IDENTIFICATION OF STRUCTURAL DAMPED VIBRATIONAL SYSTEMS

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Summary

A calculation method for coefficients of elastic structural vibration equations from vibrational test results has been developed.

Complex inertial and stiffness matrices are used whose imaginary parts describe the structural damping.

I. INTRODUCTION

The identification of a theoretical dynamic model and a physical object is discussed in[(2)–(4)]. These papers use a solution of the inverse problem of eigenvalues for a system having a finite number of degrees-of-freedom without damping (or with a damping matrix diagonal in normal coordinates).

Local inertial and elastic characteristics are found from orthonormalization conditions of eigenvalues at known natural frequencies and generalized masses.

In real structures the damping matrix is not diagonal and their modes are complex. In addition, a rather large number of modes (equal to the measurement number) has to be used. It can result in appreciable errors.

To reduce errors the present paper contains equations without friction forces proportional to velocity but having complex matrices of inertia and elasticity. All the formal mathematics for the small-vibration theory of linear systems without friction remains almost unaltered.

In spite of a traditional use of the complex elasticity matrix the complex mass matrix does not seem to be widely employed, though its application is as much valid as the use of the complex elasticity matrix.

II. PROBLEM STATEMENT AND ASSUMPTIONS VALIDATION

Let's write the vibration equations of a test structure

$$[m]\{\ddot{x}\} + [K]\{x\} = \{F\}e^{i\omega t} , \qquad (1)$$

where

$$\{x\} = \begin{Bmatrix} x \\ \vdots \\ x \end{Bmatrix} \qquad (2)$$

is the displacement vector (matrix-column).

It is further assumed that $[m]$ and $[k]$ are complex symmetrical matrices, i. e.:

$$[m] = [\mathrm{Re}\, m] + i[\mathrm{Im}\, m] \ , \quad [K] = [\mathrm{Re}\, K] + i[\mathrm{Im}\, K] \ ,$$

$$[m]' = [m] \ , \quad [\mathrm{Re}\, m]' = [\mathrm{Re}\, m] \ , \quad [\mathrm{Im}\, m]' = [\mathrm{Im}\, m] \ ,$$
$$[K]' = [K] \ , \quad [\mathrm{Re}\, K]' = [\mathrm{Re}\, K] \ , \quad [\mathrm{Im}\, K]' = [\mathrm{Im}\, K] \ . \qquad (3)$$

Deduce the matrix row as

$$\{x\}' = \{x_1, \dots x_n\} \ . \qquad (4)$$

The amplitude vector of external forces $\{F\}$ is considered complex and only at a cophased excitation is taken to be real.

The imaginary parts of elasticity matrix coefficients $[k]$ describe the structural damping[(1)] due to the hysteresis of elastic and Coulomb friction forces in joints.

The energy loss equal to the work of these forces during $T = 2\pi/\omega$ is:

$$\Delta E = \int_0^T \mathrm{Re}\left(i\omega e^{i\omega t}\{x_0\}'\right)\mathrm{Re}\left(i[\mathrm{Im}\, K]\{x_0\}e^{i\omega t}\right)dt =$$
$$= \pi\left(\{\mathrm{Re}\, x_0\}'[\mathrm{Im}\, K]\{\mathrm{Re}\, x_0\} + \{\mathrm{Im}\, x_0\}'[\mathrm{Im}\, K]\{\mathrm{Im}\, x_0\}\right) . \qquad (5)$$

ΔE is independent of ω – vibration frequency.

The imaginary coefficients of the matrix $[m]$ may be considered as coefficients of structural damping due to weak multiple collisions in rivet joints, gaps etc., and in closing crack edges as well.

Thes follows from a comparison of the energy loss expression at multiple weak unelastic collisions and from that for the force work proportional to [Im m] during a vibration period. Let us show it.

