METHODS TO ENHANCE THE ACCURACY OF FINITE VOLUME SCHEMES*

Thilo Schönfeld
CERFACS, Centre Européen de Recherche et de Formation Avancée en Calcul Scientifique
Toulouse, France

and

Peter Wilde
FFA, The Aeronautical Research Institute of Sweden
Stockholm, Sweden

Abstract
Several high-order finite volume schemes to solve the Euler equations are constructed. The stability behaviour of the methods is analyzed and the usefulness of the schemes is demonstrated.

1. Introduction

Finite volume methods are today widely spread methods to solve the Euler and Navier-Stokes equations of gasdynamics. Most of the finite volume methods which are in use today are of at most first-order accuracy in space. We will describe where discretization errors during the approximation process are coming in and how they can be decreased. The main goal of this paper is to present some new techniques to increase the accuracy of the finite volume schemes by using high-order approximations. The development of high-order schemes is important for two reasons. The first one is the increasing interest in more accurate approximations. The second one is the necessity to reduce the computer time needed to solve numerically the problem under consideration.

In the following we give a brief outline on the approximation steps usually used in finite volume methods and we will point out the sources for discretization errors.

The starting-point for every finite volume method is the integral form of the conservation law for the quantities describing the fluid flow. This conservation law has to be fulfilled in each sub-domain $V_{ijk}$ of the whole flow domain $\Omega$ under consideration. Making use of the Gauß' theorem the conservation laws follows

$$\int_{V_{ijk}} \partial_t f \, dx + \int_{\partial V_{ijk}} H(f) \, n \, ds = 0, \quad \forall V_{ijk} \in \Omega, \quad (1)$$

where $n$ is the outward pointing unit normal vector to $\partial V_{ijk}$ and the vector $f$ is defined by $f := (\rho, u^1, u^2, u^3, e)^T$.

In connection with the stability theory we will also use the primitive variable formulation

$$\int_{V_{ijk}} \partial_t g \, dx + \int_{\partial V_{ijk}} (A_1 g_x + A_2 g_y + A_3 g_z) \, n \, ds = 0, \quad (2)$$

$\forall V_{ijk} \in \Omega$, with $g := (\rho, u^1, u^2, u^3, p)^T$, and where the matrices are given by

$$A_1 := \begin{pmatrix}
    u^1 & \rho & 0 & 0 & 0 \\
    0 & u^1 & 0 & 0 & \rho^{-1} \\
    0 & 0 & u^1 & 0 & 0 \\
    0 & 0 & 0 & u^1 & 0 \\
    0 & \gamma(\gamma - 1)e & 0 & 0 & u^1 \\
\end{pmatrix},$$

$$A_2 := \begin{pmatrix}
    u^2 & 0 & \rho & 0 & 0 \\
    0 & u^2 & 0 & 0 & 0 \\
    0 & 0 & u^2 & 0 & \rho^{-1} \\
    0 & 0 & 0 & u^2 & 0 \\
    0 & \gamma(\gamma - 1)e & 0 & 0 & u^2 \\
\end{pmatrix},$$

$$A_3 := \begin{pmatrix}
    u^3 & 0 & 0 & \rho & 0 \\
    0 & u^3 & 0 & 0 & 0 \\
    0 & 0 & u^3 & 0 & 0 \\
    0 & 0 & 0 & u^3 & \rho^{-1} \\
    0 & 0 & 0 & \gamma(\gamma - 1)e & u^3 \\
\end{pmatrix}.$$

Using the mean value theorem the volume integral in (1) writes

$$\int_{V_{ijk}} \partial_t f \, dx = \partial_t f(x_{ijk}^*, t) \int_{V_{ijk}} dx, \quad (3)$$

with $x_{ijk}^* \in V_{ijk}$ as a certain unknown point in the sub-volume $V_{ijk}$.

