Abstract

Based on the kinematics of disturbance propagation from moving singularities the influence functions arising from the motion are derived for field points in an unbounded homogeneous medium. For generality, volume elements and surface elements of singularities are considered having arbitrary orientations in space and to the trajectory. Furthermore different translatory motions of the singularity elements, the field points and of the medium are admitted. The spatial and temporal influence functions show some universal relations and characteristic properties in the radiation field. Hence, for calculating the disturbance fields of moving bodies having subsonic or supersonic velocities the method can be applied directly both in the steady and unsteady cases. The solution of the field equation is obtained in the usual way by resorting to integral methods and fulfilling the kinematic boundary conditions on the actual body surface, the surface being subdivided into panel elements. The method then follows the same line as the classical panel method. For Mach number tending to zero all the expressions reduce exactly to the classical expressions for incompressible flow.

Notations

Geometric Quantities

- \( P_0 \): control surface or singularity surface in the disturbance field
- \( h_S \): radial distance of a singularity from a given trajectory
- \( h_O; h_v \): radial distance of a field point from the trajectory of a singularity (\( h_O = h_v \))
- \( H \): height of a field point normal to a panel surface
- \( l_o, l_v \): length of a singularity element and of the corresponding emission segment
- \( l_n \): lengths along panel boundary
- \( P, P_O, P_v \): location of field point, singularity and corresponding emission point
- \( r \): radial distance in spherical polar coordinate
- \( r_O \): radial distance between singularity and field point at the time instant to
- \( r^* \): radiation radius or emission radius of a spherical wave

Aerodynamic Quantities

- \( a \): local sound velocity
- \( a_w \): sound velocity in a homogeneous medium at rest
- \( D \): doublet strength
- \( F_1 \): reduced disturbance force per unit volume
- \( C_{ij} \): reduced disturbance force due to momentum exchange per unit volume
- \( I \): inducing function comprising resultant influence function at a field point
- \( I_{ij} \): inducing function of the ith panel on a field point at panel j
- \( k \): wave number (\( \omega a_w \))
- \( M_S \): Mach number of the singularity
The propagation of disturbances from space fixed or moving singularities is described by the wave equation. The classical wave equation was first formulated by J.L. d'Alembert [1] for treating the one-dimensional case of string vibrations. Thereafter the wave equation was applied extensively to various fields concerning propagation and vibration problems. The solution of the wave equation for spherical radiation of sound waves was first given by S.D. Poisson [2]. Following this result one can derive the two-dimensional solution of cylindrical wave motions, as was shown by T. Levi Civita [3], H. Lamb [4] and J. Hadamard [5]. The most general solutions of wave propagations from spatial distributions of singularities have been given by A. Cauchy [6], H.V. Helmholtz [7] and G. Kirchhoff [8], which are very useful for extensive application in the field of acoustics and aerodynamics.

Propagation of waves from moving singularities was first investigated in the field of electromagnetic radiation and propagation of light as is well known from the contributions of C. Doppler [9]. The actual mathematical theory on this topic was established later on by W. Voigt [10], H. Lorentz [11] and H. Poincaré [12]. Some lucid expositions of the physical phenomena due to wave radiation from moving sources could be given after the theory of relativity was postulated in the contributions of A. Einstein [13], H. Minkowski [14] and some corresponding works.

The perturbation fields of moving singularities in aerodynamics were first formulated when the effect of Mach number or the concept of compressible flows were introduced. Thus, steady flows were treated by O. Janzen [15], Lord Rayleigh [16], H. Glauert [17], L. Prandtl [18], J. Ackermann [19], Th. v. Kármán and N.B. Moore [20], while unsteady flows were analyzed by H. Küssner [21], C. Possio [22], and I.E. Garrick [23]. The treatment of wave propagation from moving singularities in the field of aeroacoustics was initiated through the contributions of H. Hönl [24], H. Küssner [25], N. Rott [26], H. Billing [27], H.L. Oestreicher [28], M.J. Lighthill [29] and I.E. Garrick [30].
For disturbance propagation from moving sources the linearized wave equation in the moving reference frame is equivalent to the linearized field equation of unsteady aerodynamics. The usual solution procedure for these equations, as is commonly followed, is the application of integral methods using integral transforms or Green's theorem with a suitable basic function. In both these methods the field equation is usually converted to the classical form by resorting to some mathematical transformations analogous to the Prandtl-Glaucert transformation, or using Lorentz-transformation.

