoothly and sufficiently executed without any artificial technique. The application of present method to a conventional blade designing is very easy and can be done completely. In the present paper the inverse method demonstrates the ideal cases in which the data are supposed to be known exactly and complete. Using the calculated results for the unsteady inlet transonic rotor flow and the linear cascade flows with and without air-injection, the numerical examples to reconstruct the blade suction surfaces were successfully given. The stabilities in inverse computations for both cases were very good and the code proved useful for the cases.

Introduction

The need for engineering design of turbomachines with high performance and structural durability of blade rows has motivated engineers to develop an inverse method, which accounts for unsteady, three-dimensional, compressible, and viscous effects. From an engineering point of view numerous parameters and variables appearing in governing field equations with arbitrary boundary conditions might be the objectives of everyday designing. It is well-known that inverse modelling ordinarily involves the estimation of the solution of an equation from a set of observed data. In practice, we can measure only the function of data and not derivatives and observational errors cannot be completely avoided. Then many equations of the first kind for the current problems under consideration are ill-posed and may certainly yield some difficulties. Even the unsteady three-dimensional compressible Navier-Stokes equation can only represent an aspect of flow phenomena. Coarsely simplified flow models or the discretized differential equations lead to intrinsic ambiguities, i.e., different values of the parameters in the governing equations can meet the same measured or designed values and any errors act as a perturbation on the equation. Detailed elucidation on the given or design data is essential. To reduce these ambiguities in the solution which are unstable the results of experience accumulated by engineers and scientists are inevitable. Apparently, such incorporation only, may not improve the situation, because that the governing equations might be nonlinear. While in the ideal case when the data are supposed to be known exactly and complete (the perfect data), it might be thought that an exact solution to an inverse problem would prove also useful for the practical case and only the questions of existence, uniqueness, stability and construction of the solution would be of great importance in testing the assumption behind any mathematical model. However, when solving inverse problems numerically, the solution obtained by analytic formula is usually very sensitive to the way in which the data set is completed and to errors in it. The main objective of the present paper is to discuss the information content of the governing equation that certain coefficients or the variable terms or functionals of these coefficients or the variable terms inaccessible to measurement can be determined in as stable and unique manner. Owing to the pseudo-analytic function theories, the existing direct solvers with integral operators for the unsteady three-dimensional compressible viscous cascade flows (1)-(4) can be applied to transonic flowfields with unsteady inlet conditions. Body-fitted coordinates were naturally employed. Three different kinds of computation surfaces, blade-to-blade, meridional and
cross-sectional ones those allow any deflections were introduced. The direct solvers gives the
numerical results through the iterative process and the incorporation with an inverse solution can be
easily achieved. Now the current inverse method uses a line integral and a successive iteration and is
shown especially for the cases of reconstruction of profile shapes from the velocity or temperature
distributions. The inverse-direct hybrid method is also applicable to the cases for setting of inlet flow
conditions to meet a required performance and for the considerations of inflow conditions in film
cooling problems. The admissible order of disturbance level at the inlet, the permissible order of in-
let boundary layer thickness to maintain the suit-
able cascade performance, and so on. The the-
etical concepts such as quasi-conformal mappings
and the other conventional mappings may play im-
portant roles in the present method. The present
background of the theoretical method may guar-
ante the convergence of the numerical computa-
tion without sophisticated numerical optimization
techniques or a cut and try approach. The dis-
cussed scheme here and the given numerical ex-
amples both may show that a variety of future
applications of the current method are promising.

Analysis

Principle of the Solution

Various types of inverse or design meth-
ods have been ordinarily proposed to provide ge-
ometries of cascades corresponding to a set of ob-
served or imposed data such as given blade sur-
faced velocity or pressure distributions with geo-
metric constraints. Then, the solutions of the gov-
erning equations for such inverse problems of in-
terest are inevitably ill-posed. Under the exist-
ence of strong shocks, on the other hand, the nu-
neros parameters and variable terms inaccessible
to measurement appearing in the unsteady three-
dimensional compressible viscous flow govern-
ing equations are objectives of everyday designings of
turbomachines with high performance and struc-
tural durability. A practical inverse or design solu-
tion, from the engineering point of view, should of-
fer numerical methods which can determine these
parameters and variable terms in as stable and
unique manner. Now a priori adequate infor-
mations may naturally reduce the ambiguities
pointed out in the inverse solutions. To stress
on such informations, the main object of present
study is concentrated on the ideal case with the
perfect data. After integrating with the direct
solvers, necessary and sufficient informations for
the inverse method can be afforded. The present
method also proves useful for the practical cases
mentioned above. Taking into account the entropy
variation through shocks, the present inverse tech-
nique uses the unsteady three-dimensional com-
pressible Navier-Stokes solvers. In addition
to the foregoing field equations, boundary condi-
tions must be specified at the blade surfaces and
at the inflow and outflow boundaries. For viscous
flow simulations, a non-slip condition and a pre-
scribed heat flux or wall temperature distribution
are enforced along the blade wall surfaces.

