ICAS-84-1.7.2

INDICIAL AERODYNAMIC COEFFICIENTS FOR TRAPEZOIDAL WINGS

Valter J. E. Stark

Saab Scania AB, Linköping, Sweden

Abstract

Approximating the potential jump across a wing and its wake within rectangular elements by constants, indicial aerodynamic coefficients for rigid and elastic modes of a trapezoidal wing in incompressible inviscid flow have been calculated. The results show that the indicial coefficients for different modes can be described by a single simple function - a generalized Wagner function. Knowledge of initial and steady state values and apparent mass coefficients, which have been calculated too, is thus sufficient. Laplace transformation yields a generalized Theodorsen function and transfer functions for flutter or stability analysis.

Introduction

Indicial aerodynamic coefficients are required for explicitly expressing aerodynamic forces for arbitrary motion in terms of the generalized coordinates.

Laplace transformation of expressions for arbitrary motion yields transfer functions. Usually, these functions are calculated by oscillating-surface methods, but Laplace transformation of indicial aerodynamic coefficients is an attractive alternative. If the indicial coefficients are given, the Laplace transformation can be performed with high accuracy for any real or complex value of the frequency parameter. It yields a function that is analytic at all points of the cut frequency plane.

Unfortunately, direct calculation of indicial coefficients, which was performed by Lomax et al. (Ref. 1), is very difficult for compressible flow even on the basis of the linearized theory. But for obtaining a suitable expression (Ref. 2) for approximating calculated or measured transfer functions, which is required in flutter programs, it may suffice to know the qualitative behaviour of the indicial coefficients.

For a flutter calculation in which many degrees of freedom shall be considered, one has to calculate transfer functions for many different deflection modes and reduced frequencies. In an attempt to simplify such a calculation, it is interesting to investigate if the variation of the transfer functions for different modes is significantly different.

It is known, namely, that a single function, the Wagner function, is sufficent for describing the variation of indicial coefficients for all modes of a thin wing in 2-dimensional, incompressible flow or, equivalently, that the Theodorsen function is sufficient for describing all transfer functions in this case.

In a recent investigation by Stark (Ref. 2), it was found that a single function, a generalised Wagner function, was sufficient even for rectangular wings and typical modes.

In this paper, a corresponding investigation for a trapezoidal wing is described.

The results imply that it should be sufficient for practical purposes to determine apparent mass coefficients, initial values of so called deficiency functions, and the steady state limits of the indicial coefficients plus a characteristic time for the wing considered.

Whether a corresponding simplification is possible for compressible flow remains to be investigated.

Indicial coefficients

All quantities in what follows are dimensionless and referred to typical reference quantities. Lengths are referred to the semi-root-chord L, velocities to the free-stream speed U, times to L/U, pressures to the free-stream dynamic pressure $\rho U^2/2$, and velocity potentials to UL.

It is assumed that the perturbation pressure can be calculated by the linearized theory and that the deflection of the wing can be described by the equation

$$z = \sum_{n=1}^{n_s} h_n(x,y)q_n(t) \tag{1}$$

where the functions $h_n(x,y)$ are given deflection modes and $q_n(t)$ undetermined generalised coordinates. The former depend only on the coordinates x and y (in the free-stream and the spanwise direction respectively) and the latter only on the time t.

The perturbation pressure p and the perturbation velocity potential ϕ can be resolved into components p_n and ϕ_n which correspond to the n^{th} term in Eq. (1). Since p is related to ϕ through the linearized Bernoulli equation, the pressure jump Δp_n and the potential jump $\Delta\phi_n$ across the wing shall satisfy

$$\Delta p_n = -2(\partial\Delta\phi_n/\partial x + \partial\Delta\phi_n/\partial t) \tag{2}$$

The potential ϕ_n is determined by the wave equation (referred to the moving coordinates x,

y, z), the radiation condition, the wake condition, the Kutta condition, and the boundary condition

$$\partial \phi_n/\partial z = (\partial h_n/\partial x)q_n + h_n\dot{q}_n \tag{3}$$

on the wing ($\dot{q}_n = dq_n/dt$).

