COMPUTATION OF TRANSONIC FLOW WITH
DETACHED BOW SHOCKS THROUGH
TWO-DIMENSIONAL TURBOMACHINERY CASCADES

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Abstract

A method is presented for computing steady inviscid transonic flows with and without detached bow-shocks in turbomachinery cascades, wherein both subsonic and super-sonic regions co-exist. A finite difference method based on the small disturbance formulation of the non-linear problem is used. Different finite difference equations which are appropriate to the behaviour of the differential equation at each mesh point are formulated. Weak shock structures are permitted to develop as a part of the solution process, without prior knowledge of their exact strength and position.

To take into account viscous effects, the displacement thickness of the boundary layer is added to the blade contour.

A variation of the basic procedure is used to solve the inverse problem to design airfoil contours for given pressure distribution.

The computed surface pressure distributions for given airfoil shapes agree well with experimental results.

Introduction

The design and development of turbomachines require three-dimensional solution to the flow field internal to rotating and stationary cascades. 1950, Wu proposed to solve the flow equations on separate orthogonal quasi-two-dimensional surfaces in space. (1) Solution techniques suitable for the S2 (radially oriented surfaces) appeared rather quickly. The blade-to-blade solutions have been achieved somewhat more slowly. A subsonic relaxation technique utilizing the stream function equation was proposed by Wu and developed by Katsanis and McNally. (2) The difficulty of this approach lies in the restriction to uniformly subsonic Mach numbers and that the reduction of derivatives of stream function to velocities and densities is double-valued with Mach number. No such ambiguity arises if the independent variable is selected to be the velocity potential. Murman and Cole developed a relaxation technique capable of rapidly and efficiently handling local areas of supersonic flow around isolated airfoils. (4) The method is based on the realization that one must vary the difference star to match the characteristics of the governing differential equations. Dodge used this technique to predict deviation angles and optimum angles of attack for cascades in transonic flow fields with subsonic inlet velocities. (3) Extension of the method to cascades in flow fields of upper transonic region with supersonic inlet velocities, comparison of data, and limitations of the approach are topics discussed here.

Theoretical Considerations

Assumptions

Due to the very complex nature of the real flow in a turbomachine, it is necessary to make some simplifying assumptions about the fluid and the cinematic behaviour of the flow under consideration. Therefore, the following hypothesis are made for the analysis to follow.

- the flow is considered steady relative to the blade
- the flow is two-dimensional
  - the blade-to-blade surface is a surface of revolution
  - the velocity normal to the blade-to-blade surface is zero
- the fluid is nonviscous
- the fluid is isoenergetic
- the velocity magnitude and direction is uniform across the upstream and across the downstream boundaries
- the effects of viscosity are only taken into account to calculate the boundary layer displacement thickness and the shock/boundary layer interactions
- the vorticity created by the shock wave is ignored herein.

Governing Equations

The basic differential equations of transonic flow are sufficiently well known that a derivation is unnecessary; and, therefore, they will simply be stated in
the form used.

The transonic small disturbance equation is derived as a part of a systematic expansion procedure applied to the exact equations for steady, inviscid, isentropic flow.

continuity eq. \( \text{div} \ (\rho \ \vec{V}) = 0 \)  

(1)

Euler eq. 
\[ (\vec{V} \ \text{grad}) \ \vec{V} = \frac{1}{\rho} \ \text{grad} \ p \]

(2)

isentropic rel. 
\[ p = k \ \rho^\gamma \]

(3)

where \( \vec{V} \) is the velocity vector, \( \rho \) the density, \( p \) the pressure, \( k \) a constant of proportionality, and \( \gamma \) the ratio of specific heats.

The result is the transonic small perturbation potential equation (TPE)

\[ (K - (\gamma + 1) \ \frac{\phi}{\rho}) \ \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 0 \]

or alternatively

\[ (K \ \phi_x - \frac{\gamma + 1}{2} \ \frac{\phi_x^2}{\rho}) \ \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 0 \]

(4)

with the velocity components \( u \) and \( v \) in \( x \)- and \( y \)-direction

\[ \phi_x = u \]

\[ \phi_y = v \]

and

\[ K = \frac{1 - M_{\infty}^2}{M_{\infty}^2 \ 6^{2/3}} \]

\[ \vec{y} = (M_{\infty}^2 \delta)^{1/3} \]

