THEORY AND EXPERIMENTS ON OPERATING PRINCIPLE OF 
HEMISPHERICAL RESONATOR GYRO†

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Abstract

On the basis of the thin-wall shell's theory and practical construction feature of hemispherical shell resonator (HRG), this paper establishes the HRG's math model and derives the relation between scale factor and construction parameters. Experiments were carried out with laser holographic interferometry which quite agrees with theoretical model. The results provide a reliable theoretical basis for developing the new gyro, especially for designing the hemispherical shell resonator.

I. Introduction

As compared with the conventional gyro, hemispherical resonator gyro without spinning-wheel and bearing takes advantage of mechanical resonating technique and Coriolis effect occurred as the hemispherical shell resonator rotates. It has many unique advantages, such as small size, short reaction time, long time constant, allowing operation over a wide temperature range with no temperature control and warm-up delay etc. It appears that HRG is one of the best components in the new generation of strap-down inertial navigation system. USA firstly reported that Delco System Operation had developed an inertial grade HRG performance in 1982(1), but no theoretical derivation therein. Soon after that, former USSR also reported some results studied on HRG (2). But there some considerable difference between the math model and reality, up to now.

A thin-wall hemispherical shell in resonant vibrating state, with the top free and the bottom clamped (Fig.1), is the sensing component of the HRG. When the shell rotates angle \( \psi \), about its central axis, the vibrating shape in circumferential direction of the hemispherical shell will move or precess an angle \( \psi \) relative to the shell reverse movement(Fig.2). It's so surprised that the ratio of \( \psi / \psi \) is a constant, and independent of the rotation angular velocity of the above shell. So, the angle \( \psi \) can be calculated by measuring the precession of the vibrating shape in circumferential direction. This is the operating principle of HRG.

In fact, the above precession phenomenon for the hemispherical shell, was pointed out in 1890 by G.H. Bryan, a British Scientist (3). In this paper, the precession of the vibrating shape in circumferential direction for the practical hemispherical shell is analyzed, based on the thin-wall shell's theory. And some valuable results for designing and developing HRG are proposed through the theoretical and experimental researches.

II. Vibrating Shape

Fig.3 is the sketch of the hemispherical shell, \( \vec{X} \) is the central axis, the mean radius of the shell is \( R \), the material curved surface density is \( D(\varphi, \theta) \), the bottom and top angles are \( \varphi_a \) and \( \varphi_f \) respectively. \( \varphi \) and \( \theta \) are the longitudinal and circumferential coordinates respectively.

The displacement of the arbitrary point B is

\[
\vec{r} = u \vec{e}_1 + v \vec{e}_2 + w \vec{e}_3
\]

(1)

Where \( u, v, w \) are respectively the longitudinal, circumferential and radial displacements. \( \vec{e}_1, \vec{e}_2, \vec{e}_3 \) are respectively the relevant unit moving vectors.

\[ \text{FIGURE 1. Hemispherical Shell Construction} \]

\[ \text{FIGURE 2. Precession Scheme of Shell} \]
When the hemispherical shell does not rotate, its $n$th axisymmetric vibrating shape can be written as

\begin{equation}
\begin{aligned}
u(\varphi, \theta, t) &= u(\varphi) \cos n(\theta + \psi) \cos \omega t \\
v(\varphi, \theta, t) &= v(\varphi) \sin n(\theta + \psi) \cos \omega t \\
w(\varphi, \theta, t) &= w(\varphi) \cos n(\theta + \psi) \cos \omega t
\end{aligned}
\end{equation}

Where $u(s)$, $v(s)$, $w(s)$ are respectively vibrating shape in longitudinal direction and $\omega$ is the relative natural frequency of the hemispherical shell.