We let the generalized aerodynamic forces for arbitrary motion be represented by the dimensionless coefficients

$$K_{m,n}(t) = \frac{1}{S}\iint h_m \Delta p_n dS \tag{4}$$

where S is a reference area, the wing area, and dS a surface element measured by the same unit as S.

The two indicial aerodynamic coefficients $K^1_{m,n}(t)$ and $K^2_{m,n}(t)$ which are needed for expressing the aerodynamic forces for arbitrary motion in terms of the generalized coordinates are associated with two indicial potentials ϕ^1_n and ϕ^2_n. These are defined by the same differential equation and the same conditions as ϕ_n, but the prescribed normal velocity is different. The boundary condition for ϕ^r_n on the wing reads

$$\partial \phi^r_n/\partial z = \begin{cases} (\partial h_n/\partial x)H(t) & r = 1 \\ h_n\, H(t) & r = 2 \end{cases} \tag{5}$$

where $H(t)$ is the Heaviside unit step function.

The indicial pressure jump is defined by

$$\Delta p^r_n = -2(\partial \Delta\phi^r_n/\partial x + \partial\Delta\phi^r_n/\partial t) \tag{6}$$

and the indicial coefficients by

$$K^r_{m,n}(t) = \frac{1}{S}\iint h_m \Delta p^r_n dS \tag{7}$$

Numerical method

The numerical method that has been developed is similar to Belotserkovskii's method[3]. It is applicable to a wing with unswept trailing edge, but this condition may be relaxed.

The boundary value problem for the velocity potential is solved in this method by using rectangular surface elements and by approximating the potential jump by a constant within each element. The elements, which are shown in Fig. 1 for the right wing-half, are bounded by equidistant lines in the y and x directions. These lines are defined by

$$x = x_\mu = (\mu - 1/4)2d_x \tag{8}$$

where $\mu = 1(1)(i_s+1)$ and by

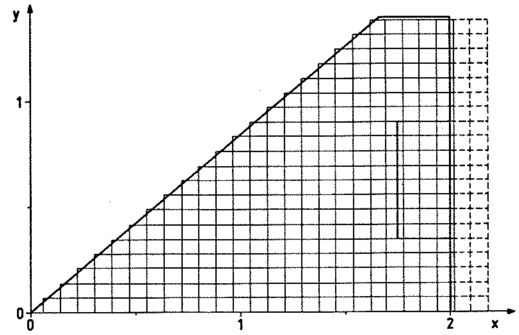

FIG. 1 WING AND DOUBLET ELEMENTS USED IN NUMERICAL CALCULATION

$$y = y_\nu = \nu 2d_y \tag{9}$$

where $\nu = 0(1)j_s$. The coordinates of the element centers, which serve as control points, are $x = X_i = x_i + d_x$ and $y = Y_j = y_j$.

The trailing edge and the wing tip are defined by $x = 2$ and $y = b$ and i_s, j_s, d_x, and d_y are chosen such that $(2i_s+1)d_x = 2$ and $(2j_s+1/2)d_y = b$. The semi-span b, the aspect ratio A, and the taper ratio c_t/c_r satisfy the relation $b = A(1+c_t/c_r)/2$.

The indicial motion starts at time $t = 0$. At this time, the approximation $\Delta\phi^{r,n}_{i,j}(t)$ to the potential jump $\Delta\phi^r_n$ within the element with center $(x,y) = (X_i, Y_j)$ can be solved from the equations

$$\sum_{\nu=1}^{j_s}\sum_{\mu=\nu}^{i_s} W^{i,j}_{\mu,\nu}\, \Delta\phi^{r,n}_{i,j}(0) = w^r_n(X_i, Y_j) \tag{10}$$

where $j = 1(1)j_s$, $i = 1(1)i_s$,

$$w^r_n(x,y) = \begin{cases} \partial h_n/\partial x & r = 1 \\ h_n & r = 2 \end{cases} \tag{11}$$

$$W^{i,j}_{\mu,\nu} = K(\mu-i, \nu-j) + sK(\mu-i, \nu+j-1) \tag{12}$$

$$K(i,j) = \lim_{z\to 0} \frac{-1}{4\pi}\int_{xi}^{xs} dx \int_{yi}^{ys} \frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)dy \tag{13}$$

$$R = (x^2 + y^2 + z^2)^{\frac{1}{2}} \tag{14}$$

$$s = \begin{cases} 1 & \text{if } h_n(x,-y) = h_n(x,y) \\ -1 & \text{"}\ \ h_n(x,-y) = -h_n(x,y) \end{cases} \tag{15}$$

$xi = (2i-1)d_x$, $xs = (2i+1)d_x$, $yi = (2j-1)d_y$, and $ys = (2j+1)d_y$.