Let v_{ij} be the velocity of mass m_i at time t_j preceeding an impact and $\eta_{ij}\, v_{ij}$ after that; η_{ij} is the velocity recovery coefficient. The energy variation is equal to

$$\Delta E_{ij} = \frac{1}{2} m_i v_{ij}^2 (1 - \eta_{ij}^2) \ . \qquad (6)$$

If each mass m_i for a period experiences k_i impacts, then one has for the whole system:

$$\Delta E = \sum_{i=1}^{N} \sum_{j=1}^{K_i} \frac{1}{2} m_i v_{ij}^2 (1 - \eta_{ij}^2) \ . \qquad (7)$$

Represent velocity v_i as a sinusoid multiplied by amplitude stepwise changing at an impact:

$$v_i = v_{i0} \sin \omega t \ , \qquad (8)$$

where

$$v_{i0} = y_{i0}\omega \exp\left(\sum_{\tau=1}^{K_i} \delta(t - t_\tau) \ln \eta_{i\tau}\right) , \qquad (9)$$

$$\delta(t-t_\tau)=\begin{cases}0 & t<t_\tau\\ 1 & t\geq t_\tau\end{cases}\quad, \tag{10}$$

then

$$v_{ij}=\omega y_{i0}\left(\prod_{\tau=1}^{j-1}\eta_{i\tau}\right)\sin 2\pi t_j/T\ . \tag{11}$$

Substituting (11) in (7) we obtain

$$\Delta E=\omega^2\sum_{i=1}^{N}\frac{1}{2}y_{i0}^2 m_i\sum_{j=1}^{K_i}(1-\eta_{ij}^2)\left(\prod_{\tau=1}^{j-1}\eta_{i\tau}\right)^2\sin^2 2\pi t_j/T\ . \tag{12}$$

In turn, the forces proportional to the imaginary part of the mass matrix lead to the energy loss during a period equal to:

$$\begin{aligned}\Delta E&=-\int_0^T \mathrm{Re}(i\omega e^{i\omega t}\{x_0\}')\,\mathrm{Re}(\omega^2 i[\mathrm{Im}\,m]\{x_0\}e^{i\omega t})dt=\\&=\pi\omega^2(\{\mathrm{Re}\,x_0\}'[\mathrm{Im}\,m]\{\mathrm{Re}\,x_0\}+\{\mathrm{Im}\,x_0\}'[\mathrm{Im}\,m]\{\mathrm{Im}\,x_0\})\ .\end{aligned} \tag{13}$$

Though (12) and (13) are derived using different computed dynamic models ($N \gg n$), the comparison shows the energy change as a function of frequency to be the same.

III. NORMAL COORDINATES

Introduce for (1) normal coordinates and express forced vibrations through these coordinates.

Substituting

$$\{x\}=\{f\}e^{\lambda t} \tag{14}$$

in the homogeneous equation

$$[m]\{\ddot{x}\}+[K]\{x\}=0\ , \tag{15}$$

will give algebraic equations for eigenvalue and eigenvector λ and $\{f\}$

$$(\lambda^2[m]+[K])\{f\}=0\ . \tag{16}$$

The characteristic equation for eigenvalues is

$$\det(\lambda^2[m]+[K])=0\ . \tag{17}$$

The eigenvalues and eigenvectors (16) can be written as:

$$\begin{aligned}&\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\ .\\&\{f_1\},\ \{f_2\},\ \ldots,\ \{f_n\}\ .\end{aligned} \tag{18}$$

Derive the orthogonality conditions for eigenvectots. Using the *transposition rule for matrix product*

$$([m]\{f\})'=\{f\}'[m]'=\{f\}'[m] \tag{19}$$

write the factor commutativity in bilinear forms with symmetric matrices

$$\{f_\tau\}'[m]\{f_s\}=([m]\{f_\tau\})'\{f_s\}=\{f_s\}'[m]\{f_\tau\}\ . \tag{20}$$

Using this property we find from (16) the orthogonality condition for two eigenvectors $\{f_r\}$ and $\{f_s\}$ with unequal eigenvalues λ_τ and λ_s

$$\{f_s\}'[m]\{f_\tau\}=0\ . \tag{21}$$

Similarly, the second orthogonality condition is found

$$\{f_s\}'[K]\{f_\tau\}=0\ . \tag{22}$$

If we introduce the modal matrix

$$[f]=\begin{bmatrix}f_{11}&\cdots&f_{1n}\\ \cdots&\cdots&\cdots\\ f_{n1}&\cdots&f_{nn}\end{bmatrix}\quad, \tag{23}$$

then conditions (21) and (22) for all n vectors are written as two matrix equalities

$$[f][m][f]'=[\diagdown\{f_s\}'[m]\{f_s\}\diagdown]\ , \tag{24}$$

$$[f][K][f]'=[\diagdown\{f_s\}'[K]\{f_s\}\diagdown]\ , \tag{25}$$

where the right parts contain diagonal matrices.

