LIFTING SURFACE APPROACH OF OSCILLATING WINGS IN WEAK SHEAR FLOW

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Abstract

The importance of investigating unsteady wing problem in shear flow is increasing. However, nonpotential property of the flow prevents using a perturbation velocity potential. In this paper, a theory of wings which are oscillating in a weak shear flow is presented. The flow is assumed to be incompressible and inviscid, and the nonuniform velocity distribution is normal to the wing surface. The potential lifting surface theory is extended into the shear flow case by the method of successive approximations. The integral equation for the lift distribution to the first order approximation is derived by the double Fourier transform, and it is solved numerically by the mode function method. Calculations regarding oscillating rectangular wings with heaving and pitching modes in a shear flow are presented as examples. Generalized forces which can be easily related with unsteady lift forces and moments are obtained. Results show that the shear flow decreases all forces in amplitudes.

I. Introduction

The problem calculating lift and pitching moment on a wing, especially in an unsteady condition, is fundamentally important for several kinds of studies of aeronautics. In almost all of the cases discussed, the flow is assumed to be uniform (potential flow). In engineering applications, however, the upstream condition is not always uniform, but rather nonuniform, being disturbed by several obstacles ahead of the wing. It is possible to divide the types of shear flow along the spanwise direction (the upper figure of Fig. 1), and the shear flow normal to the wing surface (the lower figure of Fig. 1).

For the former type, the author presented the results of calculations in which rectangular wings are submerged in the uniform shear flow along the spanwise direction; unsteady lifting surface theory was applied. Due to the linearity of the flow velocity distribution and the rectangular wing, the integral equation for the lift distribution reduced to a well known form which could be solved by the already developed method of mode functions.

For the latter type of shear flow, some theoretical works have been already published. Tsien obtained the exact solution of a uniform shear flow past a symmetrical Joukowski airfoil. Chen used the second order theory in calculating the pressure distribution over the airfoil in uniform shear flow. Chow et al. created an airfoil in nonuniform flow, and solved nonlinear differential equations numerically using the finite difference method. Yetter and Chi succeeded in obtaining the effect of boundary layer on wing surface by the lifting surface theory. Nishiyama and Hirano analyzed the unsteady characteristics of an airfoil in the shear flow.

In this paper, the latter type of shear flow is analyzed, and an unsteady lifting surface theory based on an inviscid and incompressible shear flow model is presented. Since the velocity distribution of laminar wake (see Fig. 2) is adopted, the method of Ref. 1 cannot be applied. A weak shear model is used in order to extend the linearized potential theory by successive approximations. Accordingly, an approaching flow is assumed to be slightly disturbed from the uniform flow on the plane normal to the wing surface, and then a small positive quantity representing a measure of this deviation is introduced. By a double Fourier transform, the partial differential equation for the pressure is transformed into an ordinary differential equation. The integral equation for the lift distribution is derived and solved by the mode function method. Generalized forces are calculated to the first order approximation by the collocation method.

II. Basic Equations

Let a thin wing be submerged in incompressible and inviscid flow as illustrated in Fig. 2. If it is assumed that the disturbed velocities are much smaller than the free stream...
velocity $U(z)$, the equations of motion can be
linearized by the small perturbation theory as

$$
\frac{\partial u}{\partial x} + U(z) \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad (a)
$$

$$
\frac{\partial v}{\partial y} + U(z) \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \quad (b)
$$

$$
\frac{\partial w}{\partial z} + U(z) \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} = 0 \quad (c)
$$

and the continuity equation is given by

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (2)
$$

After differentiations and algebraic manipulations between Eqs. (1) and (2), the following partial differential equation for pressure can be derived

$$
\left( \frac{\partial}{\partial x} + U(z) \frac{\partial}{\partial x} \right) p' = \frac{1}{\rho} \frac{\partial^2}{\partial x^2} \rho \frac{\partial p}{\partial z} = 0. \quad (3)
$$

Assuming simple harmonic motion, $p$ is written by

$$
p(x,y,z,t) = \tilde{p}(x,y,z)e^{i\omega t}. \quad (4)
$$

Therefore, Eq. (3) can be rewritten as

$$
\left[ i\omega + U(z) \frac{\partial}{\partial x} \right] \tilde{p} = \frac{1}{\rho} \frac{\partial^2}{\partial x^2} \rho \frac{\partial \tilde{p}}{\partial z} = 0. \quad (5)
$$