When the motion starts, a vortex sheet starts to develop at the trailing edge of the wing. This sheet moves downstream with a velocity relative to the wing that is assumed to be equal to the free-stream velocity. In order to calculate the effect of the vortex sheet, this is replaced by a planar surface with a potential jump. The velocity, that is induced by this potential jump, is calculated in the same way as the velocity, that is induced by the jump across the wing. Hence, at the time $t = t_k = k(2d_x)$ the potential jump approximations shall satisfy

$$\sum_{\nu=1}^{j_s}\sum_{\mu=\nu}^{i_s} W^{i,j}_{\mu,\nu}\,\Delta\phi^{r,n}_{\mu,\nu}(t_k) = w^r_n(X_i, Y_j) - \sum_{\nu=1}^{j_s}\sum_{\mu=i_s+1}^{i_s+k} W^{i,j}_{\mu,\nu}\,\Delta\phi^{r,n}_{\mu,\nu}(t_k) \qquad (16)$$

According to Kelvin's theorem, the strength of the vortex sheet at a point that moves with the average fluid velocity is constant and equal to the strength that was generated when the point passed the trailing edge. Hence, the potential jump approximations in the right hand member of Eq. (16) shall satisfy

$$\Delta\phi^{r,n}_{\mu,\nu}(t_k) = \Delta\phi^{r,n}_{i_s,\nu}(t_{k-\mu+i_s}) \qquad (17)$$

They can be determined successively by solving the equations (16) for $k = 1, 2, 3, \ldots$.

The derivatives in Eq. (6) may be replaced by finite differences. This yields the approximate formula

$$\Delta p^{r,n}_{i,j}(t_{k+1}) = -2\left[\Delta\phi^{r,n}_{i,j}(t_{k+1}) - \Delta\phi^{r,n}_{i-1,j}(t_k)\right]/(2d_x) \qquad (18)$$

for the pressure jump at time $t = t_{k+1}$.

For time $t \approx 0$, the linearized Bernoulli equation gives

$$\Delta p^{r,n}_{i,j}(0) = -2\,\Delta\phi^{r,n}_{i,j}(0)\,\delta(t) \qquad (19)$$

where $\delta(t)$ is the Dirac delta function.

For the indicial coefficients, we find the approximate formula

$$K^r_{m,n}(t_k) = \frac{\Delta S}{S}\sum_{j=1}^{j_s}\sum_{i=j}^{i_s} h_m(x_i, y_j)\,\Delta p^{r,n}_{i,j}(t_k) \qquad (20)$$

where $\Delta S = 4d_x d_y$. This formula yields good accuracy.

A generalized Wagner function

By means of the indicial aerodynamic coefficients, the generalized aerodynamic forces for arbitrary motion can be written

$$K_{m,n}(t) = \int_0^t K^1_{m,n}(t-\tau)\dot q_n(\tau)d\tau + \int_{0-}^t K^2_{m,n}(t-\tau)\ddot q_n(\tau)d\tau \qquad (21)$$

where $\dot q_n(\tau) = dq_n(\tau)/d\tau$.

A simplification is achieved by introducing the deficiency functions $C^r_{m,n}(t)$ and the apparent mass coefficients $D^r_{m,n}$. Together with the steady state limits $K^r_{m,n}(\infty)$, these compose the total indicial coefficient

$$K^r_{m,n}(t) = K^r_{m,n}(\infty) - C^r_{m,n}(t) + D^r_{m,n}\,\delta(t) \qquad (22)$$

Inserting this expression into Eq. (21) yields

$$K_{m,n}(t) = K^1_{m,n}(\infty)q_n + \left[K^2_{m,n}(\infty)+D^1_{m,n}\right]\dot q_n + D^2_{m,n}\ddot q_n - \int_0^t C^1_{m,n}(t-\tau)\dot q_n(\tau)d\tau - \int_{0-}^t C^2_{m,n}(t-\tau)\ddot q_n(\tau)d\tau \qquad (23)$$

The integrals in this expression are less important, in particular for low-aspect-ratio wings, than those in Eq. (21). They are also more attractive since the deficiency functions vanish for $t \to \infty$.