Using the linear transformation

$$\{x\}=[f]'\{q\}\ , \tag{26}$$

introduce a normal coordinate vector $\{q\}$.

Substituting (26) in (1), multiplying by $[f]$ and using (24) and (25), we obtain vibration equations in gentralized coordinates $\{q\}$:

$$[\diagdown\{f_s\}'[m]\{f_s\}\diagdown]\{\ddot{q}\}+[\diagdown\{f_s\}'[K]\{f_s\}\diagdown]\{q\}=[f]\{F\}e^{i\omega t}. \tag{27}$$

For s – component of the vector $\{q\}$ from (27) we find

$$\mu_{ss}\ddot{q}_s+\varkappa_{ss}q_s=\{f_s\}'\{F\}e^{i\omega t}\ , \tag{28}$$

where

$$\mu_{ss}=\{f_s\}'[m]\{f_s\}\ , \quad \varkappa_{ss}=\{f_s\}'[K]\{f_s\} \tag{29}$$

complex generalized „mass" and „stiffness" respectively.

For λ_s we have:

$$\lambda_s=\delta_s+ip_s\ , \quad \lambda_s^2=-\varkappa_{ss}/\mu_{ss}\ ,$$
$$\delta_s=\mathrm{Re}\sqrt{-\varkappa_{ss}/\mu_{ss}}\ , \quad p_s=\mathrm{Im}\sqrt{-\varkappa_{ss}/\mu_{ss}}\ , \tag{30}$$

where δ_s and p_s are the damping coefficient and the frequency S of natural vibration mode respectively.

For q_s from (28) we find

$$q_s=q_{s0}\,e^{i\omega t}\ , \tag{31}$$

where

$$q_{s0}=-\frac{\{f_s\}'\{F\}}{\mu_{ss}(\lambda_s^2+\omega^2)}\ . \tag{32}$$

For the displacement amplitude in point l at multipoint excitation:

$$x_{0e}=\sum_{s=1}^{n} f_{se}q_{s0}=-\sum_{s=1}^{n}\frac{f_{se}\{f_s\}'\{F\}}{\mu_{ss}(\lambda_s^2+\omega^2)} \tag{33}$$

and at onepoint excitation in r point:

$$x_{0e}=-\sum_{s=1}^{n}\frac{f_{se}\,f_{sr}F_r}{\mu_{ss}(\lambda_s^2+\omega^2)}\ . \tag{34}$$

Eqs. (33) and (34) coincide with similar expressions for frictionless systems with the difference that except for ω all values in these formulas are complex.

Since the vibration modes are found accurate up to the complex normalizing factor, generalized masses and stiffnesses are proportional to the normalizing factor square.

The value

$$\frac{1}{\rho_{ss}}=\frac{f_{se}\{f_s\}'\{F/F_0\}}{\mu_{ss}} \tag{35}$$

does not depend on the normalizing factor, but is dependent on the choice of a measurement point and force application points. In (35) the vector $\{F/F_0\}$ is characteristic of a force excitation distribution and the F_0 number is representative for force.

IV. HODOGRAPH

Let us examine the vibration amplitude near resonance S of a normal mode. Present (33) as

$$(\lambda_s^2+\omega^2)(u-a)=1/\rho_{ss}\ , \tag{36}$$

where

$$u=x_{0e}/F_0\ , \quad a=-\sum_{\substack{j=1\\(j\neq s)}}^{n}{}'\ \frac{f_{je}\{f_j\}'\{F/F_0\}}{\mu_{jj}(\lambda_j^2+p_s^2)}\ . \tag{37}$$

We neglect $a=a\,(\omega)$ assuming $\omega=p_s$. Introduce the notations:

$$b=\frac{i}{4\delta_s p_s \rho_{ss}}\ , \quad \frac{\psi}{2}=-\mathrm{arctg}\frac{2\delta_s p_s}{\delta_s^2\ \ p_s^2\ \ \omega^2}\ . \tag{38}$$