When the shell rotates at $\vec{O} = \vec{O}_x + \vec{O}_y$, in the inertial space (Fig.3). And in the rotation space, the corresponding axisymmetric vibrating shape can be written as

\begin{equation}
\begin{aligned}
u(\theta, t) &= u(\varphi) \cos n(\theta + \psi) \\
v(\theta, t) &= v(\varphi) \sin n(\theta + \psi) \\
w(\theta, t) &= w(\varphi) \cos n(\theta + \psi)
\end{aligned}
\end{equation}

\[\psi = \int_{t_0}^t P dt\]

Where $P$ is the precession rate of the vibrating shape related to the shell in circumferential direction. As compared with equations (2) and (3), when the shell rotates at $\vec{O}$ about its central axis $\vec{X}$ and moves angle $\psi$, $\psi = \int_{t_0}^t \Omega_s dt$, the vibrating shape retroacts at $P$ in circumferential direction, and moves angle $\psi = \int_{t_0}^t P dt$.

From the point of view of the vibrating shape, the equation (3) includes two principal vibrating shape. One is the vibration of the shell, another is the rotation of the vibrating shape of the shell. Both were represented respectively by $\cos \omega t \cos n(\theta + \psi)$ or $\sin n(\theta + \psi)$.

The actual hemispherical shell is free at the top ($\varphi_1$) and clamped at the bottom ($\varphi_r$), So, Lord Rayleigh's condition of inextensibility is satisfied. The vibrating shape in longitudinal direction should satisfy (4).

\[\begin{aligned}
u(\varphi) &= v(\varphi) \\
w(\varphi) &= -\frac{du(\varphi)}{d\varphi}
\end{aligned}\]

And when the bottom angle $\varphi_r$ is much smaller, the vibrating shape approximately is

\[u(\varphi) = v(\varphi) = C \sin \theta \sin \varphi \cos \theta \cos \varphi \]

\[w(\varphi) = -C(1 + \cos \varphi) \theta \sin \varphi \]

$C$ is the constant which is dependent on the vibration energy of the shell.

### III. Virtual Work

When the shell rotates at $\vec{O}$, the inertial force in point $B$ is

\[F = -\vec{a}D(\varphi, \theta)R^2 \sin \varphi d\varphi d\theta \]

\[\vec{a} = \vec{\omega}_0 + \vec{\omega}(\Omega_x + \Omega_y) \]

\[\vec{\omega}_0 = \frac{3}{2} \frac{u}{a^2} \vec{e}_1 + \frac{3}{2} \frac{v}{a^2} \vec{e}_2 + \frac{3}{2} \frac{w}{a^2} \vec{e}_3 \]

\[a(\Omega_x), a(\Omega_y) \] are the accelerations caused by the angular velocity $\Omega_x, \Omega_y$ respectively. So, the virtual work done by inertial $\vec{F}$ is

\[\delta T = \int S \cdot \delta \vec{V} \]

$S$ is the integrated curved surface domain. Using the relevant relationships in Reference (4), we have

\[\delta T = \delta T_0 + \delta T(\Omega_x) + \delta T(\Omega_y) + \delta T(\Omega) - \delta W(\Omega) \]

\[\delta T_0 = \omega^2 R^2 \cos^2 \varphi \int \int \int \left\{ u(\varphi) \delta u + w(\varphi) \delta w \right\} \cos^2 n(\theta + \psi) \delta \varphi + \delta \psi \right\} D(\varphi, \theta) \sin \varphi d\varphi d\theta \]

\[\delta T(\Omega_x) = R^2 \cos^2 \varphi \int \int \int \left\{ u(\varphi) \delta u + w(\varphi) \delta w \right\} \cos^2 n(\theta + \psi) \delta \varphi \]

\[\delta T(\Omega_y) = R^2 \cos^2 \varphi \int \int \int \left\{ u(\varphi) \delta u + w(\varphi) \delta w \right\} \cos^2 n(\theta + \psi) \delta \varphi \]

\[\delta T(\Omega) = 2n P \Omega \int \int \int \left\{ u(\varphi) \delta u + w(\varphi) \delta w \right\} \cos^2 n(\theta + \psi) \delta \varphi \]