(5)

(6)

(7)

The shock jump conditions must be added to the system 5 to form a complete set of equations. However, it turns out that the shock jump conditions are contained in system 5 in the following sense. If their differential equations are integrated across a jump in \( (u, v) \), the correct shock relations, to this order, result. That is, each equation in 5 is in conservation form. The corresponding surface integral forms give the shock jump relations

\[ \left[ K \phi_x - \frac{\gamma + 1}{2} \ \frac{\phi_x^2}{\rho} \right] (d\vec{y})_s - \left[ \phi_y \right] (dx)_s = 0 \]

(8)

\[ \left[ \phi_y \right] (d\vec{y})_s + \left[ \phi_x \right] (dx)_s = 0 \]

where

\[ [\ ] = \text{jump} = \text{downstream} - \text{upstream} \]

\( (\ )_s \) = element in the shock surface

The two systems of equations 5 and 8 are to be solved numerically for a cascade geometry such as is shown in figure 1 and subject to the following boundary conditions.

Boundary Conditions

The computational domain shown in figure 1, is composed of two channels between three adjacent blades. Upstream and downstream of the blading, as the stream-lines are not known, the periodic boundaries are drawn in the direction of the chord of the profile and fluid is allowed to cross these boundaries. The flow conditions at these boundaries are the same as those at the equivalent lines upstream and downstream of the central profile. Thus they may be adjusted in a correct manner step by step as the solution progresses.

The inlet boundary AB is parallel to the slope of the lower profil's detached bow shock at the upper periodic boundary. Thus its position and inclination must be updated during the iteration process to fulfill the condition of periodicity, too. (The form and position of the detached bow shock is a part of the solution). The outlet boundary is chosen to be sufficiently far away from and parallel to the outlet of the cascade so that the conditions along EF may be considered uniform.

On the rigid boundaries of the profil surfaces the wall tangency condition \( \partial \phi/\partial n = 0 \) is given in the form:

\[ \phi_y(x) = F'(x) - \frac{1}{\delta} \]

(9)

where the body shape is given as \( \vec{y} = \delta F(x) \).

If a case with circulation is considered, the Kutta condition that the flow leaves the trailing edge smoothly must be applied. This implies a jump of potential across the \( x \)-axis downstream of the profil. This jump is related to the circulation around the airfoil, which is unknown in advance.

These boundary conditions in relation with the transonic small perturbation equation define a presumably unique solution.

Axial velocity ratio

The axial velocity ratio

\[ \Omega = \frac{\rho_2 \ V_2 \ sin \ \beta_2}{\rho_1 \ V_1 \ sin \ \beta_1} \]

(10)

has an important influence on the flow-field in a turbomachinery cascade.

With the velocity

\[ V = \sqrt{\phi_x^2 + \phi_y^2} \]

(11)
the flow angle

\[ \sin \beta = \frac{\phi_y}{\sqrt{\phi_x^2 + \phi_y^2}} \]  (12)

and the density ratio

\[ \frac{p_2}{p_1} = \frac{\frac{1}{p_2}}{\frac{1}{p_1}} = \frac{\left[ 1 + \frac{Y+1}{2} M_2^2 \right]^{-\frac{1}{Y-1}}}{1 + \frac{Y+1}{2} M_1^2} \]  (13)

the small perturbation approximation of the axial velocity ratio becomes

\[ \Omega = \frac{\phi_{y2}}{\phi_{y1}} \cdot \frac{p_2}{p_1} \]

\[ = \frac{\phi_{y2}}{\phi_{y1}} \cdot \left[ 1 + \frac{Y+1}{2} M_2^2 \right]^{-\frac{1}{Y-1}} \]

During each step of iteration the values of \( \phi_{y2} \) are updated according to

\[ \phi_{y2} = \phi_{y1} \cdot \Omega/\text{const.} \]  (15)

This correction has the effect of a fictitious downstream pressure

\( (p_2 \text{ real} + \Delta p_2 \text{ fict.}) \) taking into account the pressure-forces on a volume between two non-parallel streamsurfaces (figure 3).

