FLIGHT PATH OPTIMIZATION AND SUBOPTIMAL CONTROL LAWS SYNTHESIS
FOR TRANSPORT MISSION OF MANEUVERABLE AIRCRAFT

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Abstract
The flight path optimization for transport mission of maneuverable aircraft as well as suboptimal control synthesis for the optimal flight path implementation are considered. The calculation of the extremal fields for some examples are carried out by the second order method, which presents the extension of previously known one, and by the maximum principle. The approximations of optimum solutions are proposed for the schedule design. The synthesis for a normal load factor is performed through tracking this approximations. The adaptive extremal law for the engine thrust control is proposed. Some numerical results are presented.

I. Introduction
The possibilities of modern onboard computers, as is well known, reveal the ways of optimal solutions usage for aircraft trajectory control. The transport missions for maneuverable aircraft have peculiarity related to the wide range of initial and final conditions which may belong to all range of possible flight regimes. Many papers are devoted to trajectory optimization problems for non maneuverable and maneuverable aircraft including the consideration of the optimal control solution expressed in a feedback form. One of attractive approach to the solution of such problems is related to the so-called singular perturbation method which was successfully employed for practically important tasks[1, 2, 3, 4]. A very useful approach to the solution of usual transportation task in the energy approximation was proposed in Ref.[5]. One needs to point out that the last approach to the solution of transportation task may be extended for two cruise-dash portion on trajectory for maneuverable aircraft. Another approach may be based on computation of extremals field, i.e. on flooding the state-space with extremals and control synthesis on the basis of tracking the approximations to extremals, which satisfy initial and final conditions. This approach is considered in Ref.[6] on the basis of energy formulation.

The similar approach is proposed for some problems in this paper on the basis of the second order method and on the basis of the maximum principle, and is related apparently to the approach in Ref.[7, 8]. The second order method, which was developed by one of the authors[9], provides the optimization of control for discrete nonlinear dynamic system (autonomous and non-autonomous) taking account of control/state constraints at fixed and free final time. This method may be considered as a modification or extension of the second order method[10]. There are several ways of engine thrust control. The adaptive extremal control law, which contains Lagrange multipliers, is considered for this purpose in this paper.

This paper does not pretend to describe the transportation control problem solution in full volume, but only to demonstrate some approaches and techniques that may be useful for application.

II. Transport Mission
The main transport missions or tasks involve the following functionalities: time of flight (t_f), range (L_f) and fuel consumption (w). All the transportation tasks have the initial and final conditions in terms of altitude, speed, flight path angle etc. Some of them may be free.

It is known that cost function for transport tasks to be minimized may be presented in the form

\[ J = \lambda_\omega \cdot w + \lambda_f \cdot t_f - \lambda_i \cdot L_f \]

(1)

where \( \lambda_\omega, \lambda_f \) and \( \lambda_i \) are constant non negative Lagrange multipliers.

Taking \( \lambda_f = 1, \lambda_\omega = 0, \lambda_i = 0 \), we have the problem of final time minimization. Taking \( \lambda_\omega = 1 \) and \( \lambda_f \neq 0, \lambda_i = 0 \), we have the problem of fuel consumption minimization for given \( t_f \) and \( L_f \), etc. A similar problem statement may be considered for mission when \( t_f \) and \( L_f \) correspond to desirable point of interception and it is necessary to minimize fuel consumption. Additional constraints may be imposed on the altitude and speed on some portions of trajectory.

The solution of such problems with the usage of full motion equations for arbitrary initial and final conditions is difficult. It is the reason for considering the simplified models of motion side by side with more exact ones.

III. The Main Features of the Second Order Method
The above mentioned transport tasks may be formulated as follows. Let's consider a usual discrete model of nonlinear dynamic system motion

\[ x_{k+1} = f(x_k, u_k, \Delta t_k), \]

\[ \Delta x_{k+1} = \Delta x_k, \]

(2)

that corresponds to aircraft continuous motion equations. The equation for \( \Delta \) is introduced when final time is free.

