QUASI-HOMOGENEOUS APPROXIMATIONS FOR WINGS WITH CURVED SUBSONIC LEADING EDGES AT SUPersonic SPEEDS

by

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Abstract

In this paper quasi homogeneous approximations are constructed for the flow around delta-like supersonic wings with curved subsonic leading edges. If the leading edges are given by algebraic curves and the boundary conditions are of polynomial form, the boundary value problems are reduced to completely algebraic problems. The solutions can be expressed in terms of known simple functions. The parameters defining the leading edges appear in the same way as those defining the boundary conditions. The results may be useful to acquire a qualitative insight in the possibilities of a large class of planforms.

I. Introduction

1.1 Preliminary considerations

From the viewpoint of the designer of supersonic aircraft it is essential to acquire a qualitative insight into the aerodynamic properties of wings which corresponds to the requirements to be satisfied. Conversely these requirements should be formulated in such a way that the design problem will be "well posed" in the mathematical sense and will permit a unique solution. Moreover the requirements should permit a solution that can be realized physically. During the design process, the linearized form of potential theory for supersonic wings is a powerful tool. Due to the linearity of the equations, the superposition principle applies and a systematic approach is possible. In the last phases of the design process, numerical techniques may be useful for the study of complicated geometries through the use of high-speed digital computers. In the earlier phases of the design however one is less interested in numerical values. One needs a qualitative insight which may take the form of a parameter study. Obviously, the planform of the wing deserves a high priority. Because of the theoretical difficulties, one must generally assume that the planform is fixed and one studies the effects of variations in the boundary conditions at the wing surface.

This situation is not very satisfactory. In the approximations proposed in this paper the parameters defining the planform and those defining the boundary conditions at the wing surface appear in the same way and can be given the same priority. In fact, planforms with curved, subsonic leading edges of algebraic form and boundary conditions at the wing surface of polynomial form permit a reduction of the boundary value problems to purely algebraic problems. The number of algebraic problems, which finally result, is related to the order of approximation desired.

1.2 The governing equation

In linearized steady supersonic wing theory the perturbation potential must be a solution of

$$\beta^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} = 0$$

with

$$\beta = \frac{\mu^2 + 1}{\mu + 1} \frac{1}{a}$$

(a is the velocity of sound).

The trirectangular system $(x, y, z)$ is fixed to a convenient time averaged position of the wing and moves with a constant speed $U$ in the negative $x$-direction with respect to the atmosphere.
The assumptions that lead to equation (1.1) are discussed in many textbooks and are not repeated here.

If the \((x,y,z)\) system has its origin at the apex of the wing the envelope of the disturbances is given by the Mach cone, \(P\),

\[
\sqrt{x^2 + y^2 + z^2} = \frac{V}{a} \tag{1.2}
\]

The equation (1.1) is a second order linear partial differential equation of hyperbolic type with constant coefficients. This equation occurs in mathematical physics and has been studied extensively. If one interprets \(x\) as a time variable one establishes the analogy with the two dimensional wave equation. On the other hand by putting \(\xi = \frac{x}{V}\) one establishes a formal analogy with the three dimensional Laplace equation \((\xi^2 = -1)\).

The initial value and boundary value problems however are specific for supersonic wing theory.

1.3 The boundary conditions

The boundary condition at the wing surface follows from the requirement that the flow must be tangential to the wing surface. The angle between the normal to the wing-surface and the \(x\)-axis is small. Near round leading edges this assumption is violated but the region in which this happens will, in general, be relatively small. Moreover, if the perturbation velocity is small with respect to \(U\), the boundary condition can be put in the form

\[
\omega = \varphi_x = -U\alpha(x,y), \tag{1.3}
\]

\(\alpha(x,y)\) being the local angle of incidence. If the wing surface is given by \(z = g(x,y)\) one has

\[
\varphi_x = -\alpha(x,y) \tag{1.4}
\]

The plane \(\xi = 0\) can be taken close to the wing surface so that the boundary condition can be applied at the projection, \(S\), of the wing planform.

At the envelope of the disturbances, generated by the wing, the boundary condition is \(\varphi = \sigma\).

1.4 The four types of problems

The pressure distribution is related to the perturbation velocity potential by the linearized Bernoulli equation:

\[
\rho = \sigma \tag{1.5}
\]

\(\rho\) being the density of the air.

