Abstract

It has been found that the maximum range of aircraft subject to wind effect is obtained when it flies at a lift coefficient which at every instant satisfies the following relations, respectively:

\[ C_L = C_{L0} \left(1 + \frac{w}{V + w/2}\right)^{1/2} \text{ for prop.} \]

\[ C_L = C_{L0} \left(1 + \frac{2w/V}{1 + 2w/3V}\right)^{1/2} \text{ for jet} \]

where \( C_{L0} \) is the lift coefficient giving maximum range in still air in the corresponding case, its value being

\[ C_{L0} = \left(C_{D0} \frac{\lambda}{\lambda/3}\right)^{1/2} \text{ and } C_{L0} = \frac{C_{D0}}{3} \]

respectively; they were derived from the general equation by putting \( w=0 \); \( C_{D0} \) is minimum drag coefficient of the aircraft, \( \lambda \) is the wing effective aspect ratio and \( w \) is the wind velocity component along the route, positive in the case of a tail wind.

Analysis of functions for small values of \( w/V \) shows that the airspeed for maximum range can be approximated by the following expressions:

\[ V_{Ma} = V_m - w/4 \text{ for prop.} \]

\[ V_{Ma} = V_m - w/3 \text{ for jet} \]

where \( V_m \) is the instantaneous speed for maximum range in still air in the corresponding case.

Numerical integration for jet aircraft showed: a) Favorable though negligible differences in range compared with the \( C_L \) constant method, b) A decrease in the elapsed time on route in the presence of a head wind, compared against the \( C_L \) constant case.

Constant specific fuel consumption has been assumed and no compressibility effects have been considered.

I. Introduction

Range calculations for both jet and propeller aircraft are widely known provided the flight is conducted in still air, in which case the maximum range is obtained at some particular constant value of the lift coefficient.

They were predicted for instance by the classical Breguet formula for propeller aircraft or by a similar expression for jet aircraft.

Little is found in the literature on the case when the aircraft flies subjected to a component of the wind along the route, in which case the lift coefficient for best range is no longer constant.

In this paper it is intended to determine the conditions that the lift coefficient must fulfill in order to maximize the range.

II. Propeller Aircraft

Fuel consumption by weight equals the change in total weight in an elapsed time \( \Delta t \):

\[ c(\frac{P}{\eta})dt = -dG \]  \( (1) \)

where \( P \) is the engine shaft power, \( \eta \) the specific fuel consumption (kg/kg-m), \( \eta \) the propeller efficiency and \( G \) the instantaneous weight of the aircraft.

An element of the range is given by

\[ da = (V+W)dt = - \frac{1}{c} \int_{V,0}^{V+w} C_{L0} \frac{C_L}{G} \frac{dG}{G} \]  \( (2) \)

where \( V \) and \( c \) were considered as constants.

Since \( C_L = f_1(G, V) \) and \( C_D = f_2(C_L) \), the integrand function for a given aircraft at constant altitude and wind magnitude depends only on \( G \) and \( V \):

\[ F(G, V) = \frac{V+w}{G, V} \frac{C_L}{C_D} \]  \( (3) \)

Therefore we have to find a function \( F(G, V) \) which should maximize the integral in (3), a problem of Calculus of Variations.

The first necessary condition to extremize (maximize or minimize) the integral (3) is given by the Euler-La-
Grange equation (Ref. 1) which adapted to the present nomenclature can be written as:

$$\frac{\partial F}{\partial V} + \frac{4}{\alpha g} \left( \frac{\partial F}{\partial V} \right) = 0$$  \hspace{1cm} (5)

where \( V' = \delta V / \delta \alpha \). Considering the physical nature of the problem we interpret this condition as maximizing the integral.