State variables vector \( x_k \) that has \( n \) components and control vector \( u_k \) that has \( m \) components satisfy inequality constraints

\[ \Phi^*(x_k, u_k, \Delta t_k) \leq 0, \]

(3)

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where $\Phi$ is vector with $p$ components.

It is necessary to determine the succession $u_0, \ldots, u_{N-1}$ and value of $\Delta t$ so that

$$\psi(x_N) = 0$$

and

$$J = F(x_N) + \sum_{k=0}^{N-1} L^k(x_k, u_k, \Delta t_k)$$

reaches the minimum, when initial state vector $x_0$ is given. Vector $\psi$ has $q$ components ($q \leq n-1$, if $L = 0$ and $q \leq n$, if $L \neq 0$).

Such a problem is similar to that considered in Ref. [10]. The relation is related to introduction of constraints (3) and the use of $\Delta t$ as state variable. The last permits us to consider the problems with free final time.

**Optimality conditions**

The first variation of Lagrange function gives

$$\delta J = F_x \delta x_N + \sum_{k=0}^{N-1} \left( L^{k+1}_x \delta x_k + L^{k}_u \delta u_k + L^{k}_\Delta t \delta \Delta t_k \right) +$$

$$+ \sum_{k=0}^{N-1} \lambda^{k+1}_x (g^k_{xx} \delta x_k + g^k_{xu} \delta u_k + g^k_{x\Delta t} \delta \Delta t_k - \delta \psi_k) +$$

$$+ \sum_{k=0}^{N-1} \lambda^{k+1}_u (\delta u_k - \delta \Delta t_k) + \nu^T \psi x \delta x_N +$$

$$+ \sum_{k=0}^{N-1} \lambda^{k+1}_t (\delta \Delta t_k - \delta \Delta t_k) +$$

where $\lambda$, $\mu$, $\psi$, $\nu$ are Lagrange multipliers and $\psi'$ denotes transposition. Let's introduce Hamiltonian

$$H^k = L^k(x_k, u_k, \Delta t_k) + \lambda^{k+1}_x (g^k_{xx} \delta x_k + g^k_{xu} \delta u_k + g^k_{x\Delta t} \delta \Delta t_k - \delta \psi_k) +$$

$$+ \mu^{k+1}_u (\delta u_k - \delta \Delta t_k) + \psi_k \nu^T \psi x \delta x_N$$

and adopt

$$\lambda^k = H^k, \quad \mu^k = H^k_\mu, \quad k = 0, \ldots, N-1,$$

$$\lambda^N = (F_x + \nu^T \psi)_x', \quad \mu_0 = 0,$$

then

$$\delta J = \lambda^0_0 \delta x_0 + \mu^0_0 \delta \Delta t_0 + \sum_{k=0}^{N-1} H^k_\mu \delta u_k + \omega^k \delta \psi_k.$$  

It may be shown [10] that for optimal control at $\delta x_0 = 0$ the following expressions are valid:

$$H^k_\mu = 0,$$

$$\omega_k \geq 0, \quad \phi^k = 0,$$

$$\omega_k = 0, \quad \phi^k < 0,$$

$$k = 0, \ldots, N-1,$$

$$\mu_0 = 0.$$

The determination of optimal solution demands the solution of two-point-boundary-value-problem (TPBVP), described by equations (2), (4), (5) and (6).