In the integral methods for steady or unsteady flow fields the solution procedure involves the use of aerodynamic inducing functions, depending on the nature of the singularities and their locations relative to the field points. With the evolution of the computational fluid dynamics a very flexible and well suited method has been extensively developed and is being classified as panel method. The subsonic panel method in its initial form is developed for incompressible flow as described by J.L. Hess and A.M.O. Smith [31], P.A. Woodward [32], P.E. Rubbert and G.R. Saaris [33], Th.E. Labrujere, W. Loeve and J.W. Sloof [34]. Further extensions and applications of this methods are given in the contributions of W. Kraus [35], S.R. Ahmed [36], P.A. Woodward [37] and J.L. Hess [38]. The treatment of the problem for unsteady subsonic flows were developed by E. Albano and W.P. Rodden [39], W.P. Jones and J.A. Moore [40] and W. Geißler [41], while for steady and unsteady supersonic flows the method was worked out by D.L. Woodcock [42] and W.P. Jones and K. Appa [43]. A unified treatment for steady and unsteady subsonic and supersonic flow fields has been formulated by L. Maffino, L.T. Chen and E.O. Sucio [44].

Based on the kinematics of wave propagation as described by N. Rott [26] the influence functions arising from the motion of the disturbance sources have been derived in detail in two papers by A. Das [45] [46]. For complete generality the volume elements and surface elements of the moving sources are allowed to have arbitrary orientations in space and with regard to the trajectory.

In the present paper the treatment of flow fields of moving bodies is described, where the inducing functions arising from the singularity panels of the body surface are derived directly to fulfill the kinematic boundary conditions at their pivotal points. The treatment follows similar lines as in the panel methods for incompressible flows, but without resorting to the Prandtl-Glaucert-transformation to account for the Mach number effect. Some useful reference text books are cited in [47] to [52].

II. Basic Equations for Disturbance Propagation fromingularities in Motion

The basic field equation of disturbance propagation from space fixed or moving singularities in an unbounded homogeneous medium is derived from the laws of conservation of mass, momentum and energy. If friction and heat conduction is neglected the field of small disturbance can be assumed isentropic, thus enabling the introduction of a perturbation potential. The field equation in its most general formulation appears as a wave equation and is equivalent to the classical potential equation of unsteady aerodynamics. The equation reduces to a simpler expression if the negative x-axis is made to coincide with the trajectory and the assumption of small disturbances is introduced.

2.1 The linearized field equation and the perturbation quantities

The linearized wave equation in a moving reference frame can be described as follows:

$$\nabla^2\phi - \frac{1}{a_w^2} \frac{D\phi}{Dt^2} = \eta \delta(t_o-t - \frac{r_v}{a_w}) \delta(x_v-x_o-Ma_S r_v)$$

(2.1)

with

$$\frac{D\phi}{Dt^2} = \left[ \frac{3}{9t} + \frac{3}{S \frac{3}{9x_o}} \right]^2$$

(2.2)

$\nabla^2$ as Laplace operator and $\delta$ as Dirac delta functions, defining the location of the moving disturbance source $\eta$ on the trajectory at a corresponding time. The disturbance function is given in the most general form

$$\eta = Q(t) + (v \cdot F_\perp)(t) + (v \cdot v \cdot G_\perp)(t)$$

(2.3)

where the three terms in their respective order are defined as source, dipol, and quadrapole singularities, with $Q$ as the reduced mass flux and $F_\perp$ and $G_\perp$ as the reduced force functions per unit volume of the medium.

It is often advantageous to express eq. (2.1) in a space fixed coordinate system with the medium at rest, containing the complete propagating wave system in it as shown in Fig.1. This is achieved by the Galilean transformation

$$\hat{x} = x - v \cdot t; \quad \hat{y} = y; \quad \hat{z} = z$$

(2.4)

Eq.(2.1) then reduces to the classical wave equation

$$\nabla^2\phi - \frac{1}{a_w^2} \frac{D\phi}{Dt^2} = \eta \delta(t_o-t - \frac{r_v}{a_w}) \delta(x_v-x_o-Ma_S r_v)$$

(2.5)
Although the left hand side of eq. (2.5) is identical to the wave equation of a space fixed singularity in a medium at rest, the kinematics of radiation from moving singularities are still retained in the delta functions, through which the propagation field is essentially modified. The solution of eq. (2.1) or eq. (2.5) yields the perturbation potential from which the other field quantities can easily be derived.

Fig. 1 Wave propagation from a singularity in uniform motion presented in a moving and a medium-fixed reference system interrelated by the Galilean transformation.

The pressure and density perturbations in the disturbance field are derived from the generalized Bernoulli equation yielding in the moving reference system

\[ \frac{s}{\rho_\infty} = \frac{\Delta p}{\rho_\infty} = \frac{\Delta p}{\kappa p_\infty} = -\frac{1}{a_\infty^2} \left[ \frac{3 \varphi}{3 t} + \nu_0 \frac{3 \varphi}{3 x} + \frac{(\varphi \varphi)_2}{2} \right] \] (2.6a)

If a medium fixed reference system is chosen, eq. (2.6) simplifies to:

\[ \frac{s}{\rho_\infty} = \frac{\Delta p}{\rho_\infty} = \frac{\Delta p}{\kappa p_\infty} = -\frac{1}{a_\infty^2} \left[ \frac{3 \varphi}{3 t} + \frac{(\varphi \varphi)_2}{2} \right] \] (2.6b)

Due to the linearization of the propagation problem the perturbation quantity \( s \) also fulfills the wave equation with a new reduced disturbance function \( u^* \).