Since \( \partial F / \partial V = 0 \) the condition (5) reduces itself to:

$$\frac{\partial F}{\partial V} = 0$$  \hspace{1cm} (6)

Now let us replace in (4) the explicit values of:

$$C_L = \frac{2g}{\rho SV^2} \quad \text{and} \quad C_D = C_{D_0} + \frac{C_L}{\pi \lambda}$$  \hspace{1cm} (7)

we obtain:

$$F(\alpha, V) = \frac{2(V+w)}{\rho SV^3} \left[ C_{D_0} + \frac{1}{\pi \lambda} \left( \frac{2g}{\rho SV^2} \right)^2 \right]^{-1}$$  \hspace{1cm} (8)

By performing a partial derivative of (8) with respect to \( V \), equating to zero and arranging, we obtain the condition

$$C_{D_0} (4V+6w) - \frac{(2g)}{(\rho S \lambda)} \frac{4V+2w}{\pi \lambda V^4} = 0$$  \hspace{1cm} (9)

from where:

$$g = \frac{1}{2} \rho V^2 S (C_{D_0} \pi \lambda)^{\frac{1}{2}} (1+ \frac{w}{V+w/2})^{\frac{1}{2}}$$  \hspace{1cm} (10)

Using in (10) the expression of \( C_L \) as given in (7) and solving for \( C_{L_a} \):

$$C_{L_a} = (C_{D_0} \pi \lambda)^{\frac{1}{2}} (1+ \frac{w}{V+w/2})^{\frac{1}{2}}$$  \hspace{1cm} (11)

Putting \( w = 0 \) in (11) we obtain the lift coefficient for maximum range in still air \( C_{L_o} = (C_{D_0} \pi \lambda)^{\frac{1}{2}} \), and replacing in (11) we have:

$$\frac{C_{L_a}}{C_{L_o}} = (1 + \frac{w}{V+w/2})^{\frac{1}{2}} = 0$$  \hspace{1cm} (12)

Since \( C_{L_a} / C_{L_o} = (V/V_o)^2 \), replacing in (12) we have the alternative form:

$$u = \frac{2(V-v)}{3v^2} - 1$$  \hspace{1cm} (13)

where \( u = w/V_o \) and \( v = V/V_o \).

Note that \( V_o \) is the airspeed that gives the maximum range in still air and corresponds to the constant \( C_{L_o} \) defined in the foregoing paragraph. Therefore \( V_o \) is variable during the flight according to the variation in \( g \), the instantaneous weight of the aircraft.

This result for the propeller aircraft, jointly with that of the next section for jet aircraft, will be used in Section IV for an approximate calculation of the speed for maximum range for both types of aircraft.

### III. Jet Aircraft

Following a method similar to that outlined for the propeller aircraft in the preceding section, we can express the range for the jet aircraft as:

$$a = \frac{1}{c} \int C_{D_0} \frac{V+w}{\rho S} \, dG$$  \hspace{1cm} (14)

Where \( c \) is the specific fuel consumption per unit of thrust per unit of time, (kg/kg-sec); \( q \) is the dynamic pressure \( \rho V^2 / 2 \).

Performing steps similar to those we carried out in the case of propeller aircraft we can obtain the form for the function under the integral above, as:

$$F(\alpha, V) = (V+w) \frac{\rho V^2 S}{C_{D_0} 2} + \frac{1}{2 \lambda} \rho SV^2$$  \hspace{1cm} (15)

Taking the derivative \( \partial F / \partial V = 0 \) and solving for \( G \), we obtain:

$$G = \left( \frac{\rho S \lambda^2}{3} (1+2w/\lambda V) \right)^\frac{1}{2} \rho SV^2$$  \hspace{1cm} (16)

Setting \( (C_{D_0} \pi \lambda / 3)^2 = C_{L_o} ; \quad C_{L_a} = G/QS \)
we have:

$$\frac{C_{L_a}}{C_{L_o}} = \left( \frac{1+2w/\lambda V}{1+2w/3V} \right)^\frac{1}{2}$$  \hspace{1cm} (17)

replacing \( C_{L_a} / C_{L_o} = (V/V_o)^2 \) and rearranging:

$$u = 3 \frac{v}{6V^2 - 2}$$  \hspace{1cm} (18)

Where \( u = w/V_o = V=V_o \), \( V_o \) being the airspeed that gives the maximum range in still air and \( C_{L_o} \) the corresponding lift coefficient.

