AXISYMMETRICAL RESPONSE BY A PENNY-SHAPED INTERFACE CRACK IN MULTI-LAYERED COMPOSITES

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Abstract

A rigorous theory of the scattering of normally incident longitudinal wave by a penny-shaped interface crack in multi-layered composites is presented. Made use of Hankel integral transform, a transfer matrix has been obtained, and the problem is reduced to a set of dual integral equations in matrix form, which are then reduced to a set of singular integral equations treated numerically by Jacobi polynomials. As an example, the scattering of elastic wave by a penny-shaped interface crack in one layered half space has been investigated in more detail. The scattered field is derived in far field case by means of the contour integral technique and stationary phase method. The theoretical results have been shown that at large distance from the crack, the scattered displacements in the layer are composed of Rayleigh-like mode waves predominantly, and the P wave and SV wave are predominant in the half space. The scattered amplitudes of displacements for the first two modes are plotted versus the incident wave frequency, and it is observed that the multiresonances occur at some frequency.

I. Introduction

Recently, the theoretical problem of the scattering of elastic waves by crack has received considerable attention. Authors, working in many different areas, such as applied mechanics, applied mechanics, geophysics, seismological science and quantitative non-destructive evaluation (QNDL), have contributed to the knowledge of this subject.

This paper is concerned with the scattering of time harmonic, normally incident longitudinal wave by a penny-shaped interface crack in multi-layered composites. The scattering of elastic wave by a penny-shaped crack located in an infinite isotropic medium has been considered by Mal1, Matin2, Matin and Wickham3. Srivastava et al.4 have investigated the interaction of longitudinal wave with a penny-shaped crack at the interface of two dissimilar elastic solids. The papers related to these problems are referred to those of Neerhoff; Yang and Bog4, Angel6. The problem of this paper is more complicated and difficult one.

The formulation of the problem is presented in section 2. The total field in the cracked layered media is analyzed as the superposition of the incident field and scattered field. The incident field is then given by Eing et al1, and scattered field can be changed into mixed boundary value problems.

Hankel transform is used in section 3 to obtain a suitable general solution of the wave equations. The transfer matrices are obtained, and the problem is reduced to a set of dual integral equations in matrix form.

As an example, the scattering of elastic wave by a penny-shaped interface crack in one layered half space has been considered in section 4. The discussion of numerical results are presented in section 5.

II. Formulation

Consider $n$ layers and a half space are bonded together perfectly, except in the region $0 < r < d_1$, $z = 0$, where is a penny-shaped crack, as shown in Fig.1.

Fig.1 Incidence of a longitudinal wave in multi-layered composites on a penny-shaped interface crack

Suppose the layers and half space are occupied by homogeneous isotropic materials with different properties and a incident wave is harmonic longitudinal wave which is impinging at the crack normally. The face
of the crack is free of tractions. Thus, the problem is reduced to an axisymmetric elastodynamic problem.

The total field can be divided into the sum of the incident field and scattered field according to:

\[
\{ \varphi^{(i)}, \psi^{(i)} \} = \{ \varphi^{(s)}, \psi^{(s)} \} + \{ \varphi^{(t)}, \psi^{(t)} \}
\]

in which, \( \{ \varphi^{(s)}, \psi^{(s)} \} \) represents the total field, \( \{ \varphi^{(t)}, \psi^{(t)} \} \) the incident field which is assumed to be known, and \( \{ \varphi^{(s)}, \psi^{(s)} \} \) the scattered field, i.e.,

the modification to the incident field due to the presence of crack. For the scattered field one should solve the following boundary value problem (superscription \( e \) and the time factor \( \exp(-i\omega t) \) have been suppressed):

\[
\begin{align*}
\nabla^2 \varphi_j + K_{ij}^2 \varphi_j &= 0 \\
\nabla^2 \psi_j + K_{ij}^2 \psi_j &= 0
\end{align*}
\]

where

\[
\begin{align*}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} \\
K_{ij}^2 &= \omega^2 \rho_j / (\lambda_j + 2\mu_j) \\
K_{ij}^2 &= \omega^2 \rho_j / \mu_j \\
j &= 0, 1, 2, ..., n
\end{align*}
\]

\( \rho_j \)—the density, \( \lambda_j, \mu_j \)—the lame constants, \( K_{ij}^2 \) and \( K_{ij}^2 \) are the P wave and SV wave numbers, respectively. The superscription \( j (j = 0, 1, 2, ..., n) \) represents the half space and the layers.