Now, we can transform $(x,y)$-space into $(\alpha, \beta)$-space by using a double Fourier transform expressed by

$$
\tilde{p}(x,y,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{p}(x,y,z) e^{-i(\alpha x + \beta y)} d\alpha d\beta. \quad (6)
$$

As is well known, the inverse transform of Eq. (6) is given by

$$
\tilde{p}(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{p}(\alpha, \beta, z) e^{i(\alpha x + \beta y)} d\alpha d\beta. \quad (7)
$$

Applying the double Fourier transform to both sides of Eq. (5), the following ordinary differential equation is obtained.

$$
\frac{\partial^2 \tilde{p}}{\partial x^2} - \frac{\alpha}{U(z)} \frac{\partial \tilde{p}}{\partial x} + \frac{\partial^2 \tilde{p}}{\partial z^2} = 0, \quad (8)
$$

where $R^2 = \omega^2 + \beta^2$. Eq. (8) is a linear differential equation of the second order. Here, we confine our attention to the weak shear case, in which the velocity distribution is given by

$$
U(z) = U_\infty (1 + \epsilon f(z)) \quad (9)
$$

where $\epsilon$ is a small quantity of the first order. In this case, $\tilde{p}$ may be expanded into a power series of $\epsilon$ as follows:

$$
\tilde{p} = \tilde{p}_0 + \epsilon \tilde{p}_1 + \epsilon^2 \tilde{p}_2 + \cdots, \quad (10)
$$

where the $\tilde{p}_i(z=0,1,2,\ldots)$ are functions of $z, \alpha$ and $\beta$. Substituting Eqs. (9) and (10) into Eq. (8), successive differential equations can be obtained by collecting the same order coefficients of $\epsilon$ as follows:

$$
\frac{d^2 \tilde{p}_0}{dz^2} - R^2 \tilde{p}_0 = 0 \quad : O(1) \quad (11)
$$

$$
\frac{d^2 \tilde{p}_1}{dz^2} - R^2 \tilde{p}_1 = \frac{2U_\infty \alpha f'}{\omega + U_\infty} \tilde{p}_0 \quad : O(\epsilon) \quad (12)
$$

$$
\cdots \cdots
$$

where the following approximation is made.

$$
\frac{2U_\infty \alpha f'}{\omega + U_\infty} \left( 1 - \frac{U_\infty}{\omega + U_\infty} \right) \tilde{p}_0 \quad : O(\epsilon^2). \quad (13)
$$

It is clear that Eq. (11) expresses the potential flow case, and, therefore, in order to solve Eqs. (11) and (12), we can use the method of successive approximations; i.e. (i) solve Eq. (11), first, (ii) solve Eq. (12) by substituting $\tilde{p}_0$ into the right hand side.

III. Derivation of the Integral Equation and Method of Solution

Before deriving the integral equation for the lift distribution, it is convenient to obtain the relation between $\tilde{p}$ and $\tilde{\Phi}$. The Fourier transform is applied to both sides of Eq. (11).

$$
i(\omega + U(z)\beta) \tilde{\Phi} + \frac{\partial \tilde{\Phi}}{\partial x} = 0 \quad (14)
$$

Concerning the integral equations. Substituting Eqs. (9), (10) and $\tilde{\Phi}$, expanded as

$$
\tilde{\Phi} = \tilde{\Phi}_0 + \epsilon \tilde{\Phi}_1 + \epsilon^2 \tilde{\Phi}_2 + \cdots, \quad (15)
$$

into Eq. (13), the following successive equations are obtained in a manner similar to that of Eqs. (11) and (12).

$$
i(\omega + U_\infty \beta) \tilde{\Phi}_0 + \frac{\partial \tilde{\Phi}_0}{\partial x} = 0 \quad : O(1) \quad (16)
$$

$$
i(\omega + U_\infty \beta) \tilde{\Phi}_1 + \frac{\partial \tilde{\Phi}_1}{\partial x} = 0 \quad : O(\epsilon) \quad (17)
$$

At this point, the boundary conditions can be expressed in $(\alpha, \beta)$-space as follows:

(i) $\tilde{\Phi} \to 0$ as $\alpha \to \pm \infty$, i.e. any disturbance should disappear at plus and minus infinity.

(ii) $\tilde{\Phi} (0, \beta, \pm 0) = \tilde{\Phi} (0, \beta, 0)$ (this is the condition of tangential flow.)