\[\delta W(\Omega) = 2n P \Omega \int \int \int u(\varphi) \sin \varphi \sin \theta \cos \varphi \cos \theta \theta \sin \varphi d\varphi d\theta \]

\[\delta W(\Omega) = 2n P \Omega \int \int \int w(\varphi) \sin \varphi \sin \theta \cos \varphi \cos \theta \theta \sin \varphi d\varphi d\theta \]

\[\delta W(\Omega) = 2n P \Omega \int \int \int w(\varphi) \sin \varphi \sin \theta \cos \varphi \cos \theta \theta \sin \varphi d\varphi d\theta \]

\[\delta W(\Omega) = 2n P \Omega \int \int \int w(\varphi) \sin \varphi \sin \theta \cos \varphi \cos \theta \theta \sin \varphi d\varphi d\theta \]
\[
-w(\phi)\cos\theta \cos\phi \sin^2 n(\theta + \psi) \delta \psi - (v(\phi) \sin \phi \sin \psi \sin n(\theta + \psi) + \psi \delta \psi \phi) \cdot \dot{D}(\phi, \theta) \sin \phi d\phi d\theta
\]

\[\delta T + \delta W = \delta T_{\Omega} + \delta T(\Omega_x) + \delta T(\Omega_y) + \delta T(\Omega_z) - \delta W(\Omega) \]

\[\delta W_0 = 0 \]

\[\delta W_{\Omega} \]

The definition of angle \(\beta\) is shown in Fig. 3. \(\delta W(\Omega)\) is the virtual work done by the initial elastic force which is caused by \(\Omega\). \(\delta T(\Omega)\) is the virtual work, which is independent of the rotation rate \(P\), done by the "grain inertial" force caused by \(\Omega\). Using the virtual work principle, we have

\[G_x = \int_0^{2\pi} \left\{ \int_0^{2\pi} \frac{1}{2} v(\phi) \sin 2\phi + w(\phi) \sin^2 \phi \cos n(\theta + \psi) \right\} d\phi d\theta + \int_0^{2\pi} \frac{1}{2} v(\phi) \sin 2\phi + w(\phi) \sin^2 \phi \cos n(\theta + \psi) d\theta \]

\[G_y = \int_0^{2\pi} \left\{ \int_0^{2\pi} \frac{1}{2} v(\phi) \sin 2\phi + w(\phi) \sin^2 \phi \cos (\theta + \psi) \right\} d\phi d\theta + \int_0^{2\pi} \frac{1}{2} v(\phi) \sin 2\phi + w(\phi) \sin^2 \phi \cos (\theta + \psi) d\theta \]

\[G_z = \int_0^{2\pi} \left\{ \int_0^{2\pi} \frac{1}{2} v(\phi) \sin 2\phi + w(\phi) \sin^2 \phi \cos n(\theta + \psi) \right\} d\phi d\theta + \int_0^{2\pi} \frac{1}{2} v(\phi) \sin 2\phi + w(\phi) \sin^2 \phi \cos n(\theta + \psi) d\theta \]

\[G_x, G_y, G_z\] represent the Coriolis effects caused by \(\Omega_x\) and \(\Omega_y\) respectively, and \(G_z\) represents the inertial effect for the shell.

As the curved surface density \(D(\phi, \theta)\) is constant in the circumferential direction,

\[D(\phi, \theta) = D(\phi)\]

Then the \(G_z = 0\), which indicates that the precession of the vibrating shape in circumferential direction is only caused by the angular velocity \(\Omega_z\), and independent of the \(\Omega_x\) and \(\Omega_y\). Therefore, only one directional angular message can be sensed, and there is no error of the cross influence in principle for HRG.

Then, by using the above relevant equations, we can conclude the precession factor

\[K = \frac{P}{\Omega_z} \]

\[2\int_0^{2\pi} (n - \cos \phi) \sin^2 \phi \left( \frac{1}{2} D(\phi) d\phi - \sin^2 \phi \cos \frac{\theta}{2} D(\phi) \right) \]

\[\int_0^{2\pi} \sin^2 \phi \cos^2 \left( \frac{\theta}{2} D(\phi) d\phi - \sin \phi \cos \frac{\theta}{2} D(\phi) \right) \]

Table 1 gives the results calculated by equation (22) as the curved surface density is uniform \(D(\phi) = D_0\).