Equations (1.3) and (1.5) suggest two types of problems. If the wing geometry is given one knows \(\varphi\) at the wing surface. The problem of finding the corresponding pressure distribution on the wing is called the direct problem. If the pressure distribution is prescribed and one is asked to find the geometry of the wing generating this pressure distribution, one calls it the inverse problem. The perturbation potential \(\varphi(x,y,z)\) of a flow around a wing which lies near the plane \(\xi = 0\) can be considered as the sum of an even and an odd part:

\[
\varphi = \varphi^{(e)} + \varphi^{(o)} \tag{1.6a}
\]

with

\[
\varphi^{(e)}(x,y,z) = \varphi^{(e)}(x,y,z) \tag{1.6b}
\]

\[
\varphi^{(o)}(x,y,z) = \varphi^{(o)}(x,y,z) \tag{1.6c}
\]

The first is even in \(z\) and is associated with obstacles symmetric with respect to \(Z\); it is referred to as the thickness case. The second is odd in \(Z\) and is associated with wings without thickness; it is referred to as the lifting case.

It follows that there are four types of problems:

(i) The direct thickness problem (D.T.P.).


(iii) The inverse thickness problem (I.T.P.).


1.5 Outline

If the leading edges are straight and form the same angle with the \(x\)-axis, the homogeneous flow theory of P. Germain and M. Fenain leads to a systematic treatment for a very large class of problems. It is possible to handle the solutions efficiently and therefore they are very useful for the aircraft designer. It is clearly attractive to exploit the possibilities of homogeneous flow theory to the utmost.

If the planform of the wing is delta like and the leading edges are only slightly curved the solutions will differ little from the homogeneous flow solutions. It will be shown that for quasi-conical planforms it is possible to construct quasi-homogeneous approximations which can be calculated analytically in a workable scheme. This approach seems to be the most natural one for a large class of wing planforms of practical interest, say of the Concorde type. Before proceeding to the description of this method it is necessary to explain some important features of homogeneous flow theory.
An outline of homogeneous flow theory.

P. Germain [1] generalized Busemann's conical flow theory into the theory of homogeneous flows. The problem is reduced to the construction of analytic functions. Especially M. Fenain carried out many calculations which were synthesized in [2] in a very elegant form. The solutions can be expressed in terms of functions that can be determined once and for all and in terms of coefficients that are uniquely related to the boundary conditions at the wing surface. The problem is thus reduced to a purely algebraic one.

2.1 Definition of homogeneous flows.

Germain defined a homogeneous flow of order $\eta$ as a flow in which the perturbation potential is a homogeneous function of degree $\eta$ in the variables $x_1, x_2, x_3$. This potential satisfies the equation

$$f(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda^\eta f(x_1, x_2, x_3). \tag{2.1}$$

Differentiating (2.1) with respect to $\lambda$ and putting $\lambda = 1$ one obtains Euler’s relation:

$$x_1 f_{x_1} + x_2 f_{x_2} + x_3 f_{x_3} = \eta f. \tag{2.2}$$

Equations (2.1) and (2.2) are equivalent, they can be deduced from each other.

2.2 The $\eta^h$ derivatives.

For natural numbers $\eta$, it follows that all $\eta^h$ derivatives with respect to $x_1, x_2,$ and $x_3$ are homogeneous functions of order $\eta$. They are constants on straight lines through the origin. The Mach cone $r, \chi > 1$, is considered. The $\eta^h$ derivatives (2.4) depend on $\eta$ and satisfy

$$(\chi^2 - 1) f_{\chi \chi} + \chi f_\chi + \theta f_\theta = 0. \tag{2.3}$$

With $\chi = \cosh \psi$ one obtains

$$b^2 y y + \theta_\theta = 0, \tag{2.4}$$

and the problem is reduced to the construction of harmonic functions. Analytic functions can be introduced. Two conformal mappings are of special interest:

(i) $Z = X + iY = e^\psi e^{i\theta} \tag{2.5}$

The $Z$ plane can be used with advantage for the construction of the form of the solutions.

(ii) $\tilde{Z} = \tilde{X} + i\tilde{Y} = \tilde{Z}((+Z^2)^{-1}) \tag{2.6}$

The $\tilde{Z}$ plane is more profitable for practical calculations.

The boundary conditions are specified at $x_3 = 0$ and one is primarily interested in the solutions at the wing surface. The most important $\eta^h$ derivatives are:

$$U_{n}^{n} = U_{n}^{n-1}x_{2} \frac{\partial}{\partial x_{2}}, \quad W_{n}^{n} = W_{n}^{n-1}x_{2} \frac{\partial}{\partial x_{2}} \tag{2.7}$$

with $U_{n} = \frac{\partial \phi}{\partial x_{2}}$ and $W_{n} = \frac{\partial \Omega}{\partial x_{2}}$.