First, let us consider the $O(\epsilon)$-case, Eqs. (12). Eq. (12) can be easily solved as

$$
\tilde{p}_1(\alpha, \beta, \tau) = - C_1 e^{-\rho_1 \tau} + C_0 e^{-\rho_2 \tau} \int_{-\infty}^{\infty} \tilde{\Phi}(\beta, \tau) \sin \rho_1 \tau d\tau \quad (18)
$$

$$
C_0 = \frac{2U_\infty \alpha f'}{\omega + U_\infty} \quad (18)
$$

$$
C_1 = \frac{2U_\infty \alpha f'}{\omega + U_\infty} \quad (18)
$$

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Several kinds of methods have been presented in order to solve the integral equation in lifting surface theory. Among these methods the 'mode function method' is applied in this paper. The detail of this method can be found in Ref. 2. The points different from the usual method are explained in Appendix B.

IV. Examples of Numerical Calculations and Discussion

Numerical calculations are performed by the collocation method. The wing surface is divided into many small panels as follows:

\[ x_{P_{ij}} = x_{i} (\gamma_{j}) + \frac{\phi (\gamma_{j})}{2} (1 - \cos \phi) \]  \( p = 1, \ldots, N \)

\[ \gamma_{j} = \frac{\varphi_{j}}{\varphi} \]  \( \varphi = 1, \ldots, m \)

where

\[ \frac{\varphi}{\varphi} = 2 \pi / 2N + 1 \]  \( \varphi = m + 1 \)

Then, the integral equations are transformed into a set of linear equations as follows:

\[ \begin{align*}
\frac{\omega}{\omega}(p, \gamma) &= \frac{N}{q=1} \int f(q, \gamma) q_{\rho} \frac{\omega}{\omega}(p, \gamma, \gamma_{q}) : \omega(1) \\
\frac{\omega}{\omega}(p, \gamma) &= \frac{N}{q=1} \int f(q, \gamma) q_{\rho} \frac{\omega}{\omega}(p, \gamma, \gamma_{q}) - S_{1} (p, \gamma) - S_{2} (p, \gamma) \\
\frac{\omega}{\omega}(p, \gamma, \gamma_{q}) &= \frac{N}{q=1} \int f(q, \gamma) q_{\rho} \frac{\omega}{\omega}(p, \gamma, \gamma_{q}) \end{align*} \]

where

\[ \frac{\omega}{\omega}(p, \gamma, \gamma_{q}) = \frac{N}{q=1} \int f(q, \gamma) q_{\rho} \frac{\omega}{\omega}(p, \gamma, \gamma_{q}) \]

In the above equations, \( \frac{\omega}{\omega} \), \( \frac{\omega}{\omega} \), \( \frac{\omega}{\omega} \), \( \frac{\omega}{\omega} \), \( \frac{\omega}{\omega} \), \( \frac{\omega}{\omega} \), and \( \frac{\omega}{\omega} \) take the same forms as those in Ref. 2. Therefore, if the upwash velocities \( \omega_{1} \) and \( \omega_{2} \) are given on collocation points, the simultaneous equations Eqs. (27) and (26) can be solved, and the lift distributions can be calculated. In order to obtain the unsteady response in the shear flow, it is effective to calculate the generalized force which is defined by

\[ Q_{i,j} = \frac{1}{2} \frac{\partial}{\partial x} \int x_{i,j} d(\omega_{1}) \]

where \( i \) denotes force mode, and \( j \) oscillation mode.

In this paper, rectangular wings with zero thickness are adopted as examples of numerical calculations, where the aspect ratios are 3 and 6. The number of collocation points is \( \frac{N}{\gamma} = 3 \) in the chordwise direction, and \( m = 11 \) in the spanwise direction. Some oscillation modes are given subsequently. Since a simple harmonic motion is assumed, the heaving oscillation \( 1 = 1 \), and the pitching oscillation \( 2 = 2 \) are used, i.e.

\[ \frac{\omega}{\omega} = -h, \; \frac{\omega}{\omega} = -a_{0}x \]

Both heaving amplitude \( h \) and pitching amplitude \( a_{0} \) are taken as unity in calculations. Therefore, upwash velocities are given by

\[ \frac{\omega}{\omega}(x, y) = e^{i \omega x} \frac{\partial}{\partial x} \left( \frac{\omega}{\omega} \right) + e^{i \omega x} \frac{\partial}{\partial y} \left( \frac{\omega}{\omega} \right) \]
\[ \omega(x, y) = e^{i \omega x} \cdot \left( \frac{\partial w}{\partial y} \right) + e^{-i \omega x} \cdot \left( -1 + \frac{\partial w}{\partial y} \right), \]

and force modes are given by
\[ n_{1} = -d, \quad n_{2} = -2. \]

where \( d \) is the half of the lift coefficient, and \( -2 \) a half of the nose-down pitching moment coefficient about the leading edge. Finally, the following velocity profile \( f(x) \) is assumed as
\[ f(x) = - \exp \left( - \frac{\alpha + \beta}{\pi} x \right)^{2}, \]

where \( \alpha \) is the wing position, and \( \beta \) is the degree of spread of the shear flow. This profile simulates the wake produced by an object ahead of the wing. Generalized force given by Eq. (29) are obtained for the first order approximation at