Numerical Solution

Finite Difference Schemes

The mixed differencing relaxation scheme for solving transonic boundary value problems was introduced by Murman, Cole and Krupp to treat the small disturbance formulation for nonlifting and lifting airfoils.\(^4\),\(^5\) Most of our calculations are obtained using unequally spaced grids but in the following, for simplicity, we confine our discussion to evenly spaced grids. More details are given in the cited references.\(^4\)-\(^8\)

The important type-dependent feature of the method is that, in subsonic flow regions central difference operators are used to account for the domain of dependence of elliptic equations. In supersonic flow regions, backward difference operators are used to account for the absence of upstream influence in hyperbolic equations.

For the rectangular grid structure shown in the figures 1 and 2, equation 4 can be approximated using a second-order accurate central difference operator for a point \( i,j \) in a subsonic flow region.

\[ \begin{bmatrix} K - (\gamma + 1) \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} \\ \phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j} \\ \end{bmatrix} \]

\[ \cdot \begin{bmatrix} \phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j} \\ \phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j} \\ \end{bmatrix} = 0 \]  (16)

The operator is stable by linear stability analysis if the coefficient

\[ (V_{el})_{i,j} = K - (\gamma + 1) \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} \]  (17)

is positive.

Similarly, an implicit, first-order accurate backward difference operator may be written for a point \( i,j \) in a supersonic flow region.

\[ \begin{bmatrix} K - (\gamma + 1) \frac{\phi_{i,j} - \phi_{i-2,j}}{2\Delta x} \\ \phi_{i,j} - 2\phi_{i-1,j} + \phi_{i-2,j} \\ \end{bmatrix} \]

\[ \cdot \begin{bmatrix} \phi_{i,j} - 2\phi_{i-1,j} + \phi_{i-2,j} \\ \phi_{i,j} - 2\phi_{i-1,j} + \phi_{i-2,j} \\ \end{bmatrix} = 0 \]  (18)

The operator is stable by linear stability analysis if the coefficient

\[ (V_{elh})_{i,j} = K - (\gamma + 1) \frac{\phi_{i,j} - \phi_{i-2,j}}{2\Delta x} \]  (19)

is negative.

During the iteration procedure \( (V_{el})_{i,j} \) is computed at each grid point. If \( (V_{el})_{i,j} > 0 \) the flow is subsonic and the elliptic operator (eq. 16) is used. If \( (V_{el})_{i,j} < 0 \) and \( (V_{elh})_{i,j} < 0 \), the flow is supersonic and the hyperbolic operator (eq. 18) is used.
As the flow accelerates through sonic velocity from subsonic to supersonic velocities, a point exists where \( (\text{Vele})_{i,j} < 0 \) and \( (\text{Velh})_{i,j} > 0 \) and neither equation 16 nor equation 18 is stable. A parabolic difference operator is used for such points.

\[
\frac{(\phi_{i,j}^{t+1} - 2\phi_{i,j}^{t} + \phi_{i,j}^{t-1})}{(\Delta y)^2} = 0 \quad (20)
\]

As the flow decelerates through sonic velocities from supersonic to subsonic flow, a point exists where \( (\text{Vele})_{i,j} > 0 \) and \( (\text{Velh})_{i,j} < 0 \) and both equation 16 and equation 18 are locally stable. Murman's "shock-point operator" is used for such points.

\[
(\text{Vele})_{i,j} \left( \frac{\phi_{i+1,j}^{t} - 2\phi_{i,j}^{t} + \phi_{i-1,j}^{t}}{(\Delta x)^2} \right) + (\Delta y)^2 \quad (21)
\]

\[
(\text{Velh})_{i,j} \left( \frac{\phi_{i,j}^{t} - 2\phi_{i+1,j}^{t} + \phi_{i+2,j}^{t}}{(\Delta x)^2} \right) + \frac{(\phi_{i,j}^{t+1} - 2\phi_{i,j}^{t} + \phi_{i,j}^{t-1})}{(\Delta y)^2} = 0
\]

Murman has shown that this operator, whose x-differences are the sum of the x-differences of the elliptic and hyperbolic operators, will give the correct weak solution to equation 5. He has further shown that, in the limit of vanishing grid spacing, the correct shock jumps are obtained for oblique shocks when arbitrary grid spacing is used. For shock waves that jump from supersonic to subsonic velocities the jump is spread over three mesh points. When the flow downstream of the shock is supersonic the dissipation errors dominating the first-order accurate hyperbolic operator smear the shock over six to ten mesh points.