### IV. Approximate Speed for Maximum Range

To calculate the speed which maximizes the range, it is required to solve fifth-degree equations (13) and (18).

The foregoing deductions suggest that this airspeed, \( V_o \), is placed somewhere about \( V_o \), the airspeed for maximum range in still air.

Therefore we can obtain an aproxi-
nuation to its values by expanding functions (13) and (18) in Taylor series assuming \( w/V \) to be small enough in order to represent the function with only the first two terms of the expansion:

\[
V_a = V_o + \left( \frac{dV}{dw} \right)_w \frac{dw}{du}
\]

(19)

where \( dV/dw = dV/du \) must be evaluated at \( V = V_i \), or, which is the same, at \( v = 1 \). From (13) and (18) we can obtain:

\[
\left( \frac{du}{dv} \right)_{v=1} = -4 \quad \text{for prop.} \quad (20a)
\]

\[
\left( \frac{du}{dv} \right)_{v=1} = -3 \quad \text{for jet} \quad (20b)
\]

which allow us to write:

\[
V_a = V_o - w/4 \quad \text{for prop.} \quad (21a)
\]

\[
V_a = V_o - w/3 \quad \text{for jet} \quad (21b)
\]

where \( V_o \) is the instantaneous airspeed, variable along the flight path, calculated for zero wind, and \( w \) is the wind component along that path, positive if from tail.

V. Numerical Integration for Jet Aircraft

After many attempts to integrate equations (3) and (14) in a simple analytical way without success, it was decided to attain the solution, for the time being, through a numerical method which was applied to the jet aircraft case.

To this effect the following expression, derived from (14) and combined with (15) to (18), was used, after some steps and rearrangement:

\[
a = \frac{C_L}{2 \eta C_D} \left( \frac{V_f}{V} \right)^{\frac{1}{2}} \left( \frac{2w}{V} - \frac{1}{2} \right) dV
\]

(22)

where \( Z = (3V+6w)(3V+2w) \); \( V_o \) and \( V_f \) the initial and final speeds.

For a comparison the range was also calculated for the case \( C_L = C_L = \text{const.} \) superimposed with the displacement due to the wind, \( w \), where \( t \) is the duration of the flight, the expression being:

\[
a^* = \frac{4559}{c} \left( \frac{\pi}{3} \right)^{\frac{1}{4}} \left( \frac{G_o}{G_f} \right)^{\frac{1}{4}} \left[ 1 - \left( \frac{G_f}{G_o} \right)^{\frac{1}{4}} \right]
\]

\[
\cdot \left[ 1 + \frac{1}{2} \frac{w}{V_o} \ln \left( \frac{G_o}{G_f} \right) \right]
\]

(23)

The number "4559" is the result of grouping all the numerical mathematical and physical constants (\( \pi, \rho, \text{etc.} \)).

The limits of integration of equation (22), \( V_o \) and \( V_f \), initial and final airspeeds, were obtained by solving the fifth-degree equation (18).

In both formulas the units used correspond to the Technical System (kg-force, meter, second). Accordingly fuel consumption is in kg/kg-sec (sec\(^{-1}\)) for jets and kg/kg-m (m\(^{-1}\)) for propeller aircraft.

The numerical integration has been applied, with various wind conditions, to two different aircraft representative of their respective types: one four-jet transport and one twin-jet executive, the features of which are shown in Table 1.

The principal results of these calculations are resumed in Table 2.