**The variation of solution**

Suppose that we have nominal optimal solution that satisfy (2), (4), (5) and (6). Let's consider small deviations from this solution that arise from desired small disturbances in $\delta x_0, \delta \psi_0, \delta \phi, \delta H^k_\mu$ and $\delta \mu_0$. These disturbances lead to the variations of $\delta x_k, \delta \lambda_k, \delta \mu_k, \delta \phi_k$ and $\delta v$ and $\delta u_k$ that satisfy linearized equations (2), (4), (5) and (6):

$$\delta x_{k+1} = g^k_{xx} \delta x_k + g^k_{xu} \delta u_k + g^k_{x\Delta t} \delta \Delta t_k,$$

$$\delta \Delta t_{k+1} = \delta \Delta t_k,$$

$$\delta \lambda_{k+1} = \delta \lambda_k,$$

$$\delta \psi_{k+1} = \delta \psi_k,$$

$$\delta H^k_\mu = \delta H^k_\mu,$$

$$\delta \mu_{k+1} = \delta \mu_k + \delta \phi_{k+1} \delta \phi_k +$$

$$+ \delta H^k_\mu \delta x_k + \delta H^k_\mu \delta \Delta t_k + \delta \phi_k \delta \phi_k,$$

$$\delta \psi_{k+1} = \delta \psi_k,$$

$$\delta \mu_{k+1} = \delta \mu_k + \delta \phi_{k+1} \delta \phi_k +$$

$$+ \delta H^k_\mu \delta x_k + \delta H^k_\mu \delta \Delta t_k + \delta \phi_k \delta \phi_k,$$

$$\delta \psi_{k+1} = \delta \psi_k,$$

$$\delta \mu_{k+1} = \delta \mu_k + \delta \phi_{k+1} \delta \phi_k +$$

$$+ \delta H^k_\mu \delta x_k + \delta H^k_\mu \delta \Delta t_k + \delta \phi_k \delta \phi_k,$$

$$\delta \psi_{k+1} = \delta \psi_k,$$

$$\delta \mu_{k+1} = \delta \mu_k + \delta \phi_{k+1} \delta \phi_k +$$

$$+ \delta H^k_\mu \delta x_k + \delta H^k_\mu \delta \Delta t_k + \delta \phi_k \delta \phi_k,$$

$$\delta \psi_{k+1} = \delta \psi_k,$$

$$\delta \mu_{k+1} = \delta \mu_k + \delta \phi_{k+1} \delta \phi_k +$$

$$+ \delta H^k_\mu \delta x_k + \delta H^k_\mu \delta \Delta t_k + \delta \phi_k \delta \phi_k,$$

$$\delta \psi_{k+1} = \delta \psi_k,$$

$$\delta \mu_{k+1} = \delta \mu_k + \delta \phi_{k+1} \delta \phi_k +$$

$$+ \delta H^k_\mu \delta x_k + \delta H^k_\mu \delta \Delta t_k + \delta \phi_k \delta \phi_k.$$  

The equations (7) may be rewritten in more compact form:

$$\delta y_{k+1} = A^k \delta y_k + B^k \delta u_k,$$

$$\delta \pi_k = A^k \delta \pi_{k+1} + B^k \delta y_k + B^k \delta u_k + B^k \delta \phi_k,$$

$$\delta H^k_\mu = A^k \delta H^k_{\mu k+1} + B^k \delta y_k + B^k \delta u_k + B^k \delta \phi_k,$$

$$\delta \phi_k = B^k \delta u_k + B^k \delta \phi_k,$$

$$\delta \pi_k = B^k \delta \phi_k.$$

where

$$\delta y_k = \begin{pmatrix} \delta x_k \\ \delta \lambda_k \end{pmatrix}, \quad \delta \pi_k = \begin{pmatrix} \delta \lambda_k \\ \delta \mu_k \end{pmatrix},$$