2.2 The kinematics of disturbance propagation for singularities in motion

Let \( P \) be the momentary position of the singularity \( \Phi \) moving with a constant velocity \( V_\infty \), while a disturbance signal propagating with a velocity \( a_\infty \) meets the field point \( P \) at a time instant \( t_0 \), then the location of the emission point \( P \) on the trajectory and the retarded time \( t_\nu \) of signal emission can be determined from elementary kinematic relations.

The momentum relative orientation between \( P(t_\nu) \) and \( P(t_0) \) being uniquely defined in space through the quantities \( r_\nu \) and \( \varphi_\nu \) as depicted in Fig. 2, the radiation quantities \( r_\nu, \varphi_\nu \) and \( t_\nu \) are obtained from the following expressions:

\[ (1-M_\nu^2)r_\nu^2 - 2r_\nu r_0 M_\nu \cos \varphi_\nu - r_0^2 = 0 \]

\[ r_\nu \cos \varphi_\nu - M_\nu r_\nu - r_0 \cos \varphi_\nu = 0 \]

\[ r_\nu = a_\infty(t_0 - t_\nu) \] (2.7)

The solution of eq. (2.7) yields the following relations:

\[ \frac{r_\nu}{r_0} = \frac{1}{1 - M_\nu^2} \left\{ \frac{M_\nu \cos \varphi_\nu + (-1)^{n+1} \sqrt{1 - M_\nu^2 \sin^2 \varphi_\nu}}{M_\nu \cos \varphi_\nu + (-1)^{n+1} \sqrt{1 - M_\nu^2 \sin^2 \varphi_\nu}} \right\} \] (2.8)

\[ \cos \varphi_\nu = \frac{M_\nu \cos \varphi_\nu + (-1)^{n+1} \sqrt{1 - M_\nu^2 \sin^2 \varphi_\nu}}{M_\nu \cos \varphi_\nu + (-1)^{n+1} \sqrt{1 - M_\nu^2 \sin^2 \varphi_\nu}} \] (2.9)

\[ t_\nu = t_0 - \frac{r_\nu}{a_\infty} \] (2.10)

\[ x_\nu = x_0 + M_\nu r_\nu \] (2.11)
with
\[ n = 1 \quad \text{for} \quad M_a < 1 \]
\[ n = 1/2 \quad \text{for} \quad M_a > 1 \]

Further relations connecting \( r_0 \), \( \phi_0 \) to \( r_v \) and \( \phi_v \) are:
\[ r_{vn}(\cos \phi_{vn} - M_a) = r_0 \cos \phi_0 \]
\[ r_{vn} \sin \phi_{vn} = r_0 \sin \phi_0 \]
\[ r_{vn}(1 - M_a \cos \phi_{vn}) = (-1)^{n+1} r_0 (1 - M_a^2 \sin^2 \phi_{vn})^{1/2} \]
\[ \sin c = M_a \sin \phi_0 \]  
\( \quad (2.12) \)

Using these relations the spatial influence functions arising from the motion of the singularities are derived below.

2.3. The spatial and temporal influence functions arising from the motion of the singularities

The perturbation quantities at a field point \( P \) in space at the time instant \( t \) are determined by the singularity strengths at the time of emission \( t_v \), the radiation distance \( r_v \) and the influence functions arising from the motion of the singularities. The kinematic relations established in section 2.2 show that the emission position may be such that the surface and volume elements of the singularities undergo stretching and shifting effects and the time sequences of signal emission and signal reception are influenced essentially by the Doppler effect. In the disturbance field of moving singularities one encounters four distinct influence functions.