2.3 Reduction to the construction of analytic functions.

It is usual to introduce the coordinates $r, \chi$ and $\theta$ by

$$\begin{cases}
\phi = \chi \cos \theta, \\
\chi = \sin \theta.
\end{cases} \tag{2.8}$$

In this way a one-to-one relationship is established between a pair of values of $\chi, \theta$ and a straight line through the origin. The Mach cone $r, \chi > 1$, is considered. The $\eta^h$ derivatives (2.4) depend on $\chi$ and $\theta$ only and satisfy

$$(\chi^2 - 1) f_{\chi \chi} + \chi f_\chi + \theta f_\theta = 0. \tag{2.9}$$

With $\chi = \cosh \psi$ one obtains

$$b^2 y y + \theta_\theta = 0, \tag{2.10}$$

and the problem is reduced to the construction of harmonic functions. Analytic functions can be introduced. Two conformal mappings are of special interest:

(i) $Z = X + iY = e^\psi e^{i\theta} \tag{2.11}$

The $Z$ plane can be used with advantage for the construction of the form of the solutions.

(ii) $\tilde{Z} = \tilde{X} + i\tilde{Y} = \tilde{Z}((+Z^2)^{-1}) \tag{2.12}$

The $\tilde{Z}$ plane is more profitable for practical calculations.

fig. 3.

fig. 4.
2.4 The boundary conditions.

(i) On \( \Gamma \) one has \( P^{(n-p-2)}P_{02} = 0 \), and especially \( U_{n_k} = 0 \), \( \omega_{n_k} = 0 \).

(ii) At the wingsurface, all \( \omega_{n_k} \) are known in a D.P. In an I.P. all \( \omega_{n_k} \) will be known.

(iii) Outside of the wing \( (x_2 = 0) \) one has in an L.P.: \( U_{n_k} = 0 \) and in a T.P.: \( \omega_{n_k} = 0 \).

The D.T.P. and the I.L.P. are relatively simple because the boundary conditions are known for the same functions at all points of the boundary. The D.L.P. and the I.T.P. are more difficult; they are of mixed type. In order to solve these problems the compatibility conditions are needed.

2.5 The compatibility relations.

The \( n \)-th derivatives \( \varphi^{(n-p-2)}(x_1,x_2) \) are harmonic functions which can be considered as the real parts of analytic functions \( \varphi^{(n-p-2)}(x_1,x_2) \), say of \( Z \) or \( \overline{Z} \). All \( \varphi^{(n-p-2)}(x_1,x_2) \) are derivatives of the same \( \varphi \) and are therefore not independent. In fact, one can prove the equivalence of the expressions,

\[
\frac{d\varphi^{(n-p,0)}}{dx_1} = (-i)^{\frac{n}{2}} \frac{x_1}{\sqrt{1-x_1^2}} \frac{d\varphi^{(n-p,1)}}{dx_2}, \tag{2.11}
\]

which implies that knowing one of the \( n \)-th derivatives, all the other derivatives follow with (2.11).

With \( \omega_{n_k} = R.P. U_{n_k} \),

\[
\omega_{n_k} = P + I \omega_{n_k},
\]

one may write

\[
\frac{\partial^2 U_{n_k}}{\partial x_1^2} = (-\frac{3}{2}) \frac{2}{(\sqrt{1-x_1^2})^3} \frac{d\omega_{n_k}}{dx_2}, \tag{2.12}
\]

which relates the pressure distribution to the upwashfield.

2.6 Euler's relation.

From \( n \varphi \) \( \varphi = \varphi \varphi + \varphi \varphi \varphi + \varphi \varphi \varphi \) one easily deduces:

\[
\varphi^{(n)} = (x_1 \varphi^{(n-1)} + x_1 \varphi^{(n-1)} + x_2 \varphi^{(n-1)}), \tag{2.13}
\]

with the notation

\[
\{\varphi^{(n-1)} \} \{\varphi^{(n-1)} \} \{\varphi^{(n-1)} \} = \{\varphi^{(n)} \}.
\]

\( \varphi \) can be obtained from the \( n \)-th derivatives without carrying out any integrations.

For \( x_2 = 0 \), the relations (2.13) simplify and by comparison with (2.5) one obtains:

\[
\begin{cases}
U_{n_k} = \sum_{s=0}^{n-k} \frac{x_1^{n-k-s} x_2^s}{s!} U_{n_k} \\
\omega_{n_k} = \sum_{s=0}^{n-k} \frac{x_1^{n-k-s} x_2^s}{s!} \omega_{n_k}.
\end{cases} \tag{2.14}
\]

2.7 Elementary flows.

Homogeneous flows are called "elementary" if the first derivatives of \( \varphi \), for which the boundary conditions are specified at the wing surface, are homogeneous polynomials, of degree \( (n-1) \), in \( x_1 \) and \( x_2 \). This implies that for elementary flows:

(i) in a D.P. the \( \omega_{n_k} \) are constants on the wing.

(ii) in an I.P. the \( U_{n_k} \) are constants on the wing.

Problems with boundary conditions that are odd in \( x_2 \) only make sense in cases without thickness. They are referred to as D.T.P.

The coefficients introduced into the boundary conditions are easily related to the \( U_{n_k} \) or \( \omega_{n_k} \).