In an earlier investigation[2] for a rectangular wing of aspect ratio 3, it was found that the ratios

$$\varphi(t) = C^r_{m,n}(t)/C^r_{m,n}(0) \qquad (24)$$

were almost identical for the deflection modes considered, i. e. independent of m, n, and r. It is further known that the well known Wagner function $\phi(t)$ is sufficient for describing the indicial coefficients for all deflection modes in the case of 2-dimensional incompressible flow.

There is reason to believe, therefore, that a generalized Wagner function, or rather a single generalized deficiency function $2(1-\phi(t)) = \varphi(t)$ corresponding to the normalized functions (24) can be defined for each wing and possibly not only for incompressible flow. Probably, the normalized functions are not exactly identical, but it is possible that a single function is sufficient for each wing (and Mach number) for describing the deficiency functions for all important modes in practical flutter or stability investigations.

The deficiency function for 2-dimensional incompressible flow behaves like $1/t$ for $t \to \infty$, but those for practical wings in 3-dimensional flow probably behave like $1/t^3$, which was found in the earlier investigation[2]. We assume, therefore, that a generally useful deficiency function may have the form

$$\varphi(t) = (1 + t/T)^{-3} \qquad (25)$$

where T is a characteristic time. This parameter is probably not too much different for different wings.

In terms of $\varphi(t)$, the expression for the aerodynamic coefficients for arbitrary motion may be written as

$$K_{m,n}(t) = K^1_{m,n}(\infty)q_n + \left[K^2_{m,n}(\infty)+D^1_{m,n}\right]\dot{q}_n + D^2_{m,n}\ddot{q}_n$$
$$- \int_0^t \varphi(t-\tau)\left[C^1_{m,n}(0)\dot{q}_n(\tau)+C^2_{m,n}(0)\ddot{q}_n(\tau)\right]d\tau \qquad (26)$$

A generalized Theodorsen function

Applying Laplace transformation to Eq. (23) and assuming that the initial values $q_n(0)$ are zero, we get

$$L\{K_{m,n},p\} = A_{m,n}(p)L\{q_n,p\} - D^2_{m,n}\dot{q}_n(0) \qquad (27)$$

where the quantities $A_{m,n}(p)$ are the aerodynamic transfer functions and p the dimensionless transform parameter (p = ik, k being the reduced frequency). If the normalized deficiency function is utilized, the expression for the transfer function becomes

$$A_{m,n}(p) = K^1_{m,n}(\infty) + \left[K^2_{m,n}(\infty)+D^1_{m,n}\right]p + D^2_{m,n}p^2$$
$$- \left[C^1_{m,n}(0)+pC^2_{m,n}(0)\right]2(1-C(-ip)) \qquad (28)$$

where $2(1-C(-ip)) = pL\{\varphi,p\}$. For 2-dimensional incompressible flow, the function C(-ip) is the Theodorsen function and the expression is then identical to the formulas of Theodorsen and Küssner. For plunge and pitch about 50 % chord, we then have

$$\frac{1}{\pi}\left[A_{m,n}\right] = \begin{bmatrix}0, & 2\\ 0, & -1\end{bmatrix} + \begin{bmatrix}2, & 1\\ -1, & 0\end{bmatrix}p + \begin{bmatrix}1, & 0\\ 0, & 1/8\end{bmatrix}p^2$$
$$- \left\{\begin{bmatrix}0, & 2\\ 0, & -1\end{bmatrix} + p\begin{bmatrix}2, & 1\\ -1, & -1/2\end{bmatrix}\right\}(1-C(-ip)) \qquad (29)$$

If the simple expression (25) is used for $\varphi(t)$, we get

$$2(1-C(-ip)) = TpF_3(Tp) \qquad (30)$$
$$F_k(p) = (1-pF_{k-1}(p))/(k-1) \qquad (31)$$
$$F_1(p) = e^p E_1(p) \qquad (32)$$
$$E_1(p) = -\gamma - \ln(p) - \sum_{n=1}^{\infty}(-1)^n p^n/(n(n!)) \qquad (33)$$

for $|\arg(p)| < \pi$; γ is Euler's constant, 0.577215 664901533... (See Ref. 4).