and

$$\delta \psi = \psi x \delta x_N,$$

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$$\delta \psi = \psi x \delta x_N.$$
\[
A^k = \begin{pmatrix}
\frac{\delta y}{\delta z}, & \frac{\delta y}{\delta z}, & Q_{nl}
\end{pmatrix}, \quad B^k = \begin{pmatrix}
H_{xy}^k, & H_{x\delta_t}^k, & Q_{nl}
\end{pmatrix},
\]
\[
\alpha^k = \begin{pmatrix}
\frac{\delta y}{\delta z}, & \frac{\delta y}{\delta z},
\end{pmatrix}, \quad \phi^k = \begin{pmatrix}
H_{xy}^k, & H_{x\delta_t}^k
\end{pmatrix}, \quad \delta^k = \begin{pmatrix}
\phi_{y'}, & \phi_{y''}
\end{pmatrix},
\]
\[
S_N = \begin{pmatrix}
F_{xx} + v \psi_{xx} Q_{nl} & v' \psi_x
\end{pmatrix},
\]
\[
Q_{nl} = 0, \quad Q_{nq} = 0, \quad Q_{q1} = Q_{q2}.
\]

Equation (13)

\[
\delta q_{y_k} = S_N \delta y_k + h_k
\]

substituting (8)-(12) in (14) we'll have

\[
\delta q_{y_{k+1}} = S_N \delta y_k + h_k + A_k \delta y_{k+1} + A_k \delta h_{k+1} + b_k,
\]

where

\[
A_k = \begin{pmatrix}
H_{xy}^k, & H_{x\delta_t}^k, & Q_{nl}
\end{pmatrix},
\]

and substituting (8)-(12) in (14) we'll have

\[
\delta q_{y_{k+1}} = S_N \delta y_k + h_k + A_k \delta y_{k+1} + A_k \delta h_{k+1} + b_k,
\]

Substitution of this expression in (9), (10) gives

\[
\delta q_{y_k} = \frac{Z_{yy}^k \delta y_k}{Z_{yu}^k} + Z_{yu}^k \delta u_k + \delta q_{y_k} + A_k \delta h_{k+1},
\]

\[
\delta q_{u_k} = \frac{Z_{yu}^k \delta y_k}{Z_{uu}^k} + Z_{uu}^k \delta u_k + \delta q_{u_k} + A_k \delta h_{k+1},
\]

and

\[
\delta q_{h_k} = \frac{Z_{yu}^k \delta y_k}{Z_{uu}^k} + Z_{uu}^k \delta u_k + \delta q_{u_k} + A_k \delta h_{k+1}.
\]

It is possible now to compute \( S_0, h_0 \) by means of (13) taking into account that \( h_N = 0 \). The condition (14) at \( k = 0 \) gives the possibility to compute \( \delta q_{y_0}, \delta q_{u_0}, \delta q_{h_0} \) that are components of \( \delta y_0, \delta u_0, \delta h_0 \). Thereafter, we'll get the solution of linear problem (8)-(12) using recurrent expressions (8) and (17).

Iterative procedure

Here we'll give a description of some peculiarities of iterative procedure organization, but it is brief because of restricted volume of the paper. Let's assume that we have succession \( x_k, \ k = 0, ..., N-1, \) and succession \( x_k, \ k = 0, ..., N \), which satisfy (2) and they are near to optimal solution in a sense that initial condition and (3), (4) are satisfied with small disturbances \( \Delta q_{y_0}, \Delta q_{u_0}, \Delta q_{h_0} \), \( \Delta q_{h_0} \). Then adopting \( \delta q_{y_0} = \Delta q_{y_0}, \delta q_{u_0} = \Delta q_{u_0}, \delta q_{h_0} = \Delta q_{h_0} \) it is possible to get the improving variation of solution. But the received in accordance with (8) variations of \( \delta q_{x_k} \) after adding to initial succession \( x_k \) will not satisfy equations of motion (2) exactly. That is the reason for the next proposal to compute \( \delta q_{x_k} \) (necessary to substitute in (17)) not by use of the first formula in (7), but as the difference

\[
\delta q_{x_k} = \frac{g^{k-1}(x_{k-1} + \delta q_{x_{k-1}}) + \delta q_{x_{k-1}} + \delta q_{x_{k-1}} + \Delta x_k, \Delta t + \Delta t} - x_k,
\]

where \( \delta q_{x_{k-1}} \) and \( \delta q_{x-1} \) is determined by (17).

Short description of computer program peculiarities.