For instance, in a D.L.P. with

\[
\omega_{n_k} = \sum_{s=0}^{n-k} C^{(n-k-s)} x_2^s \frac{x_1}{x_1}, \tag{2.15a}
\]

one obtains upon comparison with (2.14):

\[
U_{n_k} = \frac{(n-k-s)!}{(n-k)!} C^{(n-k-s)} x_1 x_2^s. \tag{2.15 b}
\]

There are \( n \)-coefficients \( c_{n-k-s} \).

2.8 The form of the solutions.

From the compatibility relations it is clear that in order to determine the form of the solutions it is sufficient to operate on one \( \varphi \) derivative of \( \varphi \), only, say \( \varphi_n \). In a D.L.P. for instance one obtains the result

\[
\frac{d\varphi_n}{dx_2} = \frac{\pi}{\sqrt{1-x_1^2}} \frac{d\varphi_n}{dx_2}, \tag{2.16a}
\]

There are \( n \)-coefficients \( \lambda_{n-p} \) that can be used to satisfy the boundary conditions (2.15).

The strongest singularity \( (p=n) \) that is admitted at the leading edge is a square root singularity in the first derivatives of \( \varphi_n \).

All \( \omega_{n_k} \) are imaginary on \( \Gamma \). With the compatibility relations (2.12) one verifies that all \( \omega_{n_k} \) are zero outside of the wing \( (x_2 = 0) \).

For an I.T.P. one obtains:

\[
\frac{dU_{n_k}}{dx_2} = \frac{2\pi}{\sqrt{1-x_1^2}} \frac{d\varphi_n}{dx_2} \sum_{s=0}^{n-k} C^{(n-k-s)} x_1 x_2^s. \tag{2.16 b}
\]

In the D.T.P. and the I.L.P. one admits logarithmic singularities in the first derivatives of \( \varphi_n \) and obtains

\[
\frac{d\omega_{n_k}}{dx_2} = \frac{2\pi}{\sqrt{1-x_1^2}} \frac{d\varphi_n}{dx_2} \sum_{s=0}^{n-k} C^{(n-k-s)} x_1 x_2^s. \tag{2.16 c}
\]

(1.P.)

\[
\frac{dU_{n_k}}{dx_2} = \frac{2\pi}{\sqrt{1-x_1^2}} \frac{d\varphi_n}{dx_2} \sum_{s=0}^{n-k} C^{(n-k-s)} x_1 x_2^s. \tag{2.16 d}
\]

One can add:

\[
\frac{d\omega_{n_k}}{dx_2} = \frac{2\pi}{\sqrt{1-x_1^2}} \frac{d\varphi_n}{dx_2} \sum_{s=0}^{n-k} C^{(n-k-s)} x_1 x_2^s. \tag{2.16 e}
\]

The relations between the coefficients introduced in the boundary conditions and those in (2.16) can now be determined:

In the D.T.P. and the I.L.P. the relations can be given explicitly (see section 2.10).

In a D.L.P. and an I.L.P., the problems of mixed type, the situation is more complicated. One must solve \( n \) equations with \( n \)-unknown coefficients in the D.L.P., and \( (n+1) \) equations with \( (n+1) \)-unknowns in the I.T.P.
2.9 The expression of the solutions.

The coefficients, introduced in the preceding section uniquely determine the solutions. In the D.P. for instance, the coefficient \( u_{np} \) can be determined by integration from \( r \) to a point \( r' \). Evaluation of all \( u_{np} \) involves the calculation of \( n^2 \) integrals which is very laborious.

In a D.L.P., for instance, this difficulty can be circumvented by the introduction of \( \eta = \frac{r}{r_0} \).

From (2.19) one obtains

\[
\frac{d^n}{dr^n} u_{n,i} = - \frac{1}{c_i} \frac{d}{dr} u_{n,i+1} \quad (2.18a)
\]

and by successive differentiations:

\[
\frac{d^n}{dr^n} Q_n = - \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} \quad (2.18b)
\]

The compatibility relations one obtains

\[
\frac{d}{dr} \frac{d}{dr} u_{n,i+1} = \frac{2}{\pi} \left( \frac{2r_i}{r} \right)^{i-1} \sum_{p=0}^{n} \frac{(-1)^i}{(2r_i-2)^p} \lambda_{np} \lambda_{np}^* \quad (2.19)
\]

The solution at the wing surface then becomes

\[
U_+ = - \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} \quad (2.20a)
\]

\[
\alpha = U_+ = - \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} \quad (2.20b)
\]

The terms in the solutions have thus been arranged with respect to the coefficients which are connected to the boundary conditions.