The function defined by Eqs. (30)-(33) is a generalized Theodorsen function which seems to be applicable for finite wings in incompressible flow. It is illustrated in Fig. 8. Like the original Theodorsen function and Birnbaum's[5] formulas, the generalized function contains logarithmic contributions. Due to these, the imaginary part of the function is discontinuous on the negative part of the real axis of the p plane.

Computation of indicial coefficients

The numerical method has been programmed in Fortran and run on the CRAY computer. By the present version of the program, a maximum number of 500 unknown quantities can be solved.

Results will be given for the wing that is shown in Fig. 1. This has aspect ratio 2.4 and taper ratio 0.17. The program was run for i_s = 24, j_s = 20, and 100 time steps. During this time the wing moves a little more than four root-chords.

Since there are only about 4 wing elements on the chord at the tip, the accuracy of the results for certain deflection modes may not be as high as desirable.

The reference quantities L and S are equal to half the root-chord and the wing area respectively.

The deflection modes considered are symmetric and defined by

$$h_n = \begin{cases} 1 & n = 1 \\ g(y/b) & 2 \\ x & 3 \\ xg(y/b) & 4 \\ \begin{Bmatrix} 1 & \text{on } C \\ 0 & \text{off } C \end{Bmatrix} & 5 \\ \begin{Bmatrix} x-x_c & \text{on } C \\ 0 & \text{off } C \end{Bmatrix} & 6 \end{cases} \qquad (34)$$

where x and y are coordinates with origin at the wing apex and $g(\eta) = 1.2\eta^2 - 0.2\eta^4$. C is a control-surface bounded by the leading edge x = 1.75 and the side edges y = 20b/81 and 40b/81 .

Results

Results for the apparent mass coefficients, the initial values of the deficiency functions, and the steady state limits of the indicial coefficients are given in Table 1-3 for r = 2. Results for r = 1 are not tabulated since, for the modes considered,

$$K^1_{m,n}(t) = \begin{cases} 0 & n = 1 \\ 0 & 2 \\ K^2_{m,1}(t) & 3 \\ K^2_{m,2}(t) & 4 \\ 0 & 5 \\ K^2_{m,5}(t) & 6 \end{cases} \qquad (35)$$

The results for the normalized deficiency functions are plotted in Fig. 2 - 7 for m = 1(1)5 and n = 1(1)6, and the function $\varphi(t) = (1+t/T)^{-3}$ is represented by the solid line for T = 2.55 in each figure.

The calculated values are seen to be approximated very well in most cases by $\varphi(t)$. Certain deviations are observed, but they are small and it is difficult to say, therefore, if they are due to numerical errors or if they really exist.

Table 1 Apparent mass coefficients $D^2_{m,n}$

m \ n=	1	2	3	4	5	6
1	1.6339	0.3031	2.1752	0.4625	0.1088	0.0154
2	0.3031	0.1064	0.4515	0.1722	0.0253	0.0036
3	2.1082	0.4391	3.0015	0.6911	0.1788	0.0256
4	0.4501	0.1678	0.6916	0.2767	0.0430	0.0062
5	0.0110	0.0026	0.0187	0.0045	0.0047	0.0008

Table 2 Initial values $C^2_{m,n}(0)$

m \ n=	1	2	3	4	5	6
1	0.5255	0.1324	0.9158	0.2396	0.1417	0.0266
2	0.1770	0.0468	0.3089	0.0849	0.0476	0.0089
3	0.6084	0.1562	1.0607	0.2830	0.1640	0.0307
4	0.2430	0.0650	0.4241	0.1181	0.0653	0.0122
5	.00110	.00028	.00191	.00052	.00030	.000055