The displacements and stresses can be represented in the \( \varphi_j, \psi_j \)

\[
\begin{align*}
\varphi_j &= \frac{\partial \varphi_j}{\partial r} + \frac{\partial^2 \psi_j}{\partial z^2} \\
\psi_j &= \frac{\partial \varphi_j}{\partial z} - \frac{\partial}{\partial r} \left( \frac{\partial \psi_j}{\partial r} \right) \\
\tau_{zzj} &= \lambda_j \nabla^2 \varphi_j + 2\mu_j \left( \frac{\partial^2 \varphi_j}{\partial r^2} + \frac{\partial^2 \psi_j}{\partial z^2} \right) \\
\tau_{\varphi j} &= \lambda_j \nabla^2 \psi_j + 2\mu_j \left( \frac{\partial^2 \psi_j}{\partial r^2} + \frac{\partial^2 \varphi_j}{\partial z^2} \right)
\end{align*}
\]

The boundary and continuous conditions are

\[
\begin{align*}
\tau_{zz} &= \tau_{zzn} = 0 & & z = -d_n \\
\tau_{\varphi j} &= \tau_{\varphi j-1} \quad & & \tau_{\varphi j} = \tau_{\varphi j-1} \\
\psi_j &= \psi_j-1 \quad & & \psi_j = \psi_j-1 \\
& j = 2, 3, 4, ..., & & z = -d_{j-1} \\
\tau_{zz} &= \tau_{zz0} \quad & & \tau_{zz0} = \tau_{zz0} \\
\psi_0 &= \psi_0 \quad & & \psi_0 = \psi_0 \\
& & & z = 0, 0 < r < \infty \\
\tau_{zz} &= -\tau_{zz0} \quad & & -\tau_{zz0} = -\tau_{zz0} \\
& & & z = 0, 0 \leq r < 1 & (6)
\end{align*}
\]

where \( \tau_{zz}^{(i)} \) is known.

In addition, the scattered field must satisfy the following radiation condition:

\[
\lim_{r \to \infty} R \left\{ \frac{\partial \psi_0}{\partial r} - \frac{1}{K} \psi_0 \right\} = 0 \quad (7)
\]

where \( R = (r^2 + z^2)^{1/2} \), \( K \) is wave number vector modulus.

III. Integral Equations

Applying Hankel integral transform to wave motion equations (2), and taking into account of radiation condition (7), we obtain formulae as following:

\[
\begin{align*}
\varphi_j \left( r, z \right) &= \int_0^\infty \xi t \left( \lambda_j \xi \right) \tau_{\varphi j} \left( r, z \right) d \xi \\
\psi_j \left( r, z \right) &= \int_0^\infty \xi t \left( \lambda_j \xi \right) \psi_j \left( r, z \right) d \xi
\end{align*}
\]

\[
\begin{align*}
\tau_{zzj} \left( r, z \right) &= \int_0^\infty \xi t \left( \lambda_j \xi \right) \tau_{zzj} \left( r, z \right) d \xi \\
\tau_{\varphi j} \left( r, z \right) &= \int_0^\infty \xi t \left( \lambda_j \xi \right) \tau_{\varphi j} \left( r, z \right) d \xi
\end{align*}
\]

\( j = 1, 2, ..., n \)
\[ \mathcal{D}_0(\tau, z) = \int_0^\infty \xi \cdot d_0(\xi, \tau) \cdot \frac{1}{\xi} J_0(\xi \tau) \, d\xi \]

\[ \psi_0(\tau, z) = \int_0^\infty \xi \left( \frac{1}{\xi} C_0(\xi) \right) \frac{1}{\xi} \frac{d}{dz} J_0(\xi \tau) \, d\xi \]

(9)

where \( J_0(\xi \tau) \) is zero order Bessel function, \( A_j(\xi), B_j(\xi), C_j(\xi) \) and \( D_j(\xi) \) are unknown functions.

\[ \gamma_{1j} = (K_{1j} - \xi^2)^{\frac{1}{2}}, \quad \gamma_{2j} = (K_{2j} - \xi^2)^{\frac{1}{2}} \]

\( j = 0, 1, 2, \ldots, n \)

(10)

The following two basic unknown functions \( \bar{R}_1(\xi) \) and \( \bar{S}_1(\xi) \) which are defined as:

\[ \int_0^\infty \left( \frac{\bar{R}_1(\xi)}{\xi} \right) \xi \cdot J_1(\xi \tau) \, d\xi = \begin{cases} u_{\tau 0} - w_{\tau 1} & \text{if } 0 \leq \tau < 1, z = 0 \\ 0 & \text{if } \tau > 1 \end{cases} \]

\[ \int_0^\infty \left( \frac{\bar{S}_1(\xi)}{\xi} \right) \xi \cdot J_0(\xi \tau) \, d\xi = \begin{cases} u_{\tau 0} - w_{\tau 1} & \text{if } 0 \leq \tau < 1, z = 0 \\ 0 & \text{if } \tau > 1 \end{cases} \]

(11)