From equation (2.21) one can deduce a number of recurrence relations so that only three simple integrations need to be carried out. Thus the problem is reduced to a completely algebraic one. The same can be shown for all other elementary problems by arguments analogous to those used in this section. Moreover, Fenain showed in (2) that the results can be unified by the introduction of the function \( \psi_{np} \) with

\[
\psi_{np}(t) = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} \quad (2.21)
\]

To avoid confusion of the functions to be used in the different problems, the following notations are used:

\[
\begin{align*}
\text{D.T.P.} & \quad \psi_{np} = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} & \quad \psi_{np} = \psi_{np} - \psi_{np}^* \\
\text{D.L.P.} & \quad \psi_{np} = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} & \quad \psi_{np} = \psi_{np} + \psi_{np}^* \\
\text{I.T.P.} & \quad \psi_{np} = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} & \quad \psi_{np} = - \psi_{np}^* \\
\text{D.L.P.} & \quad \psi_{np} = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} & \quad \psi_{np} = \psi_{np} - \psi_{np}^*
\end{align*}
\]

Equation (2.23) makes it possible to deduce a number of recurrence relations which allows the evaluation of all the required functions once and for all up to any desired degree of homogeneity. Only a small number of integrations has to be carried out to provide a starting point.

2.10 Summary of the results.

D.T.P. Boundary conditions:

\[
\begin{align*}
\omega_n = \sum_{i=0}^{n-s} \frac{1}{4} \psi_{n-i}^* \psi_{n-i} \quad (2.11a) \\
\text{Solution:} \quad \psi_{n-i} = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} \quad (2.11b)
\end{align*}
\]

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\]

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\text{Solution:} \quad \psi_{n-i} = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} \quad (2.13b)
\end{align*}
\]

D.L.P. This problem is formally equivalent to the D.L.P. if one replaces \( \psi_{n-i} = \psi_{n-i}^* \).

The coefficients \( \lambda_{np} \) follow from

\[
\sum_{p=0}^{n-s} \alpha_p \lambda_{np} = \psi_{n-i} \quad (2.14a)
\]

The I.T.P. Boundary conditions:

\[
\begin{align*}
\omega_n = \sum_{i=0}^{n-s} \frac{1}{4} \psi_{n-i}^* \psi_{n-i} \quad (2.15a) \\
\text{Solution:} \quad \psi_{n-i} = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} \quad (2.15b)
\end{align*}
\]

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\begin{align*}
\omega_n = \sum_{i=0}^{n-s} \frac{1}{4} \psi_{n-i}^* \psi_{n-i} \quad (2.16a) \\
\text{Solution:} \quad \psi_{n-i} = \frac{2r_i}{\pi} \frac{1}{c} \frac{d}{dr} \frac{d}{dr} u_{n,i+1} \quad (2.16b)
\end{align*}
\]
I.L.P. Boundary conditions:

\[ u_{h}^{+} = - \frac{e}{\pi} \sum_{q=0}^{n-1} \frac{c_{q}}{h - q \cdot \frac{2}{\pi}} | \frac{x_{h}}{c_{q}} |. \]  

Solution:

\[ \omega_{p} = \frac{2}{\pi} \sum_{q=0}^{n-1} \frac{c_{q}}{h - q \cdot \frac{2}{\pi}} \sum_{p=1}^{h} h_{p}^{+}, \]  

with

\[ a_{p}^{+} = \sum_{q=0}^{n-1} \frac{c_{q}}{h - q \cdot \frac{2}{\pi}} \sum_{p=1}^{h} \frac{(-1)^{p-q}(p-1)!(h+p-2)!}{(2q+2)(q+2)!} \sum_{e=1}^{h} \frac{(-1)(2q+2)(q+2)!}{(2q+2)(q+2)!}. \]  

The coefficients \( a_{p}^{+} \) and \( a_{p}^{+} \) are given in [1].

III Wings with curved subsonic leading edges

3.1 Introductory remarks

If the leading edges of the wing are curved, as in the case of gothic and ogee planforms, the flow cannot in general be represented as a superposition of homogeneous flows. In the previous outline of homogeneous flow theory it was seen that discontinuities and singularities occur at the leading edges. It is impossible, therefore, to write the solution in a form \( \sum_{p=1}^{h} \frac{c_{q}}{h - q \cdot \frac{2}{\pi}} \sum_{p=1}^{h} h_{p}^{+} \) with \( a_{q}^{+} \) as the solution for a delta wing with straight leading edges, in some way "close" to the actual leading edges, and \( c \) as a small parameter representing a measure for the "deviation from straight". A solution of that type would possess singularities and discontinuities not on but only near the leading edges. A possibility to circumvent these difficulties is the introduction of a transformation shifting all points of the curved leading edges to straight lines through the origin. Moreover it will be required that the Mach cone through the oring remains a straight circular cone. The boundaries where boundary conditions can be imposed are then of precisely the same conical nature as those occurring in homogeneous flow theory. In the next sections it will be shown that the transformed differential equation and the transformed boundary conditions can be satisfied in terms of functions which are solutions of homogeneous flow problems.