Governing Equations for Unsteady Three-
Dimensional Compressible Viscous Flows

For the practical inverse or design problems in
engineering, the governing equations should rep-
resent the possible flow phenomena in general.
The present inverse method uses the governing
equations for the unsteady three-dimensional com-
pressible viscous cascade flow with the arbitrary
inlet and wall boundary conditions written in the blade relative frame. Those are the con-
tinuity, momentum, energy, and diffusion equa-
tions, besides additional aerothermal relations
which shall be introduced, according to the objec-
tive of designing. Using the cylindrical coordi-
ates and the corresponding velocity components, the governing equations are written as follows. The continuity equation is such that:

$$\frac{\partial \log \rho}{\partial t} + \frac{\partial W_x}{\partial x} + \frac{1}{r} \frac{\partial W_\phi}{\partial \phi} + \frac{\partial W_r}{\partial r} + \frac{W_r}{r} = 0 \quad (1)$$

The momentum equations in axial, circumfer-
tential, and radial directions are such that:

$$\frac{\partial W_x}{\partial t} + W_\phi \left( \frac{1}{r} \frac{\partial W_x}{\partial \phi} - \frac{\partial W_\phi}{\partial x} \right)$$
$$-W_r \left( \frac{\partial W_x}{\partial x} - \frac{\partial W_r}{\partial r} \right) = -\frac{\partial I}{\partial x} + T \frac{\partial s}{\partial x} + \frac{\mu}{3} \frac{\Delta W_x}{\partial x} + \frac{\mu}{\rho} \nabla^2 W_x + V_{s,x} \quad (2)$$

$$\frac{\partial W_\phi}{\partial t} - W_x \left( \frac{1}{r} \frac{\partial W_\phi}{\partial \phi} - \frac{\partial W_x}{\partial x} \right)$$
$$+W_r \left( \frac{\partial W_\phi}{\partial r} - \frac{1}{r} \frac{\partial W_\phi}{\partial \phi} \right) + \frac{W_r W_\phi}{r} + 2 \omega W_r$$
$$= -\frac{1}{r} \frac{\partial I}{\partial \phi} + T \frac{\partial s}{\partial \phi} + \frac{1}{r} \frac{\partial I}{\partial \phi} + \frac{1}{r} \frac{\partial I}{\partial \phi} + \frac{\mu}{3} \frac{\Delta W_\phi}{r} + \frac{\mu}{\rho} \nabla^2 W_\phi + \frac{2}{r^2} \frac{\partial W_\phi}{\partial \phi} + V_{s,\phi} \quad (3)$$

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\[
\begin{aligned}
\frac{\partial W_r}{\partial t} - \Phi \left( \frac{\partial W_\phi}{\partial r} - \frac{1}{2} \frac{\partial W_r}{\partial r} \right) \\
+ W_r \left( \frac{\partial W_r}{\partial x} - \frac{\partial W_\phi}{\partial r} \right) - \frac{W_r^2}{r} = -\frac{\partial I}{\partial r} + \frac{T s}{\partial r} + \frac{1}{3} \frac{\mu + 2 \mu_B}{\rho} \frac{\partial \Delta}{\partial r} \\
+ \frac{\mu}{\rho} \left( \nabla^2 W_r - \frac{2}{r^2} \frac{\partial W_\phi}{\partial \phi} - \frac{W_r}{r^2} \right) + V_{s,r}
\end{aligned}
\]

where

\[
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial r} \\
\Delta &= \frac{\partial W_r}{\partial x} + \frac{1}{r} \frac{\partial W_\phi}{\partial \phi} + \frac{\partial W_r}{\partial r} + \frac{W_r}{r}
\end{aligned}
\]

\[
\begin{aligned}
V_{s,r} &= \frac{1}{\rho} \left\{ 2 \frac{\partial \mu}{\partial x} \frac{\partial W_x}{\partial x} - \frac{2}{3} \frac{\partial (\mu - \mu_B)}{\partial x} \Delta \\
&\quad + \frac{1}{r} \frac{\partial \mu}{\partial \phi} \left( \frac{\partial W_\phi}{\partial \phi} + \frac{1}{r} \frac{\partial W_r}{\partial \phi} \right) \\
&\quad + \frac{\partial \mu}{\partial r} \left( \frac{\partial W_r}{\partial x} + \frac{\partial W_\phi}{\partial r} \right) \right\}
\end{aligned}
\]