The first step toward obtaining a large coupled system of ordinary differential equations is to approximate the values $\partial_t f(x_{ijk}^*, t)$ by $\partial_t f(x_{ijk}, t)$, where $x_{ijk}$ is the center point of the cell $V_{ijk}$. This approximation is of first order accuracy in space and here we have the first source for errors. We will not say anything how to get a better approximation order for this step, but we will explain some high-order methods to solve the resulting system of equations which now is given by

$$\partial_t f(x_{ijk}, t) \int_{V_{ijk}} dx = -\int_{\partial V_{ijk}} H(f) \, n \, ds, \quad \forall V_{ijk} \in \Omega. \quad (4)$$

In the following $x_{ijk}$ denotes the center and $y_{ijk}$ the vertex of the cell $V_{ijk}$ which is the lower left corner point of the sub-surface common to the cells $V_{ijk}$ and $V_{i-1,jk}$.

* Partially supported by the Swedish National Board for Technical Development (STU), Stockholm.

Copyright (c) 1990 by T. Schönfeld and P. Wilde. Published by the American Institute of Aeronautics and Astronautics, Inc. with permission.
Now, the surface integral in (4) is usually approximated by
\[
\int_{\partial V_{ijk}} H(f) \, n \, ds \approx \sum_{i=1}^{6} \int_{S_{ij}^{i}} H(f) \, n \, ds \tag{5}
\]
\[
= \sum_{i=1}^{6} H(f(x_{i}^{i}, t)) |S_{ij}^{i}| n,
\]
where \(S_{ij}^{i}\) denote approximations to the six sub-surfaces of the cell \(V_{ijk}\), which are often calculated by \(S_{ij}^{i} = |d_1 \times d_2|/2\), with \(d_1\) and \(d_2\) as the diagonal vectors connecting opposite points on the sub-surfaces. \(x_{i}^{i}\) denotes a point on the sub-surface \(S_{ij}^{i}\). Since the values of \(H(f)\) are unknown on the sub-surfaces \(S_{ij}^{i}\), they are usually approximated by, e.g.,
\[
H(f(x_{i}^{i}, t)) \approx \frac{H(f(x_{i-1}^{i}, t)) + H(f(x_{i}^{i}, t))}{2}. \tag{6}
\]
In (5) and (6) the second and third sources for errors due to approximations arise. The approximation of the surface integrals in (5) is of at most first-order accuracy whereas the approximation in (6) is of at most second-order accuracy.

Using the common abbreviation \(f_{ij}^{i} := f(x_{ij}, t)\) the above approximations can be summarized to the following system of spatially discretized Navier–Stokes equations
\[
\partial_t f_{ij}^{i} \int_{V_{ijk}} dx = -L f_{ij}^{i}, \quad \forall V_{ijk} \in \Omega, \quad t > 0, \tag{7}
\]
where \(L\) is the operator approximating the surface integrals in the way mentioned above. Several finite volume schemes, explicit as well as implicit, based on (7) can be found, e.g., in [3].

Again, we point out that the approximation (7) is a discretization of (1) of at most first-order accuracy. The first of our new schemes is a simple upgrading of the existing finite volume methods in order to obtain second-order accuracy in space.

2. Explicit Finite Volume Approximations of High-Order

In order to construct our finite volume schemes in a first step we derive schemes to approximate the surface integrals with high-order accuracy. Next we use Taylor like expansions with respect to time, up to a certain order where the occurring time derivatives are replaced by the new spatial discretized finite volume equation (7). The idea to replace time derivatives by space derivatives was introduced by Lax and Wendroff [1,2]. Furthermore, we make the usual assumption that the matrices occurring in equation (2) are independent on both the time and space variables. It is mentioned that all the schemes are consistent.

Time step sizes are defined by \(\Delta t_l := t_{l+1} - t_l, \ l = 0, 1, 2, 3, \ldots\), and by \(\Sigma := \{t_0, t_1, t_2, \ldots\}\), with \(t_0 < t_1 < t_2 < \ldots\), we denote the corresponding partition of the time axis. The precise step sizes \(\Delta t_l\) result from the restriction for stability.