The rather complicated computer program was developed on the basis of this problem statement for three-dimensional motion of aircraft.

The list of the main points of the developed program for aircraft path optimization looks as follows:

- smooth approximation of the aerodynamic, thrust-consumption characteristics, and the usage of analytical expressions for the first and the
second derivatives, that are necessary in the second order method like Ref. [10],
* the employment of variation of the first order necessary conditions of optimality and solution of TPBVP for linearized direct and conjugate systems by the help of reverse run method in manner like Ref. [10],
* the procedure of calculation of Lagrange multipliers from the minimization of the norm of Hamiltonian derivative vector for the beginning of iteration procedure and their correction by means of neighboring-optimal technique in iterative procedure,
* the employment of the short initial trajectories with fulfillment of convexity condition for iteration procedure, which through the neighboring-optimal technique provides given of vector $x_0$ and given boundary condition $\psi(x_0) = 0$.
* the correction or improvement of the second derivatives matrix (in the neighboring-optimal technique), which must be reversed on the basis of transition to the principal axes of the appropriate quadratic form.

IV. Flight Path Optimization

The results presented in this section are obtained by the neighboring-optimal technique on the basis of the second order method optimization program described in section III. It will be shown that the optimal schedules in space "altitude - specific energy" ( $(h, E)$-space) may be sufficiently simple for approximation. The approximation of throttle control is more complicated.

Equations of motion

The usual equations of motions were used for solving some problems of optimal control in vertical plane by the help of the second order method

$$
\dot{V} = g(n_x \sin \eta) \quad \dot{\eta} = \frac{g}{V}(n_y - \cos \eta),
$$

$$
\dot{h} = \dot{V} \sin \eta \quad \dot{V} = \dot{V} \cos \eta,
$$

$$
\dot{W} = -Q_\alpha,
$$

where the tangential and the normal load factors

$$
n_x = \frac{T_{\max}(M, h) \eta}{W} \cos \alpha - \frac{qS}{W} C_D,
$$

$$
n_y = \frac{T_{\max}(M, h) \eta}{W} \sin \alpha + \frac{qS}{W} C_L,
$$

the fuel consumption per second

$$
Q_\theta = Q_\eta(\gamma, M, h),
$$

the other notation is usual.

All the dependencies such as $T_{\max}(M, h)$, $C_D(C_L, M)$, $Q_\eta(\gamma, h, M)$ are approximated by polynomials and fraction-rational functions. The control variables are $\eta$ and $C_L$ in the presence of inequality constraints:

$$
\eta_{\min} \leq \eta \leq 1, \quad C_L \leq C_{L, \max}(M), \quad \frac{qS}{W} C_L \leq n_{\max}.
$$

The inequality constraints on state variables are determined by permissible flight domain:

$$
V_{\min}(h) \leq V \leq V_{\max}(h), \quad h \geq h_{\min}.
$$

The additional constraints on flight path angle may be imposed

$$
\gamma_{\min} \leq \gamma \leq \gamma_{\max},
$$

where $\gamma_{\min}$, $\gamma_{\max}$ depend on $h$, $V$ and, maybe, on $n_x$.

The final conditions for state coordinates are given for each problem.

The transition from take-off to final value of specific energy.

The results of optimization for transition from initial condition corresponding to take-off to the given final value of specific energy $(E = h + V^2/2g)$, which correspond to the minimum fuel consumption, are presented in Figures 1-3 for various values of initial weight.

![Figure 1. Extremals in $(h, E)$-space.](image1.png)

![Figure 2. Extremals in $(h, M)$-space.](image2.png)
It is known that such so-called basic schedules may be considered as the portions of the appropriate more common schedules. Each of the computed schedules has the portion of descending the shape of which sufficiently depends on initial weight. The constructed family of schedules may be presented in form of two-dimensional function of altitude $\mu(E, W_0)$.* This function after approximation may be used for on-board implementation in control system. Figures 4–6 show how fuel consumption $\Delta w$, time $\Delta t$ and range $\Delta l$ for these schedules depend on $E$ for various values of initial weight $W_0$.