\[
\begin{aligned}
V_{s,\phi} &= \frac{1}{\rho} \left\{ 2 \frac{1}{r} \frac{\partial \mu}{\partial \phi} \left( \frac{1}{r} \frac{\partial W_\phi}{\partial \phi} + \frac{W_r}{r} \right) \\
&\quad - \frac{2}{3} \frac{\partial (\mu - \mu_B)}{\partial \phi} \Delta \\
&\quad + \frac{\partial \mu}{\partial r} \left( \frac{\partial W_r}{\partial \phi} + \frac{\partial W_\phi}{\partial r} \right) \right\}
\end{aligned}
\]

\[
\begin{aligned}
V_{s,r} &= \frac{1}{\rho} \left\{ 2 \frac{\partial \mu}{\partial r} \frac{\partial W_r}{\partial r} - \frac{2}{3} \frac{\partial (\mu - \mu_B)}{\partial r} \Delta \\
&\quad + \frac{\partial \mu}{\partial x} \left( \frac{\partial W_x}{\partial r} + \frac{\partial W_r}{\partial x} \right) \\
&\quad + \frac{1}{r} \frac{\partial \mu}{\partial \phi} \left( \frac{1}{r} \frac{\partial W_\phi}{\partial \phi} + \frac{W_r}{r} \right) \right\}
\end{aligned}
\]

Also the energy equation is such that:

\[
\begin{aligned}
\frac{\partial^2 T}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial r^2} \\
+ \left( \frac{1}{\kappa} \frac{\partial}{\partial r} - \frac{\rho C_v}{\kappa} W_r \right) \frac{\partial T}{\partial r} \\
+ \left( \frac{1}{\kappa} \frac{\partial}{\partial \phi} - \frac{\rho C_v}{\kappa} W_\phi \right) \frac{1}{r} \frac{\partial T}{\partial \phi}
\end{aligned}
\]

\[
\begin{aligned}
+ \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{\rho C_v}{\kappa} W_r \right) \frac{\partial T}{\partial r} \\
- \frac{1}{\kappa} \Psi_r + \frac{\rho C_v}{\kappa} \frac{\partial T}{\partial r} = 0
\end{aligned}
\]

(10)

where

\[
\Psi_r = 2 \mu \left[ \left( \frac{\partial W_r}{\partial x} \right)^2 + \left( \frac{1}{r} \left( \frac{\partial W_\phi}{\partial \phi} + \frac{W_r}{r} \right) \right)^2 \right]
\]

(4)

\[
+ \left( \frac{\partial W_r}{\partial r} \right)^2 + 2 \left( \frac{\partial W_\phi}{\partial r} + \frac{1}{r} \frac{\partial W_r}{\partial \phi} \right)^2
\]

\[
+ 2 \left( \frac{\partial W_\phi}{\partial r} + \frac{\partial W_r}{\partial \phi} \right)^2
\]

(11)

Here the notations, \( W \), \( \rho \), \( \mu \), \( \mu_B \), \( \omega \), \( I \), \( T \), \( s \), \( C_v \), and \( q \), indicate the relative velocity, the density, the viscosity, the bulk viscosity, the angular velocity of blades, the rothalpy, temperature, the entropy per unit mass, the heat capacity of the fluid at constant volume, per unit mass, and the energy flux, respectively. The inverse solutions for the equations hitherto mentioned are common to axial-, radial- and mixed-flow types of turbomachines. Therefore, the discussions hereafter, refer only for the axial-flow type. Under the governing equations there are no particular restrictions on boundary conditions in the present solution. Arbitrary unsteady and non-uniform inlet and outlet flow conditions and various types of wall conditions such as with or without air-injection or with designed temperature distributions and so on, are, of course, allowed in the solution. For an abbreviated description here the diffusion equation and the other aerothermodynamic relations for respective flowfields, are not referred to.

**Introduction of Complex Coordinates and Velocities**

We obtain complex forms of the given equations cited above through complex coordinates and complex velocities as follows.

Complex coordinates:

\[
z_n = \zeta_n + i \eta_n.
\]

(12)

\[
\zeta_n = \zeta_n - i \eta_n.
\]

(13)

\( n = 1, 2, 3 \)

with

\[
\zeta_1 = x, \eta_1 = \int_0^x dx, \zeta_2 = x_1, \eta_2 = r:
\]

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\[ \zeta_3 = \int r \, d\phi. \quad \eta_3 = r \]

Complex velocities:
\[ w_1 = W_r - i W_o, \quad \bar{w}_1 = W_r + i W_o; \quad (14) \]
\[ w_2 = W_r - i W_o, \quad \bar{w}_2 = W_r + i W_o; \quad (15) \]
\[ w_3 = W_o - i W_r, \quad \bar{w}_3 = W_o + i W_r. \quad (16) \]