Table 3 Steady state limits $K^2_{m,n}(\infty)$

m \ n=	1	2	3	4	5	6
1	2.7193	0.6979	4.7415	1.2650	0.7242	0.1357
2	0.7464	0.2748	1.3174	0.5071	0.1867	0.0349
3	2.8609	0.8704	5.6658	1.6667	1.0842	0.2081
4	0.9871	0.3875	1.8293	0.7408	0.2928	0.0559
5	0.0049	0.0016	0.0173	0.0049	0.0138	0.0047

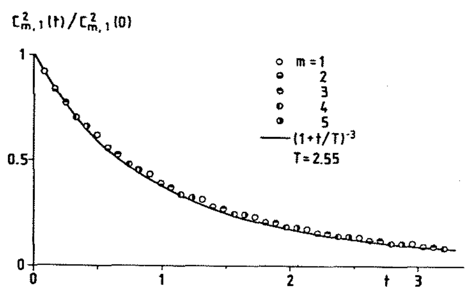

FIG. 2 NORMALIZED DEFICIENCY FUNCTIONS FOR PLUNGE

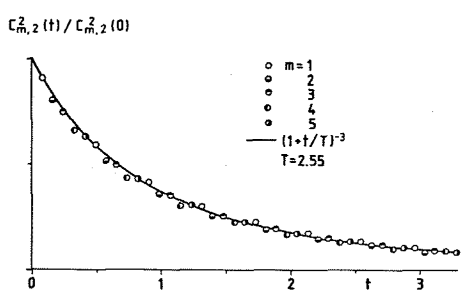

FIG. 3 NORMALIZED DEFICIENCY FUNCTIONS FOR BENDING

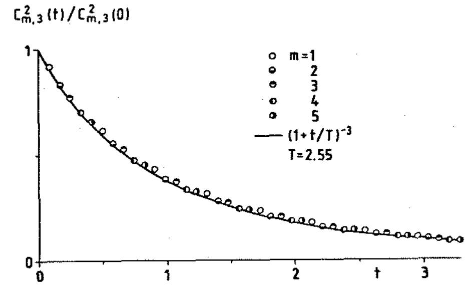

FIG. 4 NORMALIZED DEFICIENCY FUNCTIONS FOR PITCH

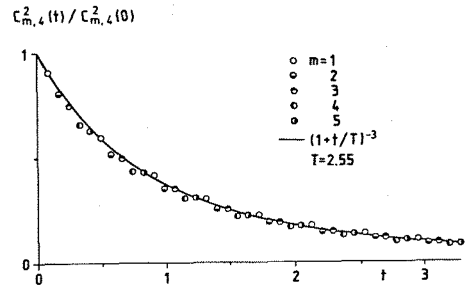

FIG. 5 NORMALIZED DEFICIENCY FUNCTIONS FOR TORSION

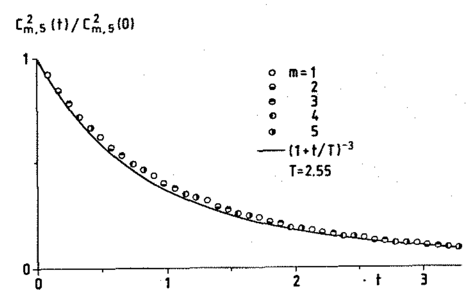

FIG. 6 NORMALIZED DEFICIENCY FUNCTIONS FOR CONTROL-SURFACE PLUNGE

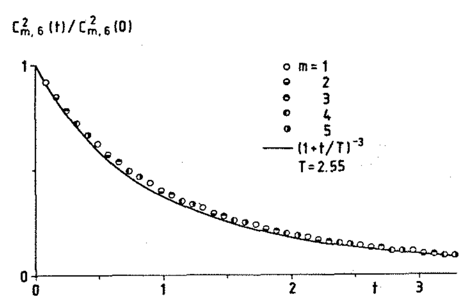

FIG. 7 NORMALIZED DEFICIENCY FUNCTIONS FOR CONTROL-SURFACE ROTATION

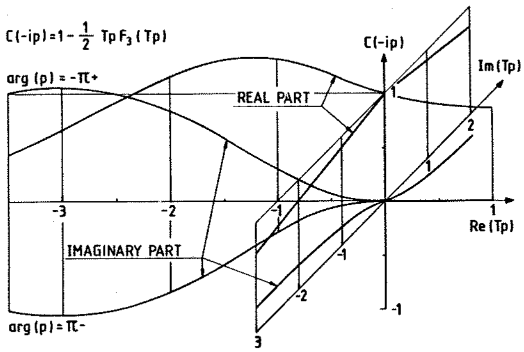

FIG. 8 THEODORSEN FUNCTION FOR TRAPEZOIDAL WING IN INCOMPRESSIBLE FLOW

Comparison

The coefficients of the expression (28) are determined (partly via the relations (35)) by the tabulated results in Table 1 - 3. By means of these and the result for the characteristic time T, it is possible to calculate the aerodynamic transfer functions $A_{m,n}(p)$ for arbitrary values of p.

Such a calculation has been performed for imaginary values of p and the result for one of the transfer functions is compared to that from a corresponding calculation by an oscillating-surface program in Fig. 9.

The agreement is seen to be close as it should be.

The oscillating-surface program that was used in the comparison is a program developed by this author by dividing the wing into a large number of trapezoidal surface elements and approximating the advanced velocity potential jump by a constant on each element. The advanced potential is obtained from the ordinary potential by replacing t by t+x.

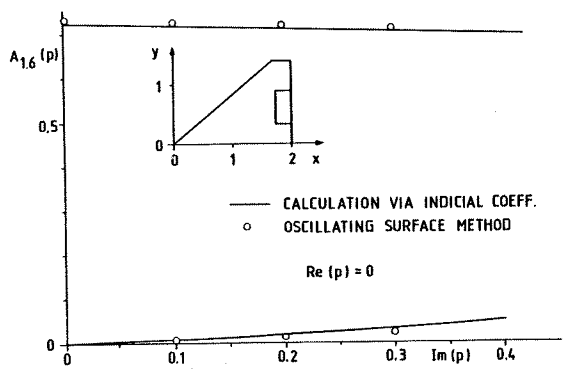

FIG. 9 TRANSFER FUNCTION FOR LIFT DUE TO FLAP ROTATION

Conclusions

A numerical method for calculation of indicial aerodynamic coefficients for trapezoidal wings in incompressible flow has been developed and applied to such a wing.