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**Fig. 3** Stiffness and Damping Derivatives against Frequency Parameter in Heaving Mode

**Fig. 4** Stiffness and Damping Derivatives against Frequency Parameter in Pitching Mode

**Fig. 5** Stiffness and Damping Derivatives against Wing Position in Heaving Mode

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\( \epsilon = 0.1 \), i.e. the maximum loss of the approaching flow velocity is 10\% of the reference velocity \( U_0 \). Complex generalized forces in the unsteady state can be rewritten as

\[
Q_{ij} = Q'_{ij} + \zeta \omega Q''_{ij} \tag{34}
\]

where \( Q' \) and \( Q'' \) are the stiffness and damping derivatives respectively (2). The former is related to the vertical and angular position of the cases \( \xi = 1 \) and \( \xi = 2 \) respectively.

In Figs. 3 and 4, these derivatives in two oscillatory modes are shown for the wing with aspect ratio 6. The broken line shows the potential flow results \((O(1)-case)\). The parameters of the shear flow are taken as \( n = 1.0 \) and \( z_0 = 0.5 \). It is clear from these figures that the loss of flux in the shear flow decreases all force coefficients except the values for \( \zeta = 1.0 \) in the heaving mode. The difference between results of the potential flow and those of the shear flow is not so affected by \( \zeta \) in the range of these calculations. As described above, subscripts \((1,1)\) and \((2,1)\) are related to a half of the lift coefficient and nose-down moment coefficient about the leading edge in the heaving respectively, and \((1,2)\) and \((2,2)\) are those in the pitching.

In Figs. 5 and 6, generalized forces are plotted against \( z_0 \) for three values of \( n \). Since they are symmetric with respect to \( z_0 = 0 \), and have a peak at \( z_0 = 0 \), the results of the positive region are illustrated. These tendencies can be predicted by the form of \( \phi_2(x,y) \) given in Eq. (24), because it includes the second order derivative \( \phi''(z_0) \), which is symmetric with respect to \( z_0 \). Although the shear flow decreases all force coefficients near the center of the wake, these coefficients take the same values at the center of the wake except for \( \zeta_1 \) and \( \zeta_2 \), and they gradually approach the potential flow values as \( z_0 \) becomes large. The larger \( \zeta \) becomes, which means the wake extends in the \( z_0 \)-plane with the constant maximum loss \( \epsilon \), the wider the influence of the shear flow becomes. It seems that not the total loss of flux but the velocity gradient with respect to \( z \) becomes dominant at small \( \zeta \).

\( S_2(x,y) \) is more affected by \( \zeta \) than by any other parameter. In Figs. 7 and 8, the spanwise aerodynamic force distributions are shown for aspect ratio 3 and 6 in the case of \( \nu = 0.5 \). \( n = 1.0 \) and \( z_0 = 0 \). By these figures, results already established in the potential flow case are obtained.

Fig. 7 Spanwise Lift Distribution in Heaving Mode

Fig. 8 Spanwise Pitching Moment Distribution in Pitching Mode
The response of oscillating wings in the weak shear flow are obtained by unsteady lifting surface theory. The flow is assumed to be inviscid and incompressible, but has no potential. The partial differential equation for the pressure is transformed into an ordinary differential equation by the double Fourier transform. In order to simplify the problem, a method of successive approximation is used, i.e., all variables are expanded into power series of \( \epsilon \), which is the degree of strength of the shear flow. Moreover, it is formulated in arbitrary modes of oscillation of wings.

Numerical calculations are performed for the velocity profile which simulates the wake caused by an obstacle ahead of the wing. Generalized forces are obtained in heaving and pitching oscillatory modes of rectangular wings with aspect ratios 3 and 6. Lift coefficients and pitching moment coefficients for each mode can be easily calculated by the generalized forces. The loss of flux in the approaching flow decreases the amplitude of all force coefficients. Unfortunately, no experimental results have been found. However, these results seem to agree well with those described in the two-dimensional steady flow case.