In the above definitions suffix \( n \) stands for the three different kinds of computation surfaces. Also we use derivatives with respect to \( z_n \) and \( \zeta_n \) such that:
\[ \frac{\partial}{\partial z_n} = \frac{1}{2} \left( \frac{\partial}{\partial z_n} - i \frac{\partial}{\partial \eta_n} \right). \quad (17) \]
\[ \frac{\partial}{\partial \zeta_n} = \frac{1}{2} \left( \frac{\partial}{\partial \zeta_n} + i \frac{\partial}{\partial \eta_n} \right). \quad (18) \]
\[ \frac{\partial^2}{\partial z_n^2} = \frac{1}{4} \left( \frac{\partial^2}{\partial z_n^2} + \frac{\partial^2}{\partial \eta_n^2} \right). \quad (19) \]

**Fundamental Equations in Complex Forms**

Using the definitions shown above, the governing equations are reduced to as follows (1)-(4). The fundamental equation for the complex velocity \( w_n \) is such that:
\[ \frac{\partial^2 w_n(z_n, \zeta_n; t)}{\partial z_n \partial \zeta_n} + A_{n,2}(z_n, \zeta_n; t) \frac{\partial w_n}{\partial z_n} + B_{n,2}(z_n, \zeta_n; t) \frac{\partial w_n}{\partial \zeta_n} + C_{n,2}(z_n, \zeta_n; t) \bar{w}_n + D_{n,2}(z_n, \zeta_n; t) w_n + F_n(z_n, \zeta_n; t) \]
\[ + E \frac{\partial w_n}{\partial t} = 0. \quad (n = 1, 2, 3) \]

Also the reduced equation of energy is such that:
\[ \frac{\partial^2 T}{\partial z_n \partial \zeta_n} + A_{n}^E(z_n, \zeta_n; t) \frac{\partial T}{\partial z_n} + B_{n}^E(z_n, \zeta_n; t) \frac{\partial T}{\partial \zeta_n} + F_n^E(z_n, \zeta_n; t) \]
\[ + E_n^E(z_n, \zeta_n; t) \frac{\partial T}{\partial t} = 0. \quad (n = 1, 2, 3) \]

On the direct solutions using the pseudo-analytic function theories the unsteady three-dimensional compressible Navier-Stokes solvers with integral operators, which can be applied to transonic flowfields with unsteady inlet conditions and a high degree of geomeric complexity were already presented(1)-(4). The operators were defined for initial boundary value problems(5)-(6). The direct solvers gives the results through the iterative process and the integration can be easily achieved.

**Reduction of Fundamental Equations on Solid Boundaries for Non-Slip Flows**

On solid boundaries the velocity distributions in the relative frame are zero or constant, except at sinks or sources. The fundamental equations for the complex velocity \( w_n \) at each time increments are then reduced to such that:
\[ \frac{\partial^2 w_n(z_n, \zeta_n; t)}{\partial z_n \partial \zeta_n} + A_{n,2}(z_n, \zeta_n; t) \frac{\partial w_n}{\partial z_n} + B_{n,2}(z_n, \zeta_n; t) \frac{\partial w_n}{\partial \zeta_n} + C_{n,2}(z_n, \zeta_n; t) \bar{w}_n + D_{n,2}(z_n, \zeta_n; t) w_n + F_n(z_n, \zeta_n; t) = 0 \]
\[ \quad (n = 1, 2, 3) \] (22)

In case of \( w_n^0 = 0 \), the coefficients, \( A_{n,2}, B_{n,2} \) and \( F_{n,2} \) are as follows.

For the blade-to-blade surfaces
\[ A_{1,2} = B_{1,2} = 0, \]
\[ F_1 = \frac{1}{4} \frac{\partial^2 w_1}{\partial t^2} + \frac{1}{4} \frac{\partial w_1}{\partial t} + \frac{1}{6} \mu + 2 \mu_B \frac{\partial \Delta'}{\partial z_1} - \frac{1}{2} r^2 \frac{\partial}{\partial \phi} - \frac{1}{2} \rho \left( \frac{\partial I}{\partial z_1} - T \frac{\partial s}{\partial z_1} \right) + \frac{1}{4} \left( V'_{s,z} - i V'_{s,\phi} \right) \]
\[ (25) \]

and for the meridional surfaces
\[ A_{2,2} = -B_{2,2} = i \frac{1}{4} \]
\[ F_2 = \frac{1}{4} \frac{\partial^2 w_2}{\partial t^2} + \frac{1}{4} \frac{\partial w_2}{\partial t} + \frac{1}{6} \mu \frac{\partial \Delta'}{\partial z_2} + \frac{1}{2} r^2 \frac{\partial}{\partial \phi} - \frac{1}{2} \rho \left( \frac{\partial I}{\partial z_2} - T \frac{\partial s}{\partial z_2} \right) + \frac{1}{4} \left( V'_{s,z} - i V'_{s,\phi} \right) \]
\[ (27) \]