3.2 Straightening the leading edges

The wing planforms considered will be symmetric with respect to the \( x \)-axis and the subsonic leading edges are given by

\[ \mathcal{F}(x, y) = 0, \quad (x = 0), \]  

\( \mathcal{F} \) being a rational function.

In [6] transformations in the form of general three dimensional power series are introduced as follows:

\[ x_{1} = x + \sum_{i+j+k=2}^{(1)} a_{i,j,k} x^{i} y^{j} z^{k}, \]

\[ x_{2} = y + \sum_{i+j+k=2}^{(2)} b_{i,j,k} x^{i} y^{j} z^{k}, \]

\[ x_{3} = 2 + \sum_{i+j+k=2}^{(3)} c_{i,j,k} x^{i} y^{j} z^{k}. \]  

In [6] these transformations are discussed systematically. The leading edges can be transformed into straight lines in the \((x, x_{1}, x_{2})\) space \( |x_{1}| = \frac{c}{\beta} x_{1}, (x_{2} = 0) \), without deforming the Mach cone.

The requirements concerning the straightening of the leading edges, without deforming the Mach cone, reduce the number of arbitrary coefficients in (3.2) but do not uniquely determine any one of them.

The requirement to retain, in some, degree the symmetry of the flow field reduces the number of arbitrary coefficients still further but not sufficiently to determine them uniquely. In [8] it is shown, that, in fact it is sufficient to stretch the \( x \)-coordinate only. An example is the transformation:

\[ x_{1} = x + \frac{\alpha \cdot \beta}{\gamma_{1}} (y^{2} + z^{2}) \]

\[ x_{2} = y - \frac{\gamma_{2}}{\gamma_{1}} (x + \gamma_{2} x^{2}) \]

\[ x_{3} = 2. \]
The formulae (3.4) straighten leading edges of the form

\[ |y| = c x + f(x), \quad (x = 0) \]

With \( f(x) = \sum a_i x^i \), (3.4) can be written in the form

\[ (i - \beta^2 x^2)^2 > \beta^2 f(x) \frac{d}{dx} \gamma_x + f(x), \quad (3.4) \]

Other possibilities and relations between the leading edges and the transformations are discussed in [6].

3.3 The transformed equation

If the \( x \)-coordinate alone is stretched the transformed differential equation becomes:

\[ \psi_{x,x} \left( \frac{\partial^2 (\beta x)}{\partial x^2} \right) - \left( \frac{\partial^2 (\beta x)}{\partial y^2} \right)^2 - \psi_{x^2, x^2} = 0 \]

3.4 Solutions of the transformed equation (\( \Phi \))

This equation admits solutions involving sums of solutions

\[ \beta \psi_{x,x} - \psi_{x^2, x^2} = 0, \quad (3.6) \]

multiplied by power series in \( x \), and \( \beta^2 (x^2 + \lambda^2) = \beta^2 \cdot \cdot \cdot \)

\[ \psi = \sum_{n=0}^{\infty} \sum_{l=0}^{n} a_{l, n} \left( x^2 + \lambda^2 \right)^{n} \sum_{i+j} \alpha_{i,j} \lambda^i \beta^j, \quad (3.7) \]

in which \( \Phi \) is a solution of (3.6).

Substitution of (3.7) into (3.5) permits a (\( p, \alpha \)) unique determination of the coefficients \( \psi_{i,j} \)

and the homogeneous flow solutions \( \psi(x, x, x) \)

3.5 Solutions of the transformed equation (\( \psi \))

Another approach to construct solutions of the transformed differential equation (3.5) consists of the following argument:

Consider a function \( \psi_n(x, y, z) \), homogeneous of degree \( n \) in \( x \), \( y \) and \( z \), which is a solution of

\[ \beta^2 \psi_{x,x} - \psi_{y,y} = \psi_{x,x} = 0. \quad (3.8) \]

The new coordinates \( (x, x, x) \) can be introduced by an inverse transformation of the form

\[ \begin{cases}
 x = x_1 + (x_2 - \beta \rho^2) \frac{\varphi}{\psi}, \\
 y = x_1, \\
 z = x_3 \end{cases} \quad (3.9) \]

Without actually calculating the transformed differential equation one obtains solutions of the transformed equation by introducing \( x, x_2 \) and \( x_3 \) into \( \psi_n(x, y, z) \). Expanding \( \psi_n(x, y, z) \) in terms of the new coordinates gives:

\[ \psi_n(x, y, z) = \psi_n \left( x_1 + (x_2 - \beta \rho^2) \frac{\varphi}{\psi}, \beta \rho^2 \right) \frac{\psi}{\psi}, \quad (3.10) \]

Because \( \psi_n(x, y, z) \) is a solution of (3.8), \( \psi_n(x_1, x_2, x_3) \) is a solution of

\[ \beta^2 \psi_{x_1, x_2} - \psi_{x_1, x_2} - \psi_{x_1, x_3} = 0. \quad (3.11) \]

The derivatives \( \frac{\partial \psi}{\partial x} \) are also solutions of (3.11). The solutions (3.10) can be shown to be equivalent to (3.7).