Taking the Hankel integral transform for the boundary and continuous conditions (5), (6), one obtain transfer matrices of \( A_j(\xi), B_j(\xi), C_j(\xi) \) and \( D_j(\xi) \):

\[ \begin{bmatrix} A_j(\xi), B_j(\xi), C_j(\xi), D_j(\xi) \end{bmatrix}^T = \begin{bmatrix} A_{j-1}(\xi), B_{j-1}(\xi), C_{j-1}(\xi) + D_{j-1}(\xi) \end{bmatrix}^T \]

\( j = 2, \ldots, n \)

(12)

\[ \begin{bmatrix} A_0(\xi) \\ C_0(\xi) \end{bmatrix} = Q_0 \begin{bmatrix} A_1(\xi) \\ B_1(\xi) \\ C_1(\xi) \\ D_1(\xi) \end{bmatrix} \]

\[ \begin{bmatrix} A_1(\xi) \\ B_1(\xi) \\ C_1(\xi) \\ D_1(\xi) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\mu} \\ 0 \\ \frac{1}{\mu} \end{bmatrix} \begin{bmatrix} \frac{1}{\xi} K_1(\mu, \xi) \xi + \frac{1}{\xi} K_3(\mu, \xi) \xi \end{bmatrix} \]

\[ \begin{bmatrix} A_0(\xi) \\ C_0(\xi) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} f(\xi) \xi \end{bmatrix} \]

(13)

where \( T_j, Q_0, Q_1, E, F \) and \( Q_n \) are known matrices, which can be referred to Ma(9).

Then, made use of the surface conditions of the crack, the integral equations in matrix are obtained as following:

\[ \int_0^\infty \left[ \begin{bmatrix} \bar{R}_1(\xi) \\ \bar{S}_1(\xi) \end{bmatrix} \right] \left[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right] \left[ \begin{bmatrix} \xi \cdot J_0(\xi \tau) \, d\xi \\ \xi \cdot J_1(\xi \tau) \, d\xi \end{bmatrix} \right] = \begin{bmatrix} \tau_{zz} \end{bmatrix} \]

\( 0 \leq \tau < 1, z = 0 \)

(14)

The equations (15) are a set of dual integral equations for the scattering of elastic wave by a penny-shaped interface crack in multilayered composites. The derivation in more detail could be referred to Ma(9).

IV. Example

As an example, the scattering of elastic wave by a penny-shaped interface crack in one layered half space has been considered, i.e., \( d_1 = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, n \). The set of dual integral equations (15) can be reduced to a set of singular integral equations by means of Abel integral transform and the results of Lowengrub and Sneddon:

\[ \begin{cases} \beta & 0 \\ 0 & -\beta \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

\[ \begin{cases} 0 & -\beta \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

\[ \frac{1}{\beta} \int_0^1 \left[ \left[ K_1(\mu, \xi) \xi + K_3(\mu, \xi) \xi \right] \right] \\
\end{cases} \]

\[ \frac{1}{\beta} \int_0^1 \left[ \left[ K_1(\mu, \xi) \xi + K_3(\mu, \xi) \xi \right] \right] \\
\end{cases} \]

\[ \begin{cases} 0 & 0 \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

\[ \begin{cases} 0 & 0 \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

\[ \begin{cases} 0 & 0 \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

\[ \begin{cases} 0 & 0 \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

\[ \begin{cases} 0 & 0 \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

\[ \begin{cases} 0 & 0 \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

\[ \begin{cases} 0 & 0 \end{cases} \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] = \int_0^1 \left[ \left[ 0 \right] \, \frac{1}{\xi + x} \right] \left[ \begin{bmatrix} \beta \xi \xi \end{bmatrix} \right] \\
\end{cases} \]

where \( a, b, c, f(x) \) and \( K_j(u, x) \) (j=1,2,3,4,5)
4) are given in \( \text{Ma}^9 \), \( R(x) \) and \( S(x) \) are the unknown functions. Following the method of Erdogan\(^\text{11}\), and expanding the unknown functions as an infinite series in Jacobi polynomials, the singular integral equations (116) can be reduced to a set of algebraic equations which are referred to \( \text{Ma}^9 \).