and for the cross-sectional surfaces
\[ A_{2,3} = i \frac{3}{4} \frac{\partial}{\partial r}, \quad B_{2,3} = i \frac{1}{4} \frac{\partial}{\partial r} \]
\[ F_3 = \frac{1}{4} \frac{\partial^2 w_3}{\partial r^2} + \frac{1}{6} \mu \frac{\partial \Delta'}{\partial z_3} - \frac{1}{2} \rho \left( \frac{\partial I}{\partial z_3} - T \frac{\partial s}{\partial z_3} \right) + \frac{1}{4} \left( V'_{s,z} - i V'_{s,\phi} \right) \]
\[ (29) \]

Similarly the fundamental equation of energy is written such that:
\[ \frac{\partial^2 T}{\partial z_n \partial \zeta_n} + A_{n}^E(z_n, \zeta_n; t) \frac{\partial T}{\partial z_n} + B_{n}^E(z_n, \zeta_n; t) \frac{\partial T}{\partial \zeta_n} + F_n^E(z_n, \zeta_n; t) \]
\[ + E_n^E(z_n, \zeta_n; t) \frac{\partial T}{\partial t} = 0 \quad (n = 1, 2, 3) \]

(30)
The variable coefficients in the above equation is omitted.

**Numerical Procedures**

The information content of the governing equation naturally dominates the stability and uniqueness in the designing. Discussions for a range of plausible and reasonable solutions in solving ill-posed flow problems numerically, are unavoidable. Moreover, the other various questions remain unsolved. Instead of the general discussion of existence, uniqueness, stability and construction in a complete solution of inverse or design problems, here, only usefulness of an exact solution to an inverse problem with perfect data shall be demonstrated for the practical cases. As was mentioned, ambiguities in the inverse solutions can be reduced by incorporating of a priori informations. Fig. 1 shows the present scheme for designing of three-dimensional high approach Mach number rotor blade contours, as an example of inverse or design method. The boundary condition on blade wall surfaces itself is the objective of the current inverse problems.

**Computation Surfaces and Paths**

For the integration of the inverse solution with the direct ones the present solver uses three different kinds of computation surfaces, blade-to-blade, meridional and cross-sectional ones those allow any deflections as shown in Fig. 2. The current inverse method likewise the direct methods uses the integral representation with body-fitted coordinates and can fix the solution through the iteration. The figure also indicates the computation paths on these surfaces. The details of computation paths on a blade-to-blade are shown in Fig. 3. The solid and broken lines in the figure indicate the computation paths for the direct and the inverse solver, respectively. In the numerical calculation, portions of both computation paths not including reference points to be designed coincide with each other. The usual design factors, the velocity distribution on blade suction surface determining the gas dynamic performance and the thickness distribution which determines the performance of the vibration and strength of blade, are the current objectives. Then the numerical method is applied to the cases of reconstruction of profile contours from the velocity or temperature distributions on the blades. In the blade passages several cross-sectional surfaces are employed.

**Correction of Blade Coordinates**

Introducing the pseudo-analytic function theories the, variety types of inverse problems can be done. As a conventional design problem, here, the method to give the unified design problem, using velocity or pressure distributions on them, is shown as follows. After obtaining the velocity distributions, the blade coordinate can be written such that:

\[
\begin{align*}
\left[ z_n \left\{ \frac{[w_n(s_n)]}{N+1} \left[ \frac{w_n(s_n)}{N+1} \right] \right\} \right]_{N+1} \\
= \left[ z_n \left\{ \frac{[w_n(s_n,0)]}{N+1} \left[ \frac{w_n(s_n,0)}{N+1} \right] \right\} \right]_{N+1} \\
+ \int_{s_n,0}^{s_n} \left\{ \frac{\partial z_n}{\partial w_n} \left[ \frac{dw_n}{ds_n} \right] \right\} ds_n \\
+ \int_{s_n,0}^{s_n} \left\{ \frac{\partial z_n}{\partial w_n} \left[ \frac{dw_n}{ds_n} \right] \right\} ds_n \quad \text{(31)}
\end{align*}
\]

where the formal derivatives:

\[
\frac{\partial z_n}{\partial w_n} = \frac{1}{2} \left( \frac{\partial z_n}{\partial W_p} - \frac{\partial z_n}{\partial W_q} \right) \\
+ \frac{i}{2} \left( \frac{\partial z_n}{\partial W_p} + \frac{\partial z_n}{\partial W_q} \right) \quad \text{(32)}
\]

\[
\frac{\partial z_n}{\partial w_n} = \frac{1}{2} \left( \frac{\partial z_n}{\partial W_p} + \frac{\partial z_n}{\partial W_q} \right) \\
+ \frac{i}{2} \left( \frac{\partial z_n}{\partial W_p} - \frac{\partial z_n}{\partial W_q} \right) \quad \text{(33)}
\]

with

\[
w_n = W_p - i \cdot W_q \quad , \quad \bar{w}_n = W_p + i \cdot W_q \quad \quad \quad \quad \text{(n = 1, 2, 3 ; p ≠ q, p, q = } \phi, \bar{\phi}, r \text{)}
\]