<table>
<thead>
<tr>
<th>TABLE 1 Assumed Characteristics for two Representative Airplanes</th>
<th>Four-Jet Transp.</th>
<th>Twin-Jet Exectv.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max.T.O.Weight, ( G_o )</td>
<td>kg</td>
<td>148325</td>
</tr>
<tr>
<td>Final Weight, ( G_f )</td>
<td>kg</td>
<td>76086</td>
</tr>
<tr>
<td>Wing Area, ( S )</td>
<td>m(^2)</td>
<td>279</td>
</tr>
<tr>
<td>Eff.Aspect R., ( \lambda )</td>
<td>-</td>
<td>5,64</td>
</tr>
<tr>
<td>Max.Wing Load.</td>
<td>kg/m(^2)</td>
<td>530</td>
</tr>
<tr>
<td>Min.Drag Coeff., ( C_D )</td>
<td>-</td>
<td>0,0243</td>
</tr>
<tr>
<td>Density ratio, ( \gamma ) at flight altitude</td>
<td>m</td>
<td>0,3098</td>
</tr>
<tr>
<td>Spec.Fuel Cons., ( c )</td>
<td>m/s (^{-1})</td>
<td>104 kg/kg-sec</td>
</tr>
</tbody>
</table>

VI. Comments on the Numerical Results

Examination of the values in Table 2, suggests the following comments.

1. When flying with a "variable \( G_f \)" according to the method of this paper, formula (18), the range is slightly greater than that predicted on the standard assumption of "constant \( G_f \)" (formula 23), regardless of whether there is a tail or head wind; but its value is practically negligible.
2. Flight time in maximum range at
variable $C_L$ shows an increase over
that of the constant $C_L$ case, when
flying in a tail wind; and a decrease
in a head wind.

We must point out that in the $C_L$
constant case it is assumed that the
flight time corresponds to the endur-
ance at that lift coefficient, and,
therefore, the value of flight time is
also constant, independent of the wind
condition. This value is indicated in
Table 2 for reference and is the same
in both methods when $w=0$.

TABLE 2. Principal Numerical Results

<table>
<thead>
<tr>
<th></th>
<th>Four-Jet Transport, $G_o = 148325$ kg.</th>
<th>Twin-Jet Executive $G_o = 12100$ kg.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w$ a a' t $V_r V'_r \Delta a \Delta t$</td>
<td>$m\ s$ km km h $km \ h$ $km \ h$ $km \ h$</td>
</tr>
<tr>
<td>+40</td>
<td>14220 14166 15.3 929 970 54 0.7</td>
<td></td>
</tr>
<tr>
<td>+20</td>
<td>13123 13112 15.0 875 898 11 0.4</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>12057 12057(14.6) 826 826 0 0</td>
<td></td>
</tr>
<tr>
<td>-20</td>
<td>11096 11096 14.1 788 760 14 -0.5</td>
<td></td>
</tr>
<tr>
<td>-40</td>
<td>10022 9947 13.6 737 681 75 -1.0</td>
<td></td>
</tr>
</tbody>
</table>

|        | $m\ s$ km km h $km \ h$ |
| +40    | 4757 4972 7.4 643 684 36 0.5 |
| +20    | 4277 4268 7.2 594 618 9 0.3 |
| 0      | 3724 3724 6.9 540 540 0 0  |
| -20    | 3283 3271 6.6 497 474 12 -0.3 |
| -40    | 2829 2773 6.0 471 402 56 -0.9 |

Notes: $h=$hour; number within paren-
thesis denote flight time for $w=0$;
$V_r$ and $V'_r$ are the average cruising
speed, ground speed.

3. From the contents of the latest
paragraph we infer that flying ac-
cording to the "variable $C_L" method is
somewhat advantageous when dealing with
a head wind, not because of the better
range, but because of the lesser time
aloft.

4. In the case of tail wind, the small
increase obtained hardly justifies the
longer time on route.

VII. Additional Comment

The assumption of the constant spe-
cific fuel consumption is a rather re-
strictive one.

It was adopted at the beginning of
the work in the hope of obtaining a sim-
er formula for educational purposes.

As this intended simplicity was not
been achieved, it was necessary to resort
to numerical methods, which now suggest
the possibility of an additional assump-
tion on the variation of the specific
fuel consumption, a proposal which can
be dealt with in a further study on this
subject.

References
1. P. Cicala—An Engineering Approach to
the Calculus of Variations—Editrice
Universitaria Levrotto & Bella-Torino
1964.

2. I.S. Sokolnoloff and H.M. Redheffer
Mathematics of Physics and Modern