2.1. A Second-Order Spatial Discretization

In this chapter we propose a simple way to calculate numerically the surface integrals in (4) with second-order accuracy. Let \(a, b, c, \) and \(d\) be the vertices common to the cells \(V_{ijk}\) and \(V_{i-1,jk}\), where \(a\) is the lower left vertex and \(a, b, c, \) and \(d\) are counted clockwise (see Figure 1). We then define two triangles \(S_{ij}^{1}\) and \(S_{ij}^{2}\) in the way that \(S_{ij}^{1}\) is the plane triangle with corner points \(a, b, \) and \(c, d\) and \(S_{ij}^{2}\) the corresponding one with corner points \(c, d, \) and \(a\). In a similar way we can define triangles \(F_{ijk}\) and \(F_{ijk}\) in the plane triangle with corner points \(a, b, \) and \(c, d, \) and \(a\).

In (7) we can now be written as
\[
\int_{\partial V_{ijk}} H(f) \, n \, ds
\]
\[
= \sum_{i=1}^{2} \left( \int_{S_{ij}^{i}} H(f) \, n \, ds - \int_{S_{ij}^{i-1}} H(f) \, n \, ds \right)
\]
\[
+ \sum_{i=1}^{2} \left( \int_{F_{ij}^{i}} H(f) \, n \, ds - \int_{F_{ij}^{i-1}} H(f) \, n \, ds \right), \tag{8}
\]
\[
+ \sum_{i=1}^{2} \left( \int_{G_{ij}^{i}} H(f) \, n \, ds - \int_{G_{ij}^{i-1}} H(f) \, n \, ds \right).
\]
Here, the surface integrals on the right-hand side can be approximated with second-order accuracy by, e.g.,
\[
32 \int_{S_{ij}^{i}} H(f) \, n \, ds \approx
\]
\[
8 \{H(f(a, t) + 2H(f(b, t) + H(f(c, t))) |S_{ij}^{i}| n(S_{ij}^{i}) \approx \}
\]
\[
(\dot{H}(f_{ij}^{i+1,j,k}) + H(f_{ij+1,j,k}^i) + H(f_{ij+1,j,k}^i) + \]
\[
H(f_{ij}^i) + H(f_{ij+1,j,k}^i) + H(f_{ij+1,j,k}^i) + \]
\[
+ H(f_{ij+1,j,k}^i) + H(f_{ij+1,j,k}^i) + \]
\[
+ 2 \{H(f_{ij}^i) + H(f_{ij+1,j,k}^i) + H(f_{ij+1,j,k}^i) + \}
\]
\[
+ H(f_{ij+1,j,k}^i) + H(f_{ij+1,j,k}^i) + \]
\[
+ H(f_{ij+1,j,k}^i) + H(f_{ij+1,j,k}^i) + \]
\[
+ H(f_{ij+1,j,k}^i) + H(f_{ij+1,j,k}^i) + \}
\]
\[
|S_{ij}^{i}| n(S_{ij}^{i}) \}
\]

Figure 1.

Instead of the common approximation, we use the exact sub-surfaces consisting of the triangles defined above. In order to calculate the surface integrals we will use the function values of \(H(f)\) at the cell vertices. The function values of \(H(f)\) at these points are unknown, but they can be approximated with second-order accuracy by using the function values at the eight cell center points surrounding the corner point under consideration. Therefore, the surface integral over \(\partial V_{ijk}\) can now be written as
\[
\int_{\partial V_{ijk}} H(f) \, n \, ds
\]
\[
= \sum_{i=1}^{2} \left( \int_{S_{ij}^{i}} H(f) \, n \, ds - \int_{S_{ij}^{i-1}} H(f) \, n \, ds \right)
\]
\[
+ \sum_{i=1}^{2} \left( \int_{F_{ij}^{i}} H(f) \, n \, ds - \int_{F_{ij}^{i-1}} H(f) \, n \, ds \right)
\]
\[
+ \sum_{i=1}^{2} \left( \int_{G_{ij}^{i}} H(f) \, n \, ds - \int_{G_{ij}^{i-1}} H(f) \, n \, ds \right).
\]