Subscript \( N \) in Eq. (31) indicates the iterative calculation on the whole flowfield. The reverse expression of formal derivative \( \partial w_n / \partial z_n \) can be numerically obtained through the following iterates process. In the first place the dependent variable \( w_n \) at each time increments in the fundamental equation (22) is changed such that:

\[
w_n(z_n, \zeta_n; t) = o_n(z_n, \zeta_n; t) \\
\exp \left\{ - \int_{\zeta_n}^{\zeta_n} A^0_{n,2}(z_n, \zeta_n; t) \text{d} \zeta_n \right\} \quad \text{(34)}
\]

Then the fundamental equation is reduced to such that:

\[
\frac{\partial^2 o_n}{\partial z_n \partial \zeta_n} + \frac{D^0_{n,2}(z_n, \zeta_n; t)}{\partial \zeta_n} \frac{\partial o_n}{\partial \zeta_n} \\
+ F^0_n(z_n, \zeta_n; t) = 0 \quad \text{(35)}
\]

where

\[
D^0_{n,2}(z_n, \zeta_n; t) = B^0_{n,2} \int_{\zeta_n}^{\zeta_n} \frac{\partial A^0_{n,2}(z_n, \zeta_n; t)}{\partial z_n} \text{d} \zeta_n \quad \text{(36)}
\]
\[ F'_{n}^{T}(z_{n}, \zeta_{n}; t) = F_{n}^{0} \exp \left\{ - \int_{\zeta_{n}, 0}^{\tilde{\zeta}_{n}} A'_{n, 0}^{0, 1}(z_{n}, \zeta_{n}; t) \, d\zeta_{n} \right\} \]  
(37)

After introducing an abbreviated notation
\[ \Phi_{n}(\zeta_{n}; z_{n}) \equiv \frac{\partial \sigma_{n}}{\partial \zeta_{n}}. \]  
(38)

Eq. (35) is rewritten such that:
\[ \frac{\partial \Phi_{n}(\zeta_{n}; z_{n})}{\partial \zeta_{n}} + D'_{n, 2} \Phi_{n}(\zeta_{n}; z_{n}) + F'^{T}_{n} = 0 \]  
(39)

Here \( \Phi_{n} \) can be given by a successive iteration as follows:
\[ \Phi_{n}^{[N+1, 0]} = \Phi_{n}^{[N]} \]  
(40)
\[ \Phi_{n}^{[N+1, M+1]} = \Phi_{n}^{[N+1, M]} + \epsilon^{[Q_{n}]} \]  
(41)
\[ \int_{z_{n, 0}}^{z_{n, M}} e^{-[Q_{n} u_{n} F'^{T}_{n} d\zeta_{n}]} d\zeta_{n} ^{[N+1, M]} = 1 \]  
(42)

where
\[ \Phi_{n}^{[N]} = \int_{z_{n, 0}}^{z_{n, M}} [D'_{n, 2}^{[0]}]_{M} \frac{d\zeta_{n}}{d\zeta_{n} ^{[N+1, M]}} d\zeta_{n} \]  
(43)

Now the formal derivative \( \partial w_{n}/\partial z_{n} \) is obtained. The other formal derivative \( \partial \bar{w}_{n}/\partial z_{n} \) can be similarly given through the conjugate form of fundamental equation such that:
\[ \frac{\partial^{2} \bar{w}_{n}(z_{n}, \zeta_{n}; t)}{\partial z_{n} \partial \zeta_{n}} + B'_{n}^{0}(z_{n}, \zeta_{n}; t) \frac{\partial \bar{w}_{n}}{\partial z_{n}} + A_{n, 2}^{0}(z_{n}, \zeta_{n}; t) \frac{\partial \bar{w}_{n}}{\partial \zeta_{n}} + F_{n}^{0}(z_{n}, \zeta_{n}; t) = 0 \]  
\( (n = 1, 2, 3) \)  
(44)

The reduced dependent variable is written such that:
\[ \bar{w}_{n}(z_{n}, \zeta_{n}; t) = \psi_{n}(z_{n}, \zeta_{n}; t) \]  
(45)
\[ \exp \left\{ - \int_{\zeta_{n}, 0}^{\tilde{\zeta}_{n}} B'_{n, 2}^{0}(z_{n}, \zeta_{n}; t) \, d\zeta_{n} \right\} \]  
(46)

The values of derivatives \( \partial w_{n}/\partial s_{n} \) and \( \partial \bar{w}_{n}/\partial s_{n} \), remaining terms in Eq. (31), can be replaced by those for the direct solution given along the adjacent computation path, as shown in Fig. 3. After substituting these derivatives in Eq. (31), the unfixed blade coordinates can be finally assumed. Just in the same way as described above the temperature distribution can give the blade coordinates. The entropy and the other parameters in the governing equations can be deservedly utilized in the designing in much the same way as the velocity and temperature.