The spatial influence functions
\[ \sigma_S = \frac{d_{1_v}}{d_{1_0}} \quad \text{Influence factor due to an effective stretching of the emitting surface- or volume-elements} \]
\[ \sigma_R = \frac{dr_v}{dr_{vo}} \quad \text{Influence shift of the emitting source} \]
\[ \sigma_{vo} = \frac{dr_v}{dr_{vo}} \quad \text{sink combinations} \]
\[ [dr_{vo} - dr_v (M_a - 0)] \]

The temporal influence functions
\[ \sigma_D = \frac{dt_v}{dt_{vo}} \quad \text{Doppler factor for signals passing through a medium fixed field point being emitted from a moving source.} \]
\[ \sigma_{D} = \frac{dt_v}{dt_{vo}} \quad \text{Doppler factor for signals passing through a moving field point being emitted from a moving source.} \]

The effective stretching factor \( \sigma \)

For a source distribution on an elementary segment \( d_1 \) at \( P \) being arbitrarily orientated to the trajectory, the derivation of the emission positions for all the signals reaching the field point simultaneously at a time instant \( t \) yields the effective emission length \( d_1 \) at the location \( P_v \). This has been derived in detail in [46]. Based on the location of the endpoints of the emission element, its effective stretching is given by
\[ \sigma_S = \frac{d_{1_v}}{d_{1_0}} = \frac{d_{1_v} \cos \psi_v}{d_{1_0}} + \frac{d_{h_{2v}}}{d_{1_0}} \sin \psi_v \]  
\[ \quad (2.15) \]

Using the partial derivatives already compiled in the eqs. (2.13) and (2.14) one obtains the exact relations:
\[ \sigma_S = 1 + \frac{M_a \cos \phi_v}{1 - M_a \cos \phi_v} \]
\[ \sigma_S = 1 \]  
\[ \quad (2.16) \]
or alternatively

\[ \sigma_S = \frac{1}{1 - M_a \cos \phi_v} \]  

(2.17)

where \( M_a = M_{a_v} \cos \chi_S \) and \( M_{a_v} = M_a / \cos(\chi_S - \chi_v) \) denote the Mach number components along \( dl_v \) and \( dl_v^* \). For \( \chi_S = 0 \) the source elements lie parallel to the trajectory and hence

\[ \sigma_S = \frac{dl_v}{dl_o} \approx \frac{dx_v}{dx_o} = \frac{1}{1 - M_a \cos \phi_v} \]  

(2.18)

\[ \zeta \text{ for } \chi_S = 0 \]

Fig. 3 The spatial influence factors \( \sigma_S \) and \( \sigma_R \) due to the effective stretching and shifting of the emission elements for singularities in motion.

For source elements placed normal to the trajectory one obtains

\[ \sigma_S = \frac{dl_v}{dl_o} \approx \frac{dh_v}{dh_o} = 1 \text{ for } \chi_S = \frac{\pi}{2} \]  

(2.19)

For the arbitrary sweep angle \( \chi \) of a two-dimensional source line, where the effect of the tangential Mach number \( M_{a_v} \) with \( \phi \) as emission angle to the field point. Therefore eq. (2.16) yields:

\[ \sigma_S = \frac{dl_v}{dl_o} = \frac{1}{1 - M_e \cos \phi_v} \]  

(2.20)

This is in complete agreement with the classical concept of sweep effects. The stretched emission length \( dl_o \) of a moving element having an arbitrary orientation to the trajectory is illustrated in Fig. 3.

In the formal solution of disturbance fields commonly given in the literature, the stretching factor \( \sigma_S \) is incorporated in the emission radius in the following way

\[ r_v^* = \frac{r_v}{\sigma_S} = r_v \left( 1 - M_e \cos \phi_v \right) \]  

(2.21)

For singularity elements lying in the plane of the trajectory i.e. for \( \chi_S = 0 \) one obtains

\[ r_v^* = r_v \left( 1 - M_a \cos \phi_v \right) \]  

(2.22)

Using an equivalent rule derived from the kinematic relations this yields:

\[ r_v^* = r_v \left( 1 - M_a^2 \sin^2 \phi_v \right)^{1/2} \]  

(2.23)

or

\[ r_v^* = \left[ (x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2 \right]^{1/2} \]  

(2.24)

In the real physical process of the radiation field, \( \sigma_S \) is a stretching factor for the surface of volume of the emitting elements arising solely from the kinematic effect, and not from the compressibility effect as is commonly assumed.

The effective shifting factor \( \sigma_R \)

If a source-sink combination is in motion, the singularity elements are displaced from the trajectory of their centroid. Thus, their relative orientation with respect to the field point differ from each other. Hence for the emitted signals reaching the field point at a time \( t \), the corresponding emitting points, as derived from the kinematic relations, undergo relative shifts compared to their original displacement from the centroid. If \( dm \) is the displacement in space of the source-sink elements at \( P \), the difference in their emission radius for the signals reaching the field point \( P \) at a time instant \( t \), can be derived easily by using the partial derivatives compiled in eq. (2.13) and eq. (2.14) and following the same line as in the previous section. This yields

\[ \frac{dr_v}{dm} = \frac{\cos \phi_v}{1 - M_e \cos \phi_v} \]  