The circulation \( \Gamma \) which satisfies the Kutta condition must be obtained as a part of the solution process for \( \phi \). A cut downstream the central profile is introduced at \( y = 0, x > 1 \) across which the potential jump must equal the circulation \( \Gamma \). Modified difference operators for \( \phi_{yy} \) are written that take into account the jump in \( \phi \) across the cut while maintaining the continuity of \( \phi_{x} \) and \( \phi_{y} \). The standard three point centered difference formulas become

\[
(\phi_{yy})_{i,0} = \frac{1}{(\Delta y)^2} \left( \frac{(\phi_{i,1} - \Gamma) - 2\phi_{i,0} + \phi_{i,-1}}{2} \right) \quad (22)
\]

\[
(\phi_{yy})_{i,1} = \frac{1}{(\Delta y)^2} \left( \phi_{i,2} - 2\phi_{i,1} + (\phi_{i,0} + \Gamma) \right) \quad (23)
\]

The points \((i, 0 +)\) are not included in the computation, since \( \phi_{i,0} \) is determined from the computed value for \( \phi_{i,0} \) and the value of \( \Gamma \). The iteration procedure for finding the proper circulation proceeds by choosing an initial circulation \( \Gamma_0 \) from which the outer-boundary solution is obtained. The circulation at the trailing edge is found from the jump in \( \phi \). On each iterative sweep of the grid a new value of \( \Gamma_{te} \) is found by linearly extrapolating the \( \phi \)'s from above and below. The new \( \Gamma_{te} \) along with \( \Gamma_0 \) is used to determine the \( \Gamma_i \) by interpolation. As the iteration proceeds, new estimates for \( \Gamma_0 \) are made, until a converged solution for \( \phi \) is obtained.

**Computational Procedure**

In the operation of a compressor cascade with slightly supersonic relative inlet Mach number, it is known that there is a unique incidence associated with the given inlet Mach number. The exit angle depends upon the blade circulation which, in turn, is fixed by the Kutta condition. Thus, neither the inlet nor the exit angle are known a priori for the usual transonic compressor operation. The only independent variables of the flowfield are the inlet Mach number and the static pressure ratio. The problem is then to find the entire flowfield for a given cascade geometry, the inlet Mach number and the static pressure ratio or the inlet-outlet velocity ratio. The inlet and outlet angles must be computed as a part of the solution.

A channel for the flow is defined as shown in figure 1. For subsonic inlet velocities the inlet boundary of the computational domain is taken to be parallel to the inlet of the cascade. The magnitude and direction of the velocity is known there. In the triangle \( \Delta AB \) the \( \phi_{x} \) and \( \phi_{y} \) values are constant.

For supersonic inlet velocities the inlet boundary is parallel to the slope of the lower profile's detached bow shock at the upper periodic boundary. Thus, an iterative process has to be used to specify this boundary as a part of the solution. To find a first guess the form of the detached bow shock is calculated for an isolated profile using Murman's method for the flowfield calculation and Levine's method for the determination of the inlet flow angle. (7),(10) This flowfield is superposed to construct the first approximation of the flowfield in the cascade. The \( \phi_{y} \)-
and $\phi_0$ - values of the upper part of the inlet boundary (line 7-B in figure 4) are found by linear extrapolation of those at the boundary A-4, which is equivalent to the bow shock section 6-10. The periodicity of the stand-off distances of the shocks and the distances between the shocks are verified at the periodic boundaries and the grid lines in front of the central profile. Every tenth step of iteration the inlet flow angle is updated. If the inlet-outlet velocity ratio is known the axial velocity can be used to determine the exit flow angle. Thus, via the Kutta condition the circulation is prescribed. For a given static pressure ratio, both axial velocity ratio and Kutta condition are necessary to calculate the magnitude and the direction of the flowfield at the outlet.

Results and Comparison with Experimental Data

The method outlined in the foregoing has been applied to the flow through several cascades. The geometries of the cascade, the profiles and the inlet and outlet conditions have been chosen so that the results can be compared with experimental data. The profile geometries as well as the cascade data are reported in detail in the references 11a - 12.