**Fitting of Given Values in Arbitrary Locations**

One may employ direct solvers in many design problems. In the direct solvers for initial boundary value problems \(^{(1)-(4)}\), arbitrary boundary conditions at the inlet and outlet can be assigned to the control surfaces. Cross-sectional computation surfaces assumed as the control surfaces can be arranged at any axial location of the flowfield. The direct solutions were given using the integral representations. The pseudo-analytic function theory \(^{(6)}\) yields a method of construction of solutions acquiring at arbitrary reference points in the flowfield some prescribed values. Through the reference points, therefore, we may set the velocity and temperature distributions together with the other aerothermodynamic parameters and variables, appearing in the governing equations, as follows \(^{(1)}\).

For the velocity distribution:
\[ \frac{\alpha}{2} \left[ w_{n}(z_{n}, \zeta_{n}; t) \right]_{N+1} = \frac{\alpha}{2} \left[ w_{n, i}(z_{n}; t) \right]_{N} + \frac{1}{2\pi i} \oint_{G_{n}} \left[ \Phi_{n}(z_{n}, \zeta_{n}; t) \right]_{N} \frac{1 - R_{n}(z_{n, 1}, \cdots, z_{n, m(n)})}{z_{n} - z_{n}} d\zeta_{n} \]  
(47)

Also for the temperature distribution:
\[ \frac{\alpha}{2} \left[ T_{n}(z_{n}, \zeta_{n}; t) \right]_{N+1} = \frac{\alpha}{2} \left[ T_{n, i}(z_{n}; t) \right]_{N} + \frac{1}{2\pi i} \oint_{G_{n}} \left[ \Phi_{n}(z_{n}, \zeta_{n}; t) \right]_{N} \frac{1 - R_{n}^{E}(z_{n, 1}, \cdots, z_{n, m(n)})}{z_{n} - z_{n}} d\zeta_{n} \]  
(48)

where the term standing for the given reference points in Eq. (47) is
\[ \hat{R}_{n}^{E} = \sum_{k=1}^{m(n)} \frac{P_{n}^{E}}{Q_{n}^{E}} \]  
(49)
with
\[ P_{n}^{E} = (z_{n} - z_{n, 1}) \cdots (z_{n} - z_{n, k-1}) * (z_{n, k} - z_{n, k+1}) \cdots (z_{n, m(n)} - z_{n, m(n)}) \]  
(50)
and
\[ Q_{n}^{E} = (z_{n, k} - z_{n, 1}) \cdots (z_{n, k} - z_{n, k-1}) * (z_{n, k} - z_{n, k+1}) \cdots (z_{n, m(n)} - z_{n, m(n)}) \]  
(51)

For the other parameters and variables the same expressions alike (45) and (46) can be introduced.
and the design values at arbitrary locations in
the flowfield may possibly be given. Using dis-
etized expressions for (45), (46) and the others,
we can numerically find the locations of the re-
currence points. Consequently, we may feasibly use
the direct solutions in designings.

**Numerical Examples**

Consideration of the questions of existence,
quickness, stability and construction in a com-
plete solution of inverse problems, is of great im-
portance in testing the assumption behind any
mathematical model. Here the present inverse
method demonstrates the ideal cases in which the
data are supposed to be known exactly and com-
plete. Using the calculated results for the un-
steady inlet transonic rotor flow and the lin-
ear cascade flows with and without air-injection,
the numerical examples to reconstruct the suction
surfaces are shown. Since the computation paths
for the direct and inverse solvers, as shown in
Fig. 3, coincided or were parallel with each other,
the informations could be easily interchanged be-
tween the solvers. In Fig. 4, the computation
paths shown with broken lines on the different
cross-sectional computation surfaces in a blade
passage were to be reconstructed. Nine sheets of
cross-sectional computation surfaces were ar-
anged in a blade passage and hundred reference
points standed for a suction side profile section
to be designed on a blade-to-blade computation
surface. Both codes were written in FORTRAN
for a HITAC S3800/480 and are applicable to un-
steady three-dimensional transonic rotor flow cal-
culations. They proved useful for the practical
cases.