(2.25)

where \( \phi \) is the angle between the emission axis \( \chi \) and the source-sink axis \( dm \) at the emission point \( P \). The shifting effect of the source-sink combination is clearly displayed in Fig. 3. The prescribed source-sink arrangement placed at \( P \), would have the following difference in emission radius
\[
\frac{dr}{\frac{dr}{d\phi}} = \cos \phi \\
\frac{dr}{d\phi} \bigg|_{M_a = 0} = \cos \phi \\
(2.26)
\]

Hence the shifting factor \( \sigma \) due to the motion of the singularity amounts to:

\[
\sigma = \frac{dr}{d\phi} \bigg|_{M_a = 0} = \frac{1}{1 - M_a \cos \phi} \\
(2.27)
\]

This is a purely kinematic effect and is formally accounted for in the classical treatment where partial derivatives of the equivalent expression for \( \sigma_* \) as given in eq.(2.24) is carried out for the singularity locations at \( P_0 \).

The Doppler-Factor \( \sigma_D \) and \( \sigma_D^* \)

When signals are emitted from moving singularities the time sequence for their passing through a field point in space usually differs from the time sequence of emission of the signals. The ratio of the time sequences \( dt_0 \) of emission and \( dt \) of reception can easily be derived from the partial derivative of eq.(2.10) with respect to \( dt_0 \), and is generally valid for a fixed or moving field point. As already derived in detail in [45] one obtains for a medium fixed field point:

\[
\sigma_D = \frac{dt}{dt_0} \bigg|_{M_a = 0} = \frac{1}{1 - M_a \cos \phi} \\
(2.28)
\]

For a moving field point, with its velocity component being \( M_a \) in the \( P_0 P_1 \)-plane, the time derivative of eq.(2.10) yields the expression

\[
\sigma_D^* = \frac{dt}{dt_0} = \frac{1}{1 - M_a^2} + \frac{M_a M_a E \cos(\phi_0 + \phi)}{1 - M_a^2} + \\
\frac{M_a \cos(\phi_0 - \phi) + M_a E \cos(\phi_0 + \phi)}{(1 - M_a^2)(1 - M_a \cos \phi)} - \\
\frac{M_a \sin(\phi_0 - \phi) \sin(\phi_0 + \phi)}{(1 - M_a^2)(1 - M_a \cos \phi)} \\
(2.29)
\]

The physical process involving the influence factors \( \sigma \) and \( \sigma^* \) is depicted in Fig.4. So for the frequency \( f \) at reception and emission the simple relation holds:

\[
\omega_F = \omega_S \cdot \sigma_D^* \\
(2.30)
\]

For \( M_a = 0 \) eq.(2.29) reduces to \( \sigma = \sigma_D \).

For field points accompanying the moving reference system eq.(2.29) yields \( \sigma_D^* = 1 \), for which \( \omega_F = \omega_S \).

![Fig.4](image)

**Fig.4** The temporal influence factors \( \sigma \) and \( \sigma_D \) due to the changed time sequence of signals passing through a fixed or moving field point in relation to the time sequence of the emitted signals from singularities in motion.

III. The Inducing Functions and Perturbation Quantities in the Radiation Field of Two- and Three-Dimensional Singularities in Motion.

The inducing functions in the radiation field of space fixed unsteady singularities in an unbounded homogeneous medium at rest are known from the solution of the classical wave equation. For unsteady disturbance sources moving slowly at a low Mach number, one can neglect the effect of the motion and treat the problem as a quasi-steady case accounting for the corresponding relative positions of the sources and the field points at different time instants. With Mach number increasing the spatial and temporal influence functions arising from the motion of the singularities become increasingly significant. The resulting induction at a field point then depends on the Mach number of the sources and the relative positions in space of the emitting elements, whose signals arrive simultaneously at the time instant \( t \). The radiation fields of some basic distributions of moving singularities is illustrated below, first for two and three-dimensional cases. The treatment will then be extended to moving bodies of arbitrary shapes.

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3.1 Radiation process and inducing functions due to moving source lines and source surfaces.

If a source line of infinite length set at an arbitrary angle $\chi$ to the trajectory is in uniform motion with a Mach number $M_a$, subsonic or supersonic, then the emission points of the signals arriving at a field point $P$ simultaneously at a time instant $t_0$, can be determined from eq. (2.11) as shown in Fig. 5. The position of the source line being known at time $t_0$, the following simple guide line relations define the emission lines $L_v$ looked for:

$$\frac{x_v-x_0}{x_v-x} = \frac{r_v}{r_0} = \frac{1}{M_a}$$

(3.1)

This equation represents a hyperbola for $M_a < 1$, a parabola for $M_a = 1$ and an ellipse for $M_a > 1$, with the field point as the focal point of these curves, situated in the plane of the source line. For arbitrary field points in space the emission lines are the envelopes of these basic curves.