To obtain good agreement with the experimental data the boundary layer displacement thickness has been calculated to correct the profile shapes. For the shock boundary layer interaction Melnik's small perturbation approximation of Reshothko-Tucker's theory was used. (13)

Figure 5 shows the computed pressure distribution for a cascade with Do-Al profiles. For this low cambered 11 percent supercritical profile excellent agreement has been obtained for subsonic inlet velocities and small deviation angles.

In figure 6 the numerical results for a high cambered, 5.9 percent DCA 48 cascade are compared with experimental data of reference 11b. For the two presented inlet angles the agreement of the results downstream of the shock suffers from the simple shock boundary layer interaction model used.

For supersonic inlet conditions a low cambered, 4.66 percent DCA 9.5 profile with 9.57$^\circ$ camber angle has been chosen. In the figures 7 to 11 results for different prescribed inlet and outlet Mach numbers are compared with the data of reference 11a. The calculated unique incidence for this cascade geometry is shown in figure 12. The axial velocity ratio used to compute the outlet angles are reported on the graphs.

For shock Mach numbers higher than 1.4 the shock boundary layer interaction is not simulated in a correct manner so that there is a small lack of agreement between experimental data and computed values downstream of the shock. But in general the agreement of the velocity and pressure distribution and the flow angle prediction is rather good.

Conclusion

The transonic relaxation method described has proved to give apparently accurate and useful results if the non-viscous calculations are combined with a boundary layer calculation. The boundary layer displacement effects must be included to obtain accurate results aft the shock and to eliminate irregularities at the trailing edge. Due to the simple shock boundary layer interaction model used the surface pressure gradients across the shock jumps are slightly steeper than in reality.

The results are obtained using between 1/20 and 1/50 the computer time of the time-dependent methods while avoiding the problem of shock smearing. The extension of the method to three-dimensional flow seems highly promising because of the small CPU-times for two-dimensional cases. Indeed, for wing-body combinations Schmidt and Vanino have already calculated three-dimensional flows using this method. (14), (15).

The major difficulty of the method rests with its restriction to thin, low-cambered profiles and in the lack of defining exact locations of oblique shocks. But, with adequate mesh refinement the location of weak oblique shocks may be sufficiently well defined.

Acknowledgment

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References


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**Figures**

![Fig. 1 Cascade Geometric and Grid Structure](image1)

![Fig. 2 Grid Structure and Associated Notation Near the Trailing Edge](image2)

![Fig. 3 Axial Velocity Ratio](image3)

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**Fig. 4** Computational Domain and Periodicity

- **M**₀ = 0.740  \( \beta_1 = 124.20^\circ \)  \( \Omega = 1.0 \)

**Fig. 5** Pressure Distributions for the Do-Al-Cascade, Comparison with Data of ref. 12

- **t/1** = 0.60  
- \( \beta_0 = 133^\circ \)  
- \( M_1 = 0.729 \)  
- \( \beta_1 = 135^\circ \)  
- \( M_2 = 0.982 \)

**Fig. 6** Velocity Distribution for the DCA-48-Cascade, Comparison with Data of ref. 11 b

- **M**₀ = 0.740  \( \beta_1 = 128.50^\circ \)  \( \Omega = 1.0 \)
Geometry of the DCA-9.5-Cascade

Profil: double circular arc

Camber angle: 9.57°
Maximum Thickness: 4.66 %
Solidity: 1.4618
Stagger angle: 141.35°
Inlet angle
(incidence = 0°): 146.14°
Outlet angle
(deviation = 0°): 136.57°

Fig.: 7 Influence of the Unique Incidence on the Velocity Distribution. DCA-9.5-Cascade, Comparison with Data of ref. 1la

M₁ = 1.050  M₂ = 0.722  Ω = 1.154

M₁ = 1.205  M₂ = 0.815  Ω = 1.061

Fig.: 8 Velocity Distribution for the DCA-9.5-Cascade, Comparison with Data of ref. 1la

M₁ = 1.266  M₂ = 0.777  Ω = 1.168

Fig.: 9 Velocity Distribution for the DCA-9.5-Cascade, Comparison with Data of ref. 1la

M₁ = 1.050  M₂ = 0.761  Ω = 1.145
Fig.: 10 Velocity Distribution for the DCA-9.5-Cascade

Fig.: 11 Velocity Distribution for the DCA-9.5-Cascade

Fig.: 12 Unique Incidence for the DCA-9.5-Cascade (data: ref.11a)