These dependencies are the functions of two arguments: $\Delta w(E, W_0)$, $\Delta t(E, W_0)$, $\Delta l(E, W_0)$ that have simpler form than basic schedules for approximation. These functions allow us to compute the fuel consumption, time and range for the flight from the point 1 to the point 2 on the schedule:

\[
\Delta w_{1,2} = \Delta w(E_2, W_0) - \Delta w(E_1, W_0),
\]

\[
\Delta t_{1,2} = \Delta t(E_2, W_0) - \Delta t(E_1, W_0),
\]

\[
\Delta l_{1,2} = \Delta l(E_2, W_0) - \Delta l(E_1, W_0).
\]

It may be necessary to go to basic schedule, when instant position is not on basic one corresponding to instant value of aircraft weight. **the superscript $D$ corresponds to desirable value**
The dependencies $W_0(E, W)$ allow us to find for instant value of $E$ the weight $W_0$ that corresponds to initial point of basic schedule and to find the basic schedule $h^0(E, W_0)$ such that it may be desirable to go to the schedule and go along it further. For this purpose a smooth additional schedule of short duration may be constructed for transition from instant point in the vicinity of desirable schedule to this one.

**Minimum time transitions in vertical plane**

The optimization for minimum time transition in vertical plane from initial condition corresponding to take-off to the final condition $h_f = \text{const}$, $V_f = 0$ for various values of final speed $V_f$ was fulfilled. The appropriate extremals field in $(h, E)$-space and $(h, M)$-space are presented in Figures 7, 8.

![Figure 7: Extremals in the $(E, h)$-space.](image)

![Figure 8: Extremals in the $(M, h)$-space.](image)

It is seen that schedules in $(h, E)$-space are simple and more convenient for approximation. The thrust is equal to maximum thrust for all trajectories. These results are analogous to those obtained in Ref. (11, 12), but have more regular character.

**The flight to cruise point with fixed range**

The flight to cruise point with the minimum fuel consumption is the first portion of flight for sufficiently large duration. The final conditions on $h, V$ at $\gamma = 0$ correspond to minimum of the fuel consumption per unit of range ($Q_{km}$) and depend on the final weight. The approximation of cruise regime in form

$$h_f - a_{11} - a_{12} W_f = 0,$$
$$V_f - a_{21} - a_{22} W_f - a_{23} W_f^2 = 0,$$
$$Q_{km} = a_{31} - a_{32} W_f,$$

gives good result. Positive coefficients $a_{ij}$ depend on aircraft characteristics, its configuration and atmospheric conditions.

The calculations for several values of range show that, after some value of range, the schedules vary insignificantly and may be easily approximated.

**V. Minimum Time Descent**

We'll consider in this Section the minimum time descent in vertical plane with the usage of maximum principle in simplified problem statement when control variables are $\eta$ and $\sin \gamma$. This approach is sufficiently simple for generation of extremals field in $(h, V)$-space for given initial conditions because the problem has only one parameter.

The application of previously described second order method is not successful for such a problem because the regular iteration procedure brings to trajectories with the control and state variables on boundaries of the admissible region. It requires the employment of complicated procedure for removing control and state variables from boundaries.

The equation of motion for this problem can be rewritten as follows

$$\dot{T} = g(\eta^0 - \sin \gamma), \quad \dot{V} = \dot{V} \sin \gamma,$$

where

$$\eta^0 = \frac{T_{max}(M, h) \eta}{W} \cos \alpha - \frac{q S}{W} C_D,$$

$$C_D = C_D(C_L, M), \quad C_L = \frac{W}{qS}, \quad W = \text{const}.$$

All the assumption in motion model are contained in these expressions.