It is shown that the \(n = 2\) is the best vibrating mode for HRG from the point of the view of measurement, wherein \(K \approx 0.3\). Moreover, whose relevant vibrating frequency is the lowest, thus easy making the shell in resonant vibrating state.

V. Error Model

For a actual hemispherical shell resonator, its curved surface density \(D(\phi, \theta)\) is a variable instead of a constant in circumferential direction. In this paper, by the way, we describe the deficiency of the hemispherical shell in circumferential direction. So, \(G_x, G_y, G_z\) are all variables versus the angle \(\psi\), which is shown as follows.

1. When \(\Omega_x = 0\), the precession rate \(P\) caused by \(\Omega_x\) is related to the angle \(\psi\) or the distribution of the vibrating shape in circumferential direction. The precession factor \(K\) is a variable, which causes the measurement
relative error for HRG. We define the precession factor $K_x$ which is only caused by the $\Omega_x$.

$$K_x = \frac{P}{\Omega_x} = \frac{-2G_x}{nG_p}$$  \hspace{1cm} (23)$$

In order to analyze the relative error, define

$$\sigma_x = \frac{K_{x_{\text{max}}} - K_{x_{\text{min}}}}{K_{x_0}}$$  \hspace{1cm} (24)$$

Where $K_{x_{\text{max}}}$, $K_{x_{\text{min}}}$, are respectively the maximum and minimum values of $K_x$, and $K_{x_0}$ is the ideal value of $K_x$ that is the $D(\phi, \theta) = D(\phi)$.

2. Since $G_{xy} \neq 0$, the $\Omega_{xy}$ may also cause the precession in circumferential direction, which is not the same as the above ideal condition that the $D(\phi, \theta) = D(\phi)$, which causes the measurement absolute error for HRG, called as the error of cross influence.

We define the precession factor $K_{xy}$, which is only caused by $\Omega_{xy}$.

$$K_{xy} = \frac{P}{\Omega_{xy}} = \frac{-2G_{xy}}{nG_p}$$  \hspace{1cm} (25)$$

Therefore, using the equations (24) and (25), we can analyze the $\sigma_x$ and $(K_{x_{\text{max}}})$ in order to analyze and determine the influence of the deficiency on the $D(\phi, \theta)$ for the precession of the actual hemispherical shell resonator in circumferential direction.

### VI. Conclusions

1. The uniformity in circumferential direction, especially in the neighborhood of the top, is the important principle for designing and selecting the hemispherical shell resonator, hereupon, the precession only caused by the $\Omega_x$ rotated about the central axis of the shell. There is no error of cross influence for HRG in principle.

2. In this paper, based on the thin-wall shell’s theory, orthogonality of the principal vibration mode and practical construction feature of the hemispherical shell resonator, we present the following formulae:

   Equation (22) for finding the precession factor $K$ versus $\phi$, $\phi_{\pi}$, $D(\phi)$, reflected the precession of the vibrating shape in circumferential direction, caused by $\Omega_x$.

   Equation (24), for finding the maximum relative error in principle.

   Equation (25), for finding the maximum absolute error in principle.

In a sense, equations (24), (25) may be determined the deficient limit in circumferential direction, or the permission variation limit of $D(\phi, \theta)$ versus the circumferential coordinate $\theta$, in principle. They may be referenced for selecting the parameters for the hemispherical shell resonator.

3. In this paper, a novel experimental project that takes advantage of the laser holographic interferometry was proposed. And the final experimental results obtained here, may provide the reliable theoretical basis for developing and designing HRG, especially for selecting the hemispherical shell resonator.
VII. Acknowledgement

The authors would like to thank the cooperation of Prof. Ding hanquan, Mr. Zhong Zhenpin and Mr. Shi Lei in experiment.

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