The results obtained for 6 typical deflection modes including control-surface deflection show that the normalized deficiency functions for these modes are almost equal and that they can be represented by a single function for practical purposes.

The representative function may have the simple form $(1+t/T)^{-3}$ where T is a characteristic time. For the wing considered, this is about 1.275 root-chords divided by the flight speed.

During this time, the deficiency functions decrease to 1/8 while the corresponding figure for a wing that travels one chord in 2-dimensional flow is 2/3.

The results imply that it should be sufficient to determine apparent mass coefficients, initial values of the deficiency functions, and steady state limits of the indicial coefficients. In addition, it is required to determine the characteristic time, but this seems to be independent of the deflection mode and only slightly different for different wings.

The representative function in the frequency domain is a generalized Theodorsen function which is obtained by Laplace transformation of $(1+t/T)^{-3}$.

The indicial coefficients or the corresponding transfer functions in the frequency domain can be used both in the flutter analysis and in the dynamic stability analysis of an airplane.

Acknowledgment

The author is indebted to the Swedish Defence Administration and to Saab Scania AB for financing the investigation.

References

1 Lomax H., Heaslet M. A., Fuller F. B., and Sluder L., "Two-and Three-dimensional Unsteady Lift Problems in High-Speed Flight," NACA Report 1077, 1952.

2 Stark V. J. E., "General Equations of Motion for an Elastic Wing and Method of Solution," AIAA Paper No. 83-0921. Accepted by AIAA J 11/22/83.

3 Belotserkovskii S. M., "Calculating the Effect of Gusts on arbitrary Thin Wing," Mekhanika Zidkosti i Gaza, Vol. 1, No. 1, 1967, pp. 34-40. Transl. in Fluid Dynamics, Vol. 1, No. 1, 1967.

4 Gautschi W. and Cahill W. F., "Exponential Integral and Related Functions," Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards Applied Mathematics Series 55, 9th printing, 1970, pp. 227-251.

5 Birnbaum W., "Das ebene Problem des schlagenden Flügels," Z. Angewandte Mathematik und Mechanik, Band 4, Heft 4, August 1924. pp. 278-292.