Using the equality

$$1-e^{i\psi}=-2\sin\frac{\psi}{2}\,e^{i(\frac{\psi}{2}+\frac{\pi}{2})}\ , \tag{39}$$

reduce (36) to

$$u=a+b(1-e^{i\psi})=a+b-|b|\,e^{i(\psi+\varphi)}, \tag{40}$$

where

$$\varphi=\pi/2-\mathrm{arctg}\ \mathrm{Im}\rho_{ss}/\mathrm{Re}\rho_{ss}\ . \tag{41}$$

Hodograph (40) is a circle with the centre $u_c=a+b$ and radius $|b|$ (fig. 1) (ψ is counted from the point $u=a$).

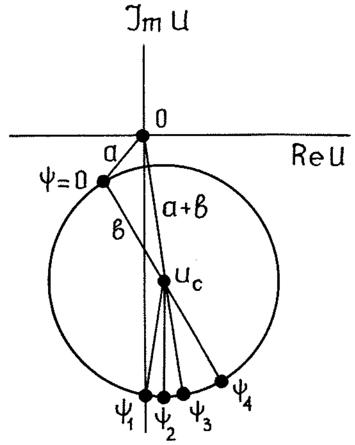

Figure 1.

Points ψ_k in fig. 1 correspond to different resonance conditions.

1. Phase resonance:

$$\mathrm{Re}\, u = 0\,, \quad \cos(\psi_1+\varphi) = \mathrm{Re}\,(a+b)\,|b|^{-1}\,.$$

2. Resonance $\mathrm{Jm}\, u$:

$$\frac{\partial\, \mathrm{Jm}\, u}{\partial \psi} = 0\,, \quad \psi_2 = \pi + \mathrm{arctg}\, \frac{\mathrm{Jm}\, \rho_{ss}}{\mathrm{Re}\, \rho_{ss}}\,.$$

3. Resonance $|u|$:

$$\frac{\partial |u|^2}{\partial \psi} = 0\,, \quad \psi_3 = -\varphi + \mathrm{arctg}\, \frac{\mathrm{Jm}\,(a+b)}{\mathrm{Re}\,(a+b)}\,.$$

4. Normal mode resonance:

$$\frac{\partial |u-a|^2}{\partial \psi} = 0\,, \quad \psi_4 = \psi_s = \pi\,, \qquad (42)$$
$$\omega_s^2 = p_s^2 - \delta_s^2\,, \quad u - a = 2b\,.$$

For systems with a complex mass matrix describe a refined method of determining the natural frequency, complex natural mode and generalized mass which is a version of the known Kennedy-Pancu method.

The method is that the perfect oscillator hodograph circle coinciding with true ones in three points corresponding to th ree discrete frequencies near resonance is inscribed in a true hodograph for a system with many degrees of freedom.

Let these three points be related to frequencies ω_1, ω_2 and ω_3 and to relative amplitudes u_1, u_2 and u_3.

The condition that three points lie on the circle of the radius $|b|$ with the centre u_c is

$$|u_1 - u_c|^2 = |u_2 - u_c|^2 = |u_3 - u_c|^2 = |b|^2\,. \qquad (43)$$

From (43) we find

$$\mathrm{Re}\, u_c = \frac{\det \begin{bmatrix} |u_1|^2 - |u_2|^2 & \mathrm{Jm}(u_1 - u_2) \\ |u_2|^2 - |u_3|^2 & \mathrm{Jm}(u_2 - u_3) \end{bmatrix}}{\det \begin{bmatrix} \mathrm{Re}\,(u_1 - u_2) & \mathrm{Jm}(u_1 - u_2) \\ \mathrm{Re}(u_2 - u_3) & \mathrm{Jm}(u_2 - u_3) \end{bmatrix}}\,,$$