It is necessary to transit from the given initial condition $h_0, V_0$ to any other for minimum time by choosing optimal schedules in $(h, V)$-space and appropriate throttle control with account of inequality constraints in the form

$$\sin \gamma_{min} \leq \sin \gamma \leq \sin \gamma_{max},$$
where \( \nu_{\text{min}} \) and \( \gamma_{\text{max}} \) are functions of \( h, V \) and \( n_x^0 \). These constraints are related to flight envelope limitations.

We'll give the common scheme of this problem solution omitting some details. The Hamiltonian for this problem may be written in form

\[
H = \lambda_V g n_x^0 + \Omega \sin \gamma,
\]

where \( \Omega = \lambda_h V - \lambda_V g \). The necessary conditions for optimality include the adjoint equations

\[
\lambda_V = -\frac{\partial H}{\partial V}, \quad \lambda_h = -\frac{\partial H}{\partial h}.
\]

The solution of these equations linearly depend on initial values \( \lambda_V^0 = \lambda_V(0) \) and \( \lambda_h^0 = \lambda_h(0) \). If the final values \( h_F, V_F \) would be given then it will be necessary to choose the values of \( \lambda_V^0 \) and \( \lambda_h^0 \) appropriately. But in our case by varying both parameters we can generate the extremals originating from point \( h_0, V_0 \) in \((h,V)\)-space. It may be shown however that there is no need in varying these parameters and it is sufficient to vary only one parameter \( \phi \) choosing the values of \( \lambda_V^0 \) and \( \lambda_h^0 \) in the following manner

\[
\lambda_V^0 = \cos \phi, \quad \lambda_h^0 = \Omega^0 + \frac{\lambda_0^0}{V} g.
\]

It follows from linear dependence of adjoint variables on \( \lambda_V \) and \( \lambda_h \) of Hamiltonian on \( \lambda_V \) and \( \lambda_h \). The optimal control then is determined from condition

\[
\min_{\eta, \sin \gamma} H < 0.
\]

The condition (19) at initial time moment gives the admissible subrange of \( \phi \) value within 0 to 360° range. The singular control for \( \sin \gamma \) arises when \( \Omega = 0 \). From this condition, it follows that

\[
r_x^0 = \frac{V}{g} \left( V \frac{\partial n_x^0}{\partial h} - g \frac{\partial n_x^0}{\partial V} \right).
\]

This expression gives the appropriate schedule in \((h,V)\)-space which depends on optimal value of \( \eta \). It is interesting to note that at \( \eta = 1 \) this schedule corresponds to minimum time energy climb. This follows from the condition

\[
\frac{d}{dV} \left( n_x(t, h, V, h(E, V), V) \right) = 0
\]

at \( E = \text{const} \).

The example of the extremals field in \((h,V)\)-space for initial point \( A \) is given in Figure 9. Several regions are shown in this field. For each region the throttle control has some peculiarities. The results may be approximated for their usage for control law synthesis when descending.

![Figure 10. Extremals in \((h,V)\)-space.](image)

**VI. Control Law for Normal Load Factor**

Let's consider the tracking of a given desirable schedule \( \lambda = \lambda(h) \) at \( E = \text{const} \) on the basis of linear equivalent models, that specify the law of compensation of the difference between desirable and current values of the flight parameters. We'll take such models in the form

\[
\frac{d}{dt} \Delta h + k_h \Delta h = 0, \quad \frac{d}{dt} \Delta \sin \gamma + k_\gamma \Delta \sin \gamma = 0,
\]

where

\[
\Delta h = \lambda - h, \quad \Delta \sin \gamma = \sin \gamma - \sin \gamma
\]

and \( k_h, k_\gamma \) — parameters.