The expansion can be expressed:

\[
\begin{align*}
R(x) &= \sum_{n=0}^{\infty} \left( \frac{t}{t^2 + \frac{\pi^2}{4}} \right) \int \frac{1}{2} \sin K_j x d\phi \frac{T_{11}}{\Delta'(K_j)} (s_f)(K_j) \\
S(x) &= \sum_{n=0}^{\infty} \left( \frac{t}{t^2 + \frac{\pi^2}{4}} \right) \int \frac{1}{2} \cos K_j x d\phi \frac{T_{11}}{\Delta'(K_j)} (s_f)(K_j)
\end{align*}
\]

where \( c_{1n} \) and \( c_{2n} \) \((n=0,1,2,\ldots)\) are undetermined constants which can be solved by the algebraic equations, and \( P_n \) \((n=0,1,2,\ldots)\) is Jacobi polynomials.

\[
W_1(x) = (1-x)^{a_1} (1+x)^{b_1}
\]

\[
W_2(x) = (1-x)^{a_2} (1+x)^{b_2}
\]

\[
a_1 = -\frac{i}{2\pi} \ln \frac{1+y}{1-y}
\]

\[
b_1 = \frac{i}{2\pi} \ln \frac{1+y}{1-y}
\]

\[
a_2 = b_1
\]

\[
b_2 = a_1
\]

\[
\sigma = \frac{\alpha}{\beta}
\]

Upon the constants \( c_{1n} \) and \( c_{2n} \) \((n=0,1,2,\ldots)\) are obtained, using the equations (8)-(14), (4), displacements can be expressed in integral form, making use of contour integral technique and the some asymptotic analysis methods, the surface scattered displacements in the layer at large distance from the crack are presented

\[
u_R^{(sP)}(R,\varnothing, t) = G_L(\varnothing, w) \frac{t(\frac{r}{r_{L0}}) R-wt}{(RK_{L0})} + \frac{1}{R} \left( \frac{1}{2} \right)
\]

\[
u_\varnothing^{(sP)}(R,\varnothing, t) = G_T(\varnothing, w) \frac{t(\frac{r}{r_{T0}}) R-wt}{(RK_{T0})} + \frac{1}{R} \left( \frac{1}{2} \right)
\]

where \( R = (r^2+z^2)^{\frac{1}{2}} \), \( \varnothing = -\cdot g^{-\frac{1}{2}} \), \( G_L(\varnothing, w) \) and \( G_T(\varnothing, w) \) are given in \( \text{Ma}^9 \).

From the formulae (19) and (21), it can be seen that the scattered displacements in the layer are composed of Rayleigh-Like Mode waves predominantly, and the \( P \) wave and \( S \) wave are predominant in the half space.
V. Discussion of Numerical Results

Numerical results are presented in Figs. 2-5. These results were computed for two groups parameters.

1) the layer is AL, and the half space is NI:
\[ d=0.3 \text{ or } d=0.6, \quad \mu_1=26.5 \times 10^9 (N/m^2), \]
\[ \rho_1=2.7 \times 10^3 (kg/m^3), \lambda_1=56.3 \times 10^9 (N/m^2) \]
\[ \mu_0=66.5 \times 10^9 (N/m^2), \quad \rho_0=8.8 \times 10^3 (kg/m^3) \]
\[ \lambda_0=108.5 \times 10^9 (N/m^2) \]

2) the layer is AU, and the half space is AL:
\[ d=0.8, \quad \mu_1=28.0 \times 10^9 (N/m^2) \]
\[ \rho_1=19.3 \times 10^3 (kg/m^3), \lambda_1=147.0 \times 10^9 (N/m^2) \]
\[ \mu_0=26.5 \times 10^9 (N/m^2), \quad \rho_0=2.7 \times 10^3 (kg/m^3) \]
\[ \lambda_0=56.3 \times 10^9 (N/m^2) \]

For the layer-half space combination—the first groups parameters which are said to "stiffen" case \( V_{t1}>V_{t0} \), there is only one scattered Rayleigh-Like-Mode wave in the layer. Figure 2 and Figure 3 show the modulus of the ratios of coefficient \( A_1 \) and \( B_1 \) to incident P wave amplitude \( A_0 \) versus the frequency, respectively. For \( w=0 \), there is no scattered field in the solid. The several resonance peaks can occur in some range of frequency. The resonance amplitude decreases as \( d \) increases.

However, for the other layer-half space combination—the second groups parameters which are said to "loading" case \( V_{t1}<V_{t0} \), there are multi-scattered Rayleigh-Like-Mode waves in the layer. The first two mode Rayleigh - waves occur with the range of the frequency of this paper. Fig. 4 and Fig. 5 represent the first mode and the second mode of Rayleigh wave, respectively. The resonance peaks in Fig.4-5 appear also. The numerical results show that the first mode amplitude is larger than that of the second mode.

The computations have been performed on the SIEMENS 7570c computer of the computing centre of Harbin Institute of Technology.

Finally, it is pointed out that the method used in this paper can be applied to the scattering of elastic waves by multi-cracks in multi-layered composites.
Fig. 4 Modulus of the ratio of coefficients $A_1$ and $B_1$ to incident $P$ wave amplitude $A_0$ for the second layer-half space combination.

Fig. 5 Modulus of the ratio of coefficients $A_2$ and $B_2$ to incident $P$ wave amplitude $A_0$ for the second layer-half space combination.

References