$$(44)$$

$$\mathrm{Jm}\, u_c = \frac{\det \begin{bmatrix} \mathrm{Re}(u_1 - u_2) & |u_1|^2 - |u_2|^2 \\ \mathrm{Re}(u_2 - u_3) & |u_2|^2 - |u_3|^2 \end{bmatrix}}{\det \begin{bmatrix} \mathrm{Re}(u_1 - u_2) & \mathrm{Jm}\,(u_1 - u_2) \\ \mathrm{Re}(u_2 - u_3) & \mathrm{Jm}\,(u_2 - u_3) \end{bmatrix}}\,,$$

$$|b|^2 = \mathrm{Re}^2(u_1 - u_c) + \mathrm{Jm}^2(u_1 - u_c)\,. \qquad (45)$$

Similarly, from (36) we have

$$\lambda_s^2 = \frac{\det \begin{bmatrix} -u_1\omega_1^2 + u_2\omega_2^2 & \omega_1^2 - \omega_2^2 \\ -u_2\omega_2^2 + u_3\omega_3^2 & \omega_2^2 - \omega_3^2 \end{bmatrix}}{\det \begin{bmatrix} u_1 - u_2 & \omega_1^2 - \omega_2^2 \\ u_2 - u_3 & \omega_2^2 - \omega_3^2 \end{bmatrix}}\,,$$

$$(46)$$

$$a = -\frac{\det \begin{bmatrix} u_1 - u_2 & u_1\omega_1^2 + u_2\omega_2^2 \\ u_2 - u_3 & u_2\omega_2^2 + u_3\omega_3^2 \end{bmatrix}}{\det \begin{bmatrix} u_1 - u_2 & \omega_1^2 - \omega_2^2 \\ u_2 - u_3 & \omega_2^2 - \omega_3^2 \end{bmatrix}}\,,$$

$$\rho_{ss} = -\frac{\dfrac{1}{u_1 - a} - \dfrac{1}{u_2 - a}}{\omega_1^2 - \omega_2^2}\,. \qquad (47)$$

Define the vibration of S – nirmal mode. The determination of a in each measurement point results in the vector:

$$\{a\} = \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} \quad . \tag{48}$$

Substituting (43) in the forced oscillation mode at resonance of S – mode obtained from (34):

$$\{x_0/F\} = \{a\} - \frac{\{f_s\}\{f_s\}'\{F/F_0\}}{\mu_{ss}(\lambda_s^2 + \omega^2)} \tag{49}$$

and finding the difference

$$\{w\} = \{x_0/F\} - \{a\} = \{f_s\} N_s \quad , \tag{50}$$

where

$$N_s = -\frac{\{f_s\}'\{F/F_0\}}{\mu_{ss}(\lambda_s^2 + \omega^2)} \quad , \tag{51}$$

we find that $\{W_s\}$ are complex natural vibrations of S mode. Since the natural vibration mode is determined accurate up to the complex factor one can assume

$$\{f_s\} = C_s \{w\} \quad , \tag{52}$$

where C_s is, generally speaking, an arbitrary normalizing factor.

From (35) we obtain the generalized mass

$$\mu_{ss} = f_{se} \{f_s\}'\{F/F_0\} \rho_{ss} \tag{53}$$

and, according to (30), the generalized stiffness

$$\varkappa_{ss} = -\lambda_s^2 \mu_{ss} \quad . \tag{54}$$

V. IDENTIFICATION

Having determined experimental generalized mass, stiffness and complex natural vibration modes and substituted them in (24) and (25) we obtain equalities to be regarded as equations for finding the coefficients of matrices $[m]$ and $[k]$:

$$[f][m][f]' = [\diagdown \mu_{ss} \diagdown] \quad , \tag{55}$$

$$[f][k][f]' = [\diagdown \varkappa_{ss} \diagdown] \quad , \tag{56}$$

or, in the vectir form:

$$\{f_s\}'[m]\{f_\tau\} = \mu_{ss}\delta_{s\tau} \quad , \tag{57}$$

$$\{f_s\}'[k]\{f_\tau\} = \varkappa_{ss}\delta_{s\tau} \quad , \tag{58}$$

$$(s = 1, \dots n \quad , \quad \tau = 1, \dots n) \quad .$$

In view of (20), independent equations in each of systems (52) and (53) are only $(n+1)n/2$ and their numder is equal to the number of $[m]$ and $[k]$ symmetric matrix coefficitnas.