Here, the surface integrals on the right-hand side can be approximated with second-order accuracy by, e.g.,
Note that the unit normal vectors \( \mathbf{n}(S_{ijk}) \) are constant on the triangles \( S_{ijk} \) and that, e.g., the normal vector on \( S_{ijk} \) is given by \( \mathbf{n}(S_{ijk}) = (c-b) \times (a-b) / ||(c-b) \times (a-b)|| \). Furthermore, the area of \( S_{ijk} \) is given by \( |S_{ijk}| = ||(c-b) \times (a-b)||/2 \). Implementing this into (4), our second-order spatial discretization can be summarized to

\[
\partial_t f_{ijk}^t + \int_{S_{ijk}} ds = -L_t f_{ijk}^t, \quad \forall V_{ijk} \in \Omega, \quad t > 0, \tag{10}
\]

where \( L_t \) is the operator approximating the surface integrals in the way mentioned above.

Denoting by \( |V_{ijk}| := \int_{S_{ijk}} dx \) the volume of the cell \( V_{ijk} \), we have the following

**Lemma 1:** If \( H(f(\cdot, t)) \in C^2(\Omega) \), \( \forall t > 0 \), then the spatial finite volume discretization

\[
|V_{ijk}| \partial_t f_{ijk}^t = -L_t f_{ijk}^t \quad \forall V_{ijk} \in \Omega, \quad t > 0, \tag{11}
\]

is a discretization of second-order accuracy of the system (4).

### 2.2. The Explicit Finite Volume Schemes EFV2*

The Taylor-like expansion

\[
f(x, t + \Delta t) = f(x, t) + \Delta t \partial_x f(x, t) + \frac{\alpha_1(\Delta t)^2}{2} \partial_x^2 f(x, t) + \frac{\alpha_2(\Delta t)^3}{6} \partial_x^3 f(x, t) + O((\Delta t)^4),
\]

which is an expansion backward in time up to a certain order \( p \), can be used to obtain an explicit finite volume scheme. When specifying the coefficients \( \alpha_r \) in the way that \( \alpha_r = \frac{1}{(r-1)!} \), one yields the usual Taylor expansion. By choosing the coefficients \( \alpha_r \) in a special way, we are able to increase the largest possible time step, but at the same time the accuracy in time is decreased by a certain order. The exponential \( p \) takes the values 2, 3, 4, or 5, depending on the choice of the \( \alpha_r \).

In the following the abbreviation \( f_{ijk}^t := f(x_{ijk}, t_n) \) is employed. Using the spatial discretized finite volume equation (11) in order to replace the time derivatives \( \partial_t f \) in the expansion (T) by the corresponding finite difference operator \( L_t \) we obtain our basic explicit finite volume scheme of second order accuracy in space, from which our schemes result by specifying the coefficients \( \alpha_1 \) in a certain way:

\[
\begin{align*}
\Delta t \partial_t f_{ijk}^t &= -L_t f_{ijk}^t, \\
\Delta t \partial_x f_{ijk}^t &= -L_x f_{ijk}^t, \\
\Delta t \partial_y f_{ijk}^t &= -L_y f_{ijk}^t, \\
\Delta t \partial_z f_{ijk}^t &= -L_z f_{ijk}^t,
\end{align*}
\]

(12)

Specifying now the coefficients \( \alpha_1 \) in (12) to \( \alpha_1 = 1/2 \), and \( \alpha_1 = 0 \), for \( i = 2, 3 \), leads to

\[
\begin{align*}
f_{ijk}^{n+1} &= f_{ijk}^n - \Delta t \frac{\partial_t f_{ijk}^n}{|V_{ijk}|} - L_t f_{ijk}^n, \\
&\quad + \frac{\alpha_1(\Delta t)^2}{2|V_{ijk}|^2} L_x^2 f_{ijk}^n, \\
&\quad + \frac{\alpha_2(\Delta t)^3}{6|V_{ijk}|^3} L_x^3 f_{ijk}^n + O(\Delta x^2) + O(\Delta y^2) + O(\Delta z^2),
\end{align*}
\]

(12)

and we note that this scheme is of third-order accuracy in time.