References


Appendix A

The double Fourier transform and its inverse transform are expressed as

\[ F[\epsilon] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\epsilon(x+iy)} dx dy \]  

\[ \text{(A.1)} \]

\[ F^{-1}[\epsilon] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\epsilon(x+iy)} dx dy \]  

\[ \text{(A.2)} \]

respectively. Therefore, inverse transform of \( \sigma' \) and \( R/\pi \cdot 2(\omega' \delta) \) can be obtained as follows:

\[ F^{-1}[\sigma' \delta?(\omega')] = 2 \delta(\omega) \left\{ \sigma(x) - \frac{\omega}{2} \int_{-\infty}^{\infty} \frac{e^{i\omega \delta}}{\omega^2 + \epsilon^2} \right\} \]  

\[ \text{(A.3)} \]

\[ F^{-1}[\epsilon^{-2\pi \delta}] = \frac{1}{\pi} \left\{ \epsilon \cos \frac{\pi \epsilon}{2\pi} \right\} \]  

\[ \text{(A.4)} \]

\[ F^{-1}[\epsilon^{-2\pi \delta}] = \frac{1}{\pi^2} \left\{ \epsilon \cos \frac{\pi \epsilon}{2\pi} \right\} \]  

\[ \text{(A.5)} \]

The right hand side of Eq.(A.5) is the kernel function of the integral equation for the lift distribution in potential flow.

Next, the convolution formula of the inverse transform is given by

\[ F^{-1}[\xi \cdot \delta'] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-\epsilon, y-\epsilon') \delta(x', y') dx' dy' \]  

\[ \text{(A.6)} \]

Using Eqs.(A.3), (A.4), (A.5) and (A.6), following results can be obtained.

\[ F^{-1}[\sigma' \delta?(\omega')] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x', y') dx' dy' \]  

\[ \text{(A.7)} \]

\[ F^{-1}[\epsilon^{-2\pi \delta}] = \frac{1}{\pi} \left\{ \epsilon \cos \frac{\pi \epsilon}{2\pi} \right\} \]  

\[ \text{(A.8)} \]

Appendix B

Let \( \gamma \) be expanded into following series.

\[ \gamma(\delta', \epsilon') = \sum_{n=1}^{N} \epsilon^{n} \gamma(\delta', \epsilon') \]  

\[ \text{(B.1)} \]

where

\[ \gamma(\delta', \epsilon') = \cos(\epsilon') \delta' + \cos \phi \epsilon' \]  

\[ \text{(B.2)} \]
Substituting Eq. (24) into Eq. (22), the integral equation is rewritten as

\[
\frac{\zeta_1(x, \eta)}{u_{\infty}} + f(0) \frac{\zeta_3(x, \eta)}{u_{\infty}} + S_1(x, \eta) + S_2(x, \eta)
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} \int_{-\infty}^{\infty} f(x, \eta) f(x', \eta') d\eta' \tag{B.3}
\]

where

\[
S_1(x, \eta) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.4.4}
\]

\[
S_2(x, \eta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.4.5}
\]

In order to calculate the right hand side of Eq. (B.5), it is convenient to divide into four parts as follows:

\[
T(x, \eta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{f(x, \eta')}{1 + (\eta')^2} d\eta' \tag{B.6}
\]

where

\[
T_1(x, \eta, \eta') = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.7.1}
\]

\[
T_2(x, \eta, \eta') = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.7.2}
\]

\[
T_3(x, \eta, \eta') = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.7.3}
\]

\[
T_4(x, \eta, \eta') = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.7.4}
\]

and

\[
F(x, \eta, \eta') = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.7.5}
\]

Accordingly, \(S_2\) is rewritten as

\[
S_2(x, \eta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.8.1}
\]

where

\[
S_2(x, \eta, \eta') = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.8.2}
\]

Logarithmic singularity included in \(S_1'\) is removed by

\[
S_1'(x, \eta, \eta') = S_1(x, \eta, \eta') - c_1(x, \eta) \log |\eta - \eta'| \tag{B.11.1}
\]

\[
c_1(x, \eta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.11.2}
\]

\[
f_2(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.11.3}
\]

\[
f_2(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.11.4}
\]

Finally, \(S_2\) is written as

\[
S_2(x, \eta) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.12.1}
\]

\[
S_1'(x, \eta, \eta') = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.12.2}
\]

\[
S_1'(x, \eta, \eta') = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{1 + (\eta')^2} f(x, \eta') f(x', \eta') d\eta' \tag{B.12.3}
\]