![Diagram](image)

Fig. 5 The emission lines $L_v$ for signals arriving at a field point $P(x_0,0,0)$ at the time instant $t_0$, originating from infinite line sources of different sweep angles $\chi$ and Mach numbers $M_a$.

For source lines set at an angle $\chi$ to the trajectory the emission lines $L_v$ are no longer symmetrical about the $x$-axis through the field point, and for $M_a > 1$ the length of the effective source line lying within the Mach fan cones of the field point will be altered essentially. If a $x_1$-axis is chosen passing through $P$ and normal to the source line, then all the emission lines reveal complete symmetry about it. The solution procedure in the classical literature resorts to the use of this property.

In the sonic and supersonic case with $M_a \ge 1$ a similar procedure would violate the physics, as the signals reaching $P$ from the emission lines $L_v$ would then be attributed to source elements lying outside the fore-cone as can be seen from Fig. 5. Thus, the resolution of the motion $M_a$ in normal and tangential components to the source line, as is done in classical treatment is no more than an artificial means to arrive at the correct solution valid only for infinite source lines. For finite source lines with varying source intensity such a procedure will violate extremely the physics of disturbance propagation. The radiation phenomena from moving sources as mentioned in section 2, will give rise to the spatial and temporal influence functions. As the relative positions in space of the source line, of the field point, and of the emission lines are known all the influence factors $\sigma_s$, $\sigma_r$ and $\sigma_d$ are completely defined. According to eqs. (2.20), (2.27) and (2.28)

$$\sigma_s = \frac{1}{1 - M_a \cos \phi_e},$$

$$\sigma_r = \frac{1}{1 - M_a \cos \phi_o},$$

$$\sigma_d = \frac{1}{1 - M_a \cos \phi_d},$$

(3.2)

where $M_a = M_a \cos \chi$, and $\chi = \chi$, the sweep angle ranging from 0 to $\pi/2$. For an infinite source line moving longitudinally but with the source elements placed perpendicularly to the trajectory ($\chi = \pi/2$) their stretching factor reduces to $\sigma_s = 1$ for signals meeting any field point in space. If in contrast, a source line moves in the longitudinal direction but with the source elements aligned to it ($\chi = 0$), as occurring in the axisymmetric case, then the propagation process exhibits some remarkable properties in the region close to the trajectory ($x$-axis). Eqs. (2.18) and (2.22) show that the effective emission radius for signals meeting a field point at $P(x,0,0)$ simply becomes

$$r_v^* = \frac{r_v}{\sigma_s} = r_0$$

(3.3)

with $\phi_e = 0$ or $\pi$. This means that the Mach number dependency of the inducing functions, contained essentially in $r_v$ and $\sigma_s$ as given in eqs. (2.8) and (2.18) drops off exactly in this case. This is the true physical explanation of the slender body effect, as is exactly proven from the kine-
matics of the radiation process, without resorting to any transformations or assumptions as are commonly called on in the literature. The validity of eq.(3.3) is illustrated in the three examples shown in Fig.6 comprising subsonic and supersonic motion of the sources.

For a line segment $\Delta l$ containing infinite singularity lines with constant strengths all over, the inducing function at a field point is given by

$$I_S = -\int_{-\Delta l/2}^{+\Delta l/2} \ln \left( \frac{1}{(x-x_o)^2 + e^2 \Delta x_o^2} \right) dx_o$$

(3.6)

for $Ma_S < 1$ and

$$I_S = -\frac{1}{2} \int_{-\Delta l/2}^{+\Delta l/2} d\Delta x_o$$

(3.7)

with $(x, \Delta x)$ denoting $(x, \Delta z)$ and $\Delta x_o = \Delta x_o / \cos \gamma$ affirming that the line segment and the source lines lie in the Mach fore cone of the field point $P(x,y,z)$, formed with the source line Mach number $Ma_S > 1$. Hence for $Ma_S < 1$ one obtains

$$I_S = -\ln \left( \frac{1}{(x-x_o)^2 + e^2 \Delta x_o^2} \right)$$

$$\Delta x_o = -\frac{\Delta l}{2}$$

$$-2\frac{e}{\Delta x_o} \left[ \text{arc tan} \left( \frac{\Delta x_o}{\Delta x_o} \right) \right]$$

$$\Delta x_o = -\frac{\Delta l}{2}$$

(3.8)

and for $Ma_S > 1$

$$I_S = -\ln \left( \left. \frac{1}{(x-x_o)^2 + e^2 \Delta x_o^2} \right|_{\Delta x_o} \right)$$

$$\Delta x_o = -\frac{\Delta l}{2}$$

(3.9)

These expressions of the inducing functions given by the eqs. (3.8) and (3.9) are valid for two dimensional steady disturbance fields. Equivalent expressions for the unsteady cases can be derived in a similar way. For treating the disturbance fields of arbitrary singularity distributions in space the inducing functions of the individual surface elements are to be determined.