If the transformation (3.9) straightens a class of leading edges, the solutions (3.10) can be used to solve the transformed problem. If a homogeneous function \( \psi_n(x_1, x_2, x_3) \) admits a singularity in a point \( (x_1, x_2, x_3) \) then the same type of singularity occurs for all points \( (\lambda x_1, \lambda x_2, \lambda x_3) \).

Thus the solutions (3.10) can be used to locate the singularities in the solutions at the transformed leading edges.
It is easily verified that the inverse transformation (3.9) straightens leading edges of the form:
\[ \chi = \left[ \frac{y^2}{x} \right] \left( -\frac{\partial^2}{\partial \chi^2} - \frac{\partial^2}{\partial \chi^{\prime 2}} \right) \frac{y}{x} \frac{\partial^2}{\partial \chi^{\prime 2}} = D. \]
The expression (3.12) can be put in the form (3.12a)
\[ \frac{y^2}{x} \left( \frac{\partial^2}{\partial \chi^2} - \frac{\partial^2}{\partial \chi^{\prime 2}} \right) \frac{y}{x} \frac{\partial^2}{\partial \chi^{\prime 2}} = D. \]

The too strong singularities can be partly removed as follows:
If \( \psi_n \) is a solution of (3.11), homogeneous of degree \( n \) in \( \chi, \chi', \chi^2 \), a solution of (3.11), homogeneous of degree \( (n+1) \), is obtained by
\[ \psi_{n+1} = -B n + \int \psi_n \chi' + (\chi^2/\beta^2) \psi_n \chi. \]
The singularity in \( \psi_{n+1} \) is stronger than in \( \psi_n \)
if there is one. (3.15) is a recurrence relation and can be used to generate solutions that can be substituted into (3.10). They can be multiplied by constants, chosen in such a way that the strongest singularities are removed. In general, however, too strong singularities remain present and must be removed by another technique, based on a detailed study of the behaviour of the functions \( f(x, y, t) \) near the leading edges, defined in equation (2.23).

The boundary conditions at the Mach cone present no difficulties and are automatically satisfied if the \( \psi_n(\chi, \chi', \chi^2) \) are homogeneous flow solutions (the \( \chi^{(k)} \) derivatives being zero on \( \partial \)).

3.7 Concluding remarks

In the preceding sections it was seen that the transformed equation admits solutions in which the homogeneous flow solutions occur in a simple way. Orders of approximation can be associated with the number of terms included.

The terms in the solutions can be arranged with respect to ascending degrees of homogeneity and solved successively by the same techniques as those applied in Témain's theory. Up to a certain order of approximation, the problem can be reduced to a completely algebraic one. In these problems the parameters defining the leading edges can be made to occur in the same way as those defining the boundary conditions.

IV Applications

4.1 Slightly curved leading edges

Leading edges of the form
\[ y = -\frac{c}{\alpha_0} \chi + \epsilon \frac{\partial}{\partial \chi} \left( \chi = 0 \right), \]
with \( f(x) = \sum a_i \chi^i \) are straightened by the transformation
\[ x_1 = x + \epsilon \frac{\chi^2}{\chi^{(2)}} \frac{\partial}{\partial \chi}, \]
\[ y = \epsilon \frac{\chi^2}{\chi^{(2)}} \frac{\partial}{\partial \chi}. \]

For \( 0 < \alpha \leq 1 \), with fixed coefficients \( a_i \), the leading edges will be slightly curved for sufficiently small \( \epsilon \).

The inverse of (4.2) is, including terms of \( O(\epsilon) \):
\[ x = x_1 - \epsilon \frac{\chi^2}{\chi^{(2)}} \frac{\partial}{\partial \chi^2} + O(\epsilon), \]
\[ y = x_2, \]
The solution of the transformed problem can be put in the form:

$$\psi = \psi_0 + \varepsilon \psi_1.$$  \hfill (4.5)

As an example, consider a D.L.P.