This theory being a generalization for the case of complex natural forms of the results published in[4] may be used to identify a full-scale structure and its computatuinal as well as elastic dynamical models.

The magnitudes of coefficients $[m]$ and $[k]$ are dependent on measurement point locations, a number of considered modes, the choice of calculation points on the hodograph and on other factors. Therefore, they differ from actual local mass and stiffness characteristics and from damping characteristics as well. Nevertheless, if we have defined them by the same procedure on the full-scale structure, and by prediction, then by comparing these coefficients we can establish a reciprocal check.

When appreciable cracks, stability loss of a structural element, impacts from a separated element or an equipment unit etc. occur, the coefficients of matrices $[k]$ and $[\mathrm{Im}\, m]$ corresponding to the flaw point may be altered much more than such integral characteristics as generalized mass, frequency and oscillation decrement. Hence, the present identification method may offer certain promise for vibration flaw detection, particullarly for those structural elements which are unaccessible by conventional flaw detection methods or for an approximate flaw detection.

VI. EXAMPLES

Fig. 2 and tables 1 and 2 show processed results of vibration tests for a composite swept wing model. The location of the stiffness axis necessary to represent the wing as a beam working in bending and torsion was found. To that end, the $[k]$ matrix coeffocoents were used from:

$$[k] = [f]^{-1}[\diagdown \varkappa_{ss} \diagdown]([f]^{-1})' \quad , \tag{59}$$

which follows from (56).

Further, compliance coefficient matrix $[r]$ was defined

$$[r] = [\mathrm{Re}\, K]^{-1} \tag{60}$$

and the next experssion was formulated for a wing displacement in two points when applying forces Φ_i and Φ_j

$$x_i = \tau_{ii}\Phi_i + \tau_{ij}\Phi_j \quad , \quad x_j = \tau_{ji}\Phi_i + \tau_{jj}\Phi_j \quad . \tag{61}$$

If the points lie on an undeformed section and the force moment relative to the stiffness axis (flexural centre) is equal to zero, i. e.:

$$\Phi_i b_i = - \Phi_j (l_i - b_i) \quad , \tag{62}$$

then the section will displace translationally and

$$x_i = x_j \quad . \tag{63}$$

Eqs. (56), (57) and (58) give the following expression for the stiffness axis location

$$\frac{b_i}{l_i} = \frac{\tau_{ii} - \tau_{ij}}{\tau_{ii} - \tau_{jj}} \quad , \tag{64}$$

where l_i is the distance between force application points.

Fig. 3 and tables 3 and 4 prese nt processed results of vibrational tests for two similar compressor blades. One of the blades has a 10 cm flaw located on 45% of the span.

The greatest discrepancy is contained in coefficient r_{23}.

REFERENCES

(1). R. L. Bisplinghoff, H. Ashley and R. L. Halfman, „Aeroelasticity", Addison – Wesley Publishing Company Inc., Cambridge 42. Mass., 1955.

(2). S. R. Ibrahim, „Computation of Normal Modes from Identified Complex Modes", AIAA Journal, vol. 21, N 3, March 1983.

(3). S. R. Ibrahim, „Dynamic Modeling of Structures from Measured Complex Modes", In 23 rd. Structural Dynamics and Materials Conferens. Part 2 AIAA/ASME/ASCE/AHS 1982–0770, p. 564–573.

(4). A. Berman, W. G. Flannelly, „Theory of Incomplete Models of Dynamic Structures", AIAA Journal, vol. 9, N 8, 1971, p. 1481–1487.