Now considering an element of the source surface in space moving uniformly with Mach number $Ma_e$, the trajectory being along the negative $x$-axis and the surface inclination to it being $\chi$. The kinematic relations of Sec. 2 are used to define the relative locations of the source surface at $P_e(t_e)$ and the emission surface at $P_o(t_o)$ for signals reaching the field point $P$ at a time instant $t$. The radiation process from the moving source surface is displayed in Fig.7.

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The shift as illustrated in Fig. 7 amounts to

$$\Delta H = M_a r_v \{ \sin x_S - \cos x_S \tan (x_S - x_v) \}$$

(3.13)

For surface elements with $x_S = 0$, lying in the plane of the trajectory no shift $\Delta H$ of a panel is necessary and the effective emission radius $r_v^* = r_v$ is identical with the values given by Prandtl-Glauert transformation. For surface elements lying normal to the trajectory with $x_S = \pi/2$ the stretching of the emission element vanishes as is confirmed through eq. (3.11) and eq. (3.12) yielding $r_v^* = r_v$. In this case the Prandtl-Glauert transformation would yield an effective emission radius $r_v^*$ differing largely from $r_v$ and $r_v'$ and hence does not conform to the physical process.

3.2 Inducing Functions due to Moving Source and Doubt Panel

Having established the basic radiation process and the inducing functions of source surface elements it is a simple matter to derive the total inducing effect of a moving source or doublet-panel at a field point in space. A moving source panel 1-2-3-4 having arbitrary orientation to the trajectory induces a perturbation potential at a medium fixed field point $P$ in space.

![Geometrical relations used in the calculation of the inducing functions of a moving source panel for a given field point in space.](image)

The shape of the panel and its geometric relations to the field point is completely defined and $H$ is the vertical height of $P$ from the plane of the panel surface. The local reference system of the panel surface will be denoted by the coordinates $x_1, y_1, z_1$, with the footpoint of $H$ as the origin. $R$ is a normal to the corresponding panel edge as shown. An elementary surface of the panel is denoted as
\( dF_O = R_1 dR_1 d\theta \)  \( (3.14) \)

If the panel surface is displaced parallel to itself to the height \( H_1 \), so that the effective emission radius of the centroid is matched to the exact value given by the radiation process, then the radial distance of an element from the field point amounts to

\[ r_O = \sqrt{R_1^2 + H_1^2} \]  \( (3.15) \)

and the effective emission radius of the element becomes

\[ r^*_n = \left[ \frac{2}{R_1(1-Ma_T^2 \sin^2 \theta)} + \frac{2}{R_1^2 H_1^2} \right]^{1/2} \]  \( (3.16) \)

with \( \beta_T = \sqrt{1-Ma_T^2} = \sqrt{1-Ma_s^2 \cos^2 \lambda} \)

For panel surface containing the \( x \)-axis i.e. for \( x=0 \) one has \( Ma_m = Ma_s \) and \( H_1 = H \) and thus \( r^*_n \) reduces to the usual expression with the panel surface on the body.

The above considerations suggest that the inducing function of a source panel for a given field point can be presented in the general form:

\[ I_s = \int_0^R \int_0^{2\pi} \frac{R_1 dR_1 d\theta}{\left[ \frac{\beta_T}{R_1^2 H_1^2} \right]^{1/2}} \]  \( (3.17) \)

Carrying out the integrations with respect to \( R \) and \( \theta \) one obtains

\[ I_S = \sum_n \left[ \frac{R}{A} \ln \left( 1 - \frac{B}{A} + \sqrt{\left( 1 - \frac{2B}{A} \right)^2 + \left( \frac{C+D}{A} \right)^2} \right) \right]^{n_2}_{n_1} \]

\[ + \sum_n \left[ \arctan \left( \frac{\sqrt{D/A}}{\sqrt{(C/A)-(B/A)^2}} \frac{1 - B/A}{\sqrt{1 - 2B/A + \left( C+D/A \right)^2}} \right) \right]^{n_2}_{n_1} \]

\[ + \sum_n H_1 \left[ \arctan (\beta_T, \theta) \right] \]  \( (3.18) \)

where

\[ A_n = 1 - Ma_T^2 \cos^2 \lambda \]

\[ B_n = Ma_T^2 \beta_T \cos \lambda \sin \lambda \]

\[ C_n = \beta_T^2 (1 - Ma_T^2 \sin^2 \lambda) \]

\[ D_n = \beta_T^2 H_1^2 \]  \( (3.19) \)

Following the same procedure as in [31] the panel geometry can be defined in terms of the corner point coordinates \((x_{n1}, y_{n1})\),