These models correspond to smooth transition processes and it is possible to agree the long period motion quality and short one by choosing \( k_h, k_\gamma \). It is follows from (20) and (18) that

\[
\sin \gamma = \frac{d}{dt} \left( n_x^0 \frac{d \lambda}{dt} + k_h \Delta h \right),
\]

\[
n_x^0 = \frac{V}{g \cos \gamma} \left( \frac{d}{dt} \sin \gamma + k_\gamma \Delta \sin \gamma + \cos \gamma \right).
\]

The tangential load factor \( n_x^0 \) depends on \( n_x^0 \). The system of two equations (21) contains two unknown parameters \( \sin \gamma \) and \( n_x^0 \). It is possible to find the solution of this system presenting the dependence of \( n_x^0 \) upon \( n_x^0 \) by means of appropriate rank in the vicinity of instant value of \( n_x^0 \). It is important for on-board implementation. For usual trajectory computation it is sufficient to use (21) adopting that \( n_x^0 = n_x \).
VII Adaptive Control Law for Throttle

If the specific energy varies monotonously

\( \dot{E} > 0 \) or \( \dot{E} < 0 \) functional (1) may be rewritten in form

\[
J = \int f dE,
\]

\[
\Gamma = \frac{\lambda_W Q_3 + \lambda_V - \lambda_I V \cos \gamma}{n_x V},
\]

(22)

where \( \Gamma \) is the given realized schedule in \((h, E)\)-space, and \( \lambda_W \), \( \lambda_V \), \( \lambda_I \) are constant non-negative Lagrange multipliers that may be computed when the full optimization problem is solving. Assuming that dependence \( T = T(h, V, Q_3) \) is known and monotonous on \( Q_3 \), we'll find the optimal thrust control from minimum (at \( \dot{E} > 0 \)) or maximum (at \( \dot{E} < 0 \)) on \( Q_3 \) of function \( f \) in (22).

The function \( f \) in (22) depends on fast \( Q_3 \), \( n_y \) and slow \( h, V, \gamma \) variables. Therefore it is possible to search desirable value of \( Q_3 \) on-line by means of adaptive procedure like

\[
\dot{T} = \frac{Q_3}{n_x V} \frac{\partial T}{\partial Q_3}.
\]

(23)

In such approach we neglect the interrelation of \( Q_3 \) and \( n_y \) related to the necessity of tracking of the given schedule \( \Gamma \).

Assuming that the motion proceeds exactly along the given schedule and \( n_x = (T - D)/W \) it follows from (23) that the adaptive control law synthesis for throttle may have, for example, the form

\[
\dot{T} = \frac{Q_3}{n_x V} \frac{\partial T}{\partial Q_3}.
\]

(24)

where \( k_T \) is a positive control law parameter. It is necessary to introduce additionally to (24) the limitations on \( \dot{T} \) and \( T \).

The usage of such control law demands the information about instant values of \( n_x, Q_3, W, V, \gamma \) and estimation of the derivative \( \partial T / \partial Q_3 \) that may be introduced by means of its approximation. If the schedule in \((h, E)\)-space for \( \dot{E} > 0 \) corresponds to minimum fuel consumption then \( \lambda_W = 1, \lambda_V = 0, \lambda_I = 0 \), the expression (24) may be written as follows

\[
\dot{T} = -k_T \left( n_x V - \frac{Q_3}{n_x V} \frac{\partial T}{\partial Q_3} \right).
\]

For minimum time we have \( \lambda_T = 1, \lambda_W = 0, \lambda_I = 0 \),

\[
\dot{T} = k_T \frac{1}{W} \frac{\partial T}{\partial Q_3} > 0
\]

and for this reason \( T = T_{\text{max}} \). For minimum fuel consumption when time and range are given or free this control law provides acceptable results. The results of computations show that for climb up the usage of such adaptive control law gives the fuel consumption that does not exceed the optimal one more than 0.2% when all information is exact. In more complicated examples this excess reaches 2%.

Conclusions

The examples described in this paper indicate that considered technique of flight path optimization gives the acceptable results for the extremals fields generation. Since the regularities of such fields may be understood there are possibilities for approximation of extremals and appropriate control synthesis. There are ways of simplification of the computed schedules without significant change of results for the worse. But this topic is not discussed in this paper.

References