The boundary conditions can be put in the form

$$\frac{d}{d s} \psi_j = \sum_{s_0}^{s_1} \psi_j \frac{\partial \psi}{\partial s} \bigg|_{s=s_0}^{s=s_1},$$  \hfill (4.6)

and the leading edges in the form:

$$\gamma_j = \gamma_0 + \varepsilon \gamma_1.$$  \hfill (4.7)

If $$\psi$$ will be homogeneous of degree $$j$$ and we write

$$\psi_j = \psi_0 + \varepsilon \psi_1.$$  \hfill (4.8)

If $$\psi$$ will be homogeneous of degree $$(i+j-1)$$ and we write

$$\psi_j = \psi_0 + \varepsilon \psi_1.$$  \hfill (4.9)

For the flat plate solution, or the incidence dependent part of the solution, one finds:

$$\psi_j = -\frac{2 \pi}{7} \gamma_j \frac{d^j}{d s^j} - \sum_{i=1}^{j} \lambda_i d_j^i (i) \frac{d^i}{d s^i}.$$  \hfill (4.10)

For Wings at ideal angle of attack

The solutions without square root singularities in the velocity field can be obtained for $$i > j$$ by taking $$\psi_j = 0$$ in (4.8). From (4.13) one has

$$\psi_j = \frac{2 \pi}{7} \gamma_j \frac{d^j}{d s^j} + \sum_{i=1}^{j} \lambda_i d_j^i (i) \frac{d^i}{d s^i}.$$  \hfill (4.11)

Wings at ideal angle of attack

The solutions that can be constructed in this way may be interesting for the design of wings under cruising conditions.

For all other types of boundary value problems similar expressions can be derived [6].
4.2 Comparison with other results.

4.2.1 Consider a flat plate solution with leading edges. \[ \|y\| = c + e \alpha_1 x^2. \] (4.19)

The solution can be expressed as follows:

\[ \psi(x, y, \theta) = \frac{1}{\omega} \left[ (c + e \alpha_1 x^2) \right] \left( \frac{1}{E'} + e \alpha_1 \beta \rho x' + \beta^2 f' \right), \] (4.20)

with \( \frac{p}{p} = \frac{2E'(\kappa^3 - \lambda_1 x')}{(\kappa^2 - \kappa_1^2)E' \left( \left( -2 \kappa_1 \right) E' + \kappa_1^2 K' \right)}, \) (4.21)

with \( \kappa = \beta c'. \)

\( E' \) and \( K' \) have modulus \( \sqrt{1 - \alpha_k^2}. \) The factor \( p \) has been plotted and compared with other results:

![Graph showing comparison with other results](image)

In the limit \( k = 1, \) the leading edges are nearly sonic. The factor \( \frac{p}{p} \) approaches the corresponding factor extracted from the first reflection integral, as it should. The slender body theory predicts \( \frac{p}{p} = 0 \) in the limit \( k \to 0. \) The not-so-slim body theory predicts values close to \( \frac{p}{p} \), for small values of \( k. \) Carafoli's result is valid for \( k = 0 \) only.

4.2.2 Wing at ideal angle of attack.

The formula derived for the D.L.P. can be used to construct wings which sustain an interesting form of pressure distribution. In these cases the integral representation of the upwash field permits an easy numerical evaluation. \( (\beta = 1) \)

Wings with a planform \( \{y\} \) \( \delta, \theta \), \( \alpha \), \( \gamma^2 \), \( \left( \alpha x \times \alpha \right) \) are considered. The pressure distribution is prescribed in the form

\[ \psi_x = (D_{+} + D_{-} x) \sqrt{\omega x - \alpha \times \gamma} \psi_2 \] (4.22)

The corresponding \( \psi_z \) is calculated in the form

\[ \psi_z = \lambda_1 x + \lambda_2 y^2 + \lambda_3 x^2 y + \lambda_4 x y^2 + \lambda_5 y^4 \cdots (4.23) \]

The analytic expressions for the coefficients have been calculated including those associated with the third and fourth degree terms. For \( x = 0.4; \ x = 0.6; \ x = 0.8; \ x = 1.0 \), \( \|z\| \) is plotted for two cases:

\[ \begin{cases} D_{-} = 1, \quad D_{+} = 0, \quad (4.24) \\ D_{-} = 1, \quad D_{+} = -1. \end{cases} \]

The deviation of straight is not small and so for \( x > 0.6 ... \) requirement (3.41) is not satisfied. Solutions in the form (3.7) are used.

![Graph showing pressure distribution](image)
In this case the deviation of straight is not small. Near \( y = 0 \) there is good convergence. Near the tips in case (i), more terms are required. In case (ii) the convergence is very rapid. This may be due to the fact that there is little loading of the wing near the tips. The accuracy of the numerical approximation was found to be good, except near the leading edges, by comparison with exact homogeneous flow solutions. The accuracy and the speed of convergence does not only depend on the leading edges but also on the boundary conditions.

4.3. Another possibility which should be mentioned is the following. In (3.10) one may take into account only two coefficients, say \( g_{20} \) and \( g_{30} \), and calculate the solutions including terms of say \( O(\alpha^4) \). This leads to a workable scheme for a large class of planforms \([6]\). It can be expected to be sufficiently accurate to obtain some qualitative insight. The calculation of the terms of \( O(\alpha^4) \) can be carried out along the same lines that led to the terms of \( O(\alpha^2) \).

References.


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