In order to obtain our second explicit finite volume scheme (EFV2b) we specify the parameters \( \alpha_1 \) and \( \alpha_2 \) in (12) to \( \alpha_1 = 1/2 \) and \( \alpha_2 = 1/6 \) obtaining

\[
f_{ijk}^{n+1} := f_{ijk}^n - \Delta t \frac{\partial_t f_{ijk}^n}{|V_{ijk}|} - L_t f_{ijk}^n + \frac{(\Delta t)^2}{2|V_{ijk}|^2} L_x^2 f_{ijk}^n - \frac{\alpha_2(\Delta t)^3}{6|V_{ijk}|^3} L_x^3 f_{ijk}^n, \tag{12}
\]

and we note that this scheme is of fourth-order accuracy in time.

### 3. Von Neumann Stability Analysis of the Schemes EFV2*

Since we are interested in von Neumann stability assertions for the finite-volume schemes (EFV2*), we now restrict ourselves to rectangular grids. A typical gridpoint \( x_{ijk} \) is then defined by

\[
x_{ijk} := (i \cdot \Delta x, j \cdot \Delta y, k \cdot \Delta z)
\]

the volume \( |V_{ijk}| \) of a sub-domain \( V_{ijk} \) can be expressed by

\[
|V_{ijk}| = |\Delta x \cdot \Delta y \cdot \Delta z|
\]

and the area of, e.g., \( S_{ijk} \) is given by \( |S_{ijk}| = |\Delta y \cdot \Delta z|/2 \).

Now, by straightforward calculations it follows that in the case of the Euler equations and equi-spaced orthogonal grids the operator \( L_t \) comes out to be

\[
2L_t f_{ijk}^n = \Delta x \Delta y \Delta z \left( A_1 \delta_x (4+2\gamma_j + 2\gamma_k + \gamma_{\gamma \gamma}) + \right)
\]

\[
A_2 \delta_y (4+2\gamma_k + 2\gamma_l + \gamma_{\gamma \gamma}) + A_3 \delta_z (4+2\gamma_j + 2\gamma_l + \gamma_{\gamma \gamma}) \right) f_{ijk}^n
\]

where the operators \( \delta_x \) and \( \gamma_{\gamma \gamma} \) are defined by

\[
\delta_x f_{ijk}^n := \frac{f_{i+1,j,k}^n - f_{i-1,j,k}^n}{2\Delta x}, \quad \gamma_{\gamma \gamma} f_{ijk}^n := \frac{f_{i+1,j,k}^n + f_{i-1,j,k}^n}{2\Delta x},
\]

\[
\delta_y f_{ijk}^n := \frac{f_{i,j+1,k}^n - f_{i,j-1,k}^n}{2\Delta y}, \quad \gamma_{\gamma \gamma} f_{ijk}^n := \frac{f_{i,j+1,k}^n + f_{i,j-1,k}^n}{2\Delta y}.
\]

In order to study the stability of the scheme (EFV2a) it is applied on a typical Fourier mode, given by

\[
G = G_0 e^{(k_x x + k_y y + k_z z)},
\]

where \( k_1, k_2, k_3 \in \mathbb{R} \) with \( k_1^2 + k_2^2 + k_3^2 = 1 \), and where \( G_0 \) is a constant vector, and one obtains

\[
G^{n+1} = Q_n^* G^n.
\]