\((x_{n2}, y_{n2})\) and the intersection point \((x^*_O, y^*_O)\) of \( H \) on the panel surface:

\[ d_n = \sqrt{(x_{n2} - x_{n1})^2 + (y_{n2} - y_{n1})^2} \]

\[ S_n = \sin \lambda_n \frac{x_{n2} - x_{n1}}{d_n} ; \]

\[ C_n = \cos \lambda_n \frac{y_{n2} - y_{n1}}{d_n} \]

\[ \hat{r}_n = \left[ (x_o - x_{n1}) C_n - (y_o - y_{n1}) S_n \right] \]

\[ \hat{r}_n = r_s \frac{S_r}{x_s} \]  \( (3.20) \)

\[ l_{n1} = -(x_{n2} - x_o) S_n + (y_{n2} - y_o) C_n \]

\[ l_{n2} = -(x_{n2} - x_o) S_n + (y_{n2} - y_o) C_n \]

For \( Ma_m = 0; \]

\( A_s = 0; \]

\( C_s = \frac{H^2}{2} ; \]

\( D = H^2 \)

and consequently eq. (3.18) reduces exactly to the expression given by Hess and Smith [31].

For a moving doublet panel the inducing function is derived in a similar way by resorting to a contour integration along the panel sides.

\[ \text{Fig.9 Geometrical relations used in the calculation of the inducing functions of a moving doublet panel for a given field point in space.} \]

With the source-sink axis perpendicular to the panel surface the derivative of the source function of eq. (3.17) with respect to \( Z_1 \) or \( H_1 \) leads to

\[ I_D = \int_{-\pi}^{\pi} \int_0^{H_1 R_1 dR_1 d\theta} \]  \( (3.21) \)

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The integration is straightforward, yielding for the inducing function of a doublet panel:

\[
I_D = \sum \text{arc tg} \left[ \frac{\sqrt{D}}{\sqrt{C - \frac{B}{A} \left( \frac{C+D}{A} \right)}} \right]^{n_2} \left( \frac{\text{arc tg} (B \cdot \text{tg} \theta)}{n_1} \right)^{n_2} \tag{3.22}
\]

with \( A, B, C \) and \( D \) being identical with the expressions of eq. (3.19).

\[\int_0^{\pi} \left( \frac{3 \varphi}{3n} \right) (P) = -\dot{n}_i \cdot \ddot{V}_m \tag{3.24}\]

Dividing the body surface into panels of surface areas \( F_{ij} \), the inducing functions of the panels on a field point can now be introduced into eq. (3.23) in the following manner:

\[
\sum_{j=1}^{N} \ddot{n}_i \cdot \left[ \int_{F_{ij}} \left( \frac{1}{r_{ij}^2} \right) \text{d}F_{ij} \right] \frac{S_j}{4nV_m} + \sum_{k=1}^{M} \ddot{n}_i \cdot \left[ \int_{F_{ik}} \left( \frac{3 \varphi}{3n} \right) \text{d}F_{ik} \right] \frac{D_j}{4nV_m} = -\dot{n}_i \cdot \ddot{V}_m \tag{3.25}\]

The integral expressions for each panel element have already been evaluated as \( I_S \) and \( I_D \) in eqs. (3.18) and (3.22).

The bracketed terms being purely functions of the body geometry i.e. of the relative locations of field points and the panels, they can be computed once and for all.

Denoting as \( A_{ij} \) and \( B_{ik} \) the coefficients of the system of equal number of unknowns:

\[
\sum_{j=1}^{N} \ddot{n}_i \cdot A_{ij} X_i + \sum_{k=1}^{M} \ddot{n}_i \cdot B_{ik} Y_i = -\dot{n}_i \cdot \ddot{V}_m \tag{3.26}\]

Here \( X_i \) and \( Y_i \) denote the unknown source and doublet singularities. The whole procedure then follows the same line as the classical panel method. Having determined \( X_i \) and \( Y_i \) all the aerodynamic coefficients of the body or in the field around the body can be determined without difficulty.

The panel method for supersonic flows can be dealt with in a similar way.

IV. Conclusion

In the panel method outlined in this paper a direct treatment is formulated for arbitrary bodies in compressible flows - subsonic and supersonic. In order to incorporate the effect of the transitory motion of the panels the inducing coefficients are rederived including Mach number terms. The underlying physical principles comprising the basic kinematics of disturbance propagation are outlined extensively. The effects due to the Mach number originate from the spatial and temporal stret-
chin effects in the process of emission and propagation of the disturbance signals. The resultant inducing functions of source- and doublet panels for field points in space are derived in closed form. The calculation of the perturbation field by using integral methods follows the same line as the panel method for incompressible flows. In the limit of Mach number tending to zero, all the influence coefficients reduce to the classical expressions known from literature, thus including the standard panel method as a special case.

V. Literature


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