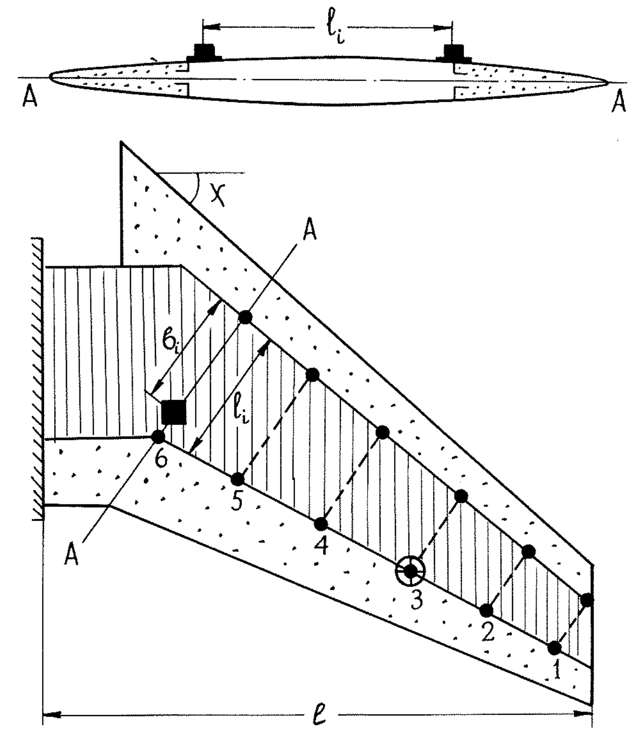

Figure 2. Dynamic model of the wing with a torsion box from anisotropic fiber-glass plastic

▲, ● – measurement point; ⊕ – excitation point; [foam] – foam core; [hatched] – anisotropic fiberglass plastic; ■ – flexion center b_i semispan $l = 750$ cm, $\chi = 45°$.

i	1	2	3	4	5	6
	I Flex.	II Flex.	I Tors.	II Tors.	III Flex.	III Tors.
ν	11,35	36,4	55,5	102,8	114,5	134,3
θ	0,09	0,08	0,09	0,07	0,08	0,12

Table 1. Frequencies and decrements of experimental modes used in the calculation

Sec. number i	1	2	3	4	5	6
b_i/l_i		0,91	0,65		1	
b_i/l_i	0,32	0,78	0,53			
b_i/l_i	0,49				0,87	0,95
b_i/l_i static	1,17	0,96	0,805	0,765	0,75	0,83

Table 2. Flexion center location for different section choices. Comparison of b_i/l_i obtained by dynamic and static methods

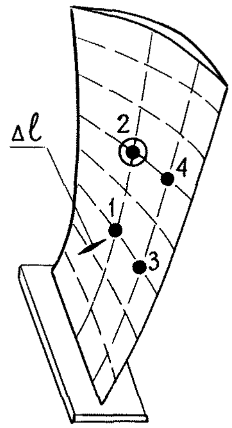

Figure 3. Compressor blade; length $l = 110$ cm; chord $b = 55$ cm; crack length $\Delta l = 10$ cm; ⊕ – excitation point; ● – measurement point.

Modes	I Flexion		I Torsion		II Flexion		II Torsion	
	Frequency, ν (Hz)	Decrem. θ log.	ν	θ	ν	θ	ν	θ
Blade N° 1 (with no crack)	52,7	0,004	179,3	0,053	234,7	0,015	418,9	0,046
Blade N° 2 (with no crack)	53,6	0,003	180,5	0,06	238,1	0,020	414,5	0,051
Blade N° 3 (with a crack)	47,2	0,0037	167,3	–	223,0	0,027	406,5	0,052

Table 3. Frequencies and decrements of experimental modes used in the calculation

$0{,}123 \cdot 10^{-6}$	$0{,}267 \cdot 10^{-6}$	$0{,}170 \cdot 10^{-6}$	$0{,}362 \cdot 10^{-7}$
	$0{,}683 \cdot 10^{-6}$	$0{,}357 \cdot 10^{-6}$	$0{,}123 \cdot 10^{-7}$
		$0{,}357 \cdot 10^{-6}$	$0{,}111 \cdot 10^{-6}$
			$0{,}997 \cdot 10^{-7}$

Table 4. Matrix $[r]$ of influence coefficients for blade N° 1

$0{,}129 \cdot 10^{-6}$	$0{,}18 \cdot 10^{-6}$	$0{,}14 \cdot 10^{-6}$	$0{,}388 \cdot 10^{-7}$
	$0{,}455 \cdot 10^{-6}$	$0{,}12 \cdot 10^{-6}$	$0{,}148 \cdot 10^{-7}$
		$0{,}268 \cdot 10^{-6}$	$0{,}867 \cdot 10^{-7}$
			$0{,}982 \cdot 10^{-7}$

Table 5. Matrix $[r]$ for blade N° 3