**Unsteady Inlet Transonic Rotor Flow**

A numerical result to stress on the important
roles of the unsteadiness and nonuniformity de-
tected at the inlet, was given for the axial trans-
sonic flow through the 23 bladed overhung ro-
tor without inlet guide vanes installed in MIT
Blow Down Facilities. In the computation of the
results, reflecting the experimental circum-
cstances, some simplifications such as Prandtl
number of 1.0 and the sinusoidal oscillation of the inlet
Mach number with 2.5 percent of an amplitude of
deviation, were assumed without loss of generality.
The tip clearance effect was omitted for the sim-
plification of the computation. The experiment-
tally supposed conditions such as the tip relative
Mach number of 1.30, radially constant stagnation
temperature rise, uniform inlet conditions and the
shockless through flow, were also introduced.
For the reconstruction of profile contour, we
assumed that the pressure sides of different profile
sections were fixed and only the suction sides were
flexible, except for the sections at hub and tip.
Fig. 5 (a) and (b) show the first approximations
of the profile sections on the suction side near at
the tip and at the hub, respectively. The original
blade contour was constructed, using 6 cross sec-
tions from the hub to the tip and the hub ratio
was 0.5 at the leading edge. For the modification
of blade thickness the adjacent blade section was
referred and 30 percent of the difference in thick-
ness between the sections was added or subtracted
to the original thickness. The code in the direct
problem, for the non-uniform case, consumed a
memory of about 600 Mega bytes as working vol-
ume and approximately 3 hours of CPU time.

Using the detailed results from the direct prob-
lem, the current run of inverse code for the recon-
struction required 100 Mega bytes and about 15
minutes of CPU time, additionally. To check the
sensitivity of solution to the given data set and to
errors in it, the same code was applied to a con-
ventionally transformed cascade geometry shown
in Fig. 6 for the reconstruction of the pressure
side. The contours on pressure side at different
sections resemble each other for this case. Fig.
7 shows the original profile contour and its first
approximation in the run. Using the given data
and the first approximations, the code also recon-
structed the original profile with five digits. For
this case CPU time of approximately 10 minutes
was required.

**Linear Cascade Flows With and Without
Air-Injection**

The other numerical result to show the match-
ing between the non-uniformity at the inlet and
the assumed velocity distributions of film-cooling
inflows was also used. Fig. 8 shows the original
profile contour and the location of the slit. Ac-
cording to the experimental data, the result was
calculated for a linear cascade in an annular flow-
field under the simplified condition that the in-
jected air and the main stream were at room tem-
perature and the radially varied but azimuthally
uniform total pressure loss distribution at the in-
let, were assumed. The test Reynolds number
based on the mass-averaged cascade outlet and the
blade chord was about 1.5 x 10^5 and the test inci-
dence at the mid span was -2.8 degrees. The cal-
culated total pressure loss distribution maps with
and with air-injection were plotted with 10 per-
cent intervals, in Figs. 9 and 10, respectively. For
both cases the same first approximation of profile
contour on the suction surface shown in Fig. 8
was used. The profile was given such that 20 per-
cent of additional thickness was added to the or-
iginal suction surface and the pressure surface pro

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file was unchanged. Employing the given results from the direct solution and the first approximation, the above mentioned code was applied to the flowfields without and with air-injection. The reconstruction of the original profile was successfully done both without and with air-injection cases. For the direct problem the code required a memory of about 400 Mega bytes as working volume and approximately 20 minutes of CPU time. For the inverse calculation additional 80 Mega bytes as working volume was necessary. For both cases the additional 10 minutes of run gave the converged results with five digits of accuracy.

Conclusions

An inverse-direct hybrid Navier-Stokes solver using pseudo-analytic function theories, which can deal with objectives of everyday designing in engineering, is presented. The incorporation and interchange of necessary informations between direct and inverse methods can be smoothly and completely done. The present paper stress on the ideal cases with the perfect data. The application of present method to a conventional designing of blade contours was very easy and could be done completely. The numerical examples to reconstruct the blade suction surfaces for the unsteady inlet transonic rotor flow and the linear cascade flows without and with air-injection, were successfully shown. The stabilities in inverse computations for both cases were very good and the code proved useful for the cases.

References


Fig. 1 Scheme for designing of the blade contour
Fig. 2 Computation paths

Fig. 3 Computation paths on a blade-to-blade surface

Fig. 4 Cross-sectional surfaces and computation paths

Fig. 5 Reconstruction of profile sections
Fig. 6 Transformation of flowfield

(a) Existing cascade geometry

(b) Transformed cascade geometry

Fig. 7 Reconstruction of profile section on pressure side

Fig. 8 Reconstruction of blade contour with air-injection
Fig. 9 Total pressure loss maps (without air-injection)

Fig. 10 Total pressure loss maps (with air-injection)