Defining the matrix \( E_1 \) by

\[
E_1(\kappa_n) := - \kappa_n \sin \theta_0 \cos^2(\theta_2/2) \cos^2(\theta_z/2) A_1
\]

\[
+ \kappa_n \sin \theta_0 \cos^2(\theta_2/2) \cos^2(\theta_z/2) A_2
\]

\[
+ \kappa_n \sin \theta_0 \cos^2(\theta_2/2) \cos^2(\theta_z/2) A_3,
\]

with the used abbreviations \( \theta_z := k_1 \Delta x, \theta_y := k_2 \Delta y, \theta_z := k_3 \Delta z, \kappa_n := \kappa_n \Delta x, \kappa_n := \kappa_n \Delta y, \kappa_n := \kappa_n \Delta z, \) and \( n \) := \( \kappa_n \sin \theta_0 \cos^2(\theta_2/2) \cos^2(\theta_z/2) \), the amplification matrix \( Q_n \) for the \( n \)-th time step is given by

\[
G^{n+1} = Q_n^* G^n,
\]

and we note that this scheme is of fourth-order accuracy in time.
\[ Q_n := I + i \frac{1}{2} E_1(\kappa_n) - \frac{1}{8} E_1^2(\kappa_n). \]  \hspace{1cm} (14)

The matrices \( A_1, A_2, \) and \( A_3 \) can simultaneously be symmetrized by using matrices consisting of the left and right eigenvectors to a linear combination of \( A_1, A_2, \) and \( A_3, \) see [3]. The sets of eigenvalues are given by \( EV(E_1) := \{ A_j^I \}_{j=1}^5 := \{ -\kappa \cdot u, -\kappa \cdot u, -\kappa \cdot u, -\kappa \cdot u - ||\kappa||\sqrt{\gamma(\gamma - 1)}e/\rho, \}
\]
\[ -\kappa \cdot u + ||\kappa||\sqrt{\gamma(\gamma - 1)}e/\rho, \}
\]

Since the eigenvalues and the linearly independent eigenvectors of the matrices \( E_1(\kappa_n) \) are already known [3], the eigenvalues of the matrices \( I + \frac{1}{2} E_1(\kappa_n) - \frac{1}{8} E_1^2(\kappa_n) \) are known, too. Therefore, the amplification matrices \( Q_n \) can be diagonalized [3],
\[ P_n^{-1} Q_n P_n = \text{diag}\{1 + \frac{1}{2} \Lambda(n)_i - \frac{1}{8} \Lambda(n)_i^2\} =: \text{diag}\{\mu_i(n)\}, \]

where the \( \Lambda(n)_i, i = 1, ..., 5, \) are the eigenvalues of the matrix \( E_1(\kappa_n). \)

Since the Euler equations are of hyperbolic type, it follows that the von Neumann condition is necessary and sufficient for stability of the scheme (EFV2a). Therefore, denoting by \( \rho(Q_n) \) the spectral radius of the matrix \( Q_n, \) the condition \( \rho(Q_n) \leq 1, \) is necessary and sufficient for the stability of the scheme (EFV2a). So, the condition to be satisfied is
\[ |\mu_i(n)| = \left|1 + \frac{1}{2} \Lambda(n)_i - \frac{1}{8} \Lambda(n)_i^2\right| \leq 1, \text{ for } i = 1, ..., 5, \forall n \geq 1, \]

from which one yields \( |\Lambda(n)_i| \leq 2, l = 1, ..., 5, n = 0, 1, 2, 3, ... \)

With the notation \( |u^a|, e^n, \) and \( \rho^n \) being the maximum values of the solution on the \( n \)-th time level, one gets the following

**Lemma 2:** The explicit finite volume scheme (EFV2a) is stable, if
\[ \Delta t_n \leq \frac{2 \Delta x \Delta y}{\sqrt{\Delta y \Delta z \sin^2 \theta_a + \Delta x \Delta z \sin^2 \theta_a + \Delta x \Delta y \sin^2 \theta_a}}, \]

where the abbreviation
\[ \Gamma := \sqrt{\Delta y \Delta z \sin^2 \theta_a + \Delta x \Delta z \sin^2 \theta_a + \Delta x \Delta y \sin^2 \theta_a} \]

has been used.

Note that this estimate is conservative.

**Remark 1:** For \( \Delta x = \Delta y = \Delta z, \) the above lemma simplifies to
\[ \Delta t_n \leq \frac{2 \Delta x}{|u^a| + \sqrt{3} \sqrt{\gamma(\gamma - 1)}e/\rho/\sin \theta_a|}, \]

where the 1-norm of a vector \( u^a \) is defined by \( |u^a| := |u^a_1| + |u^a_2| + |u^a_3|. \)

In [3] we obtained the same estimate but for a finite volume scheme which was of about first order accuracy in space. The most favorable advantage with our scheme can be described in a simplified way as follows. Assume we have to solve a boundary value problem on the unit interval \( x \in [0, 1]. \) In order to get an error of order e.g. \( 10^{-6} \) one has to use \( N \approx 10^{-3} = 1000 \) gridpoints when using a finite difference method of second-order accuracy. This corresponds to a system of 1000 equations. Note, that for a corresponding problem in two dimensions we would have to solve a system of \( 10^8 \) equations. Using a method of fourth-order accuracy we only need \( N \approx 10^{-6/4} \approx 31 \) gridpoints, which correspond to a system of only 31 equations.

Next we study the stability of the scheme EFV2b. The corresponding amplification matrix is given by
\[ Q_n := I + i \frac{1}{2} E_1(\kappa_n) - \frac{1}{8} E_1^2(\kappa_n) - \frac{i}{48} E_1^3(\kappa_n). \]

However, the amplification matrix \( Q_n \) can be diagonalized by \( P_n^{-1} Q_n P_n = \text{diag}\{1 - \frac{1}{8} \Lambda(n)_i^2 + i(\frac{1}{2} \Lambda(n)_i - \frac{1}{48} \Lambda(n)_i^2)\} =: \text{diag}\{\mu_i(n)\}. \]

The condition for stability of the scheme comes out to be
\[ |1 - \frac{1}{8} \Lambda(n)_i^2 + i(\frac{1}{2} \Lambda(n)_i - \frac{1}{48} \Lambda(n)_i^2)| \leq 1, \text{ for } i = 1, ..., 5, \]

\[ \forall n \geq 1, \]

from which follows that \( |\Lambda(n)_i| \leq 4, l = 1, ..., 5, n = 0, 1, 2, 3, ... \)

Therefore, for the largest possible time step we have the following

**Lemma 3:** The explicit finite volume scheme (EFV2b) is stable, if
\[ \Delta t_n \leq \frac{4 \Delta x \Delta y}{\sqrt{\Delta y \Delta z \sin^2 \theta_a + \Delta x \Delta z \sin^2 \theta_a + \Delta x \Delta y \sin^2 \theta_a}} \]

where the abbreviation
\[ \Gamma := \sqrt{\Delta y \Delta z \sin^2 \theta_a + \Delta x \Delta z \sin^2 \theta_a + \Delta x \Delta y \sin^2 \theta_a} \]

has been used.

**Remark 2:** For \( \Delta x = \Delta y = \Delta z, \) the above lemma simplifies to
\[ \Delta t_n \leq \frac{4 \Delta x}{|u^a| + \sqrt{3} \sqrt{\gamma(\gamma - 1)}e/\rho/\sin \theta_a|}. \]

4. Future Work.

In this proceeding we have given a simple upgrading of the existing finite volume methods in order to obtain second-order accuracy in space. We have started to develop schemes of high-order accuracy (higher than order two) by using overlapping control volumes. These schemes together with actual numerical results obtained for the three-dimensional Euler equations will be presented during the conference.

5. References