POST-BUCKLING COMPUTATION OF LARGE STIFFENED STRUCTURES

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Abstract

Non-linear numerical analysis of large stiffened structures subjected to local buckling has a computational cost that prevents its achievement, especially in an industrial framework. This paper deals with strategies to enable such challenging computation. Existing domain decomposition and model order reduction methods, with their contribution to post-buckling analysis, are first recalled. A domain decomposition strategy with non-linear reduced localization for post-buckling analysis is then proposed in order to address large refined models without compromising computational performance. A comparative study is finally presented in the case of flat stiffened panel and the computing gains are discussed.

1 Introduction

One of the main non-linearities that arise in aeronautical structures such as airplane fuselages is the local buckling of the skin between stiffeners. Whereas this phenomenon does not lead to the global failure, the non-linear response implies a stress redistribution in the structure. Therefore the detection of the local damages is linked to the prediction of local post-buckling. The debonding of the stiffeners is for instance one of the local damages that may occur as a consequence of local post-buckling [3].

The finite element (FE) analysis of the large stiffened structures subjected to local post-buckling and the evaluation of the local damages criterion [4] requires a refined modeling, resulting in a high number of degrees of freedom (d.o.f.). In addition, the non-linear behavior increases the cost of the solution procedure, thus going beyond the capacity of the hardware resources in terms of data storage and/or of allowable CPU time.

On one hand, the buckling phenomenon is partially predictable in the sense that eigenvalue analysis supplies the critical load and the approximated shape of each buckling mode. On the second hand, the localness of the post-buckling of large stiffened structures concentrates the main non-linearities in some areas. These two characteristics led engineers and researchers to the development of computational strategies based on either the use of the knowledges of the phenomenon or its localness. Classical global-local approaches enable the computation of refined models but the complex interactions between the local and global scales prevent their use without compromising the accuracy of the solution. For this reason this paper focuses on the computational strategies that guarantee the accuracy of the solution.

In a first section, the existing computational strategies for post-buckling analysis of large structures are presented. Attention is paid especially to those based on domain decomposition and model reduction techniques which principles are recalled to make this paper self-contained.

From an analysis of the main weaknesses of the aforementioned strategies, a new one is proposed in a second section.

In the last section, behaviour and performances are investigated in the application of the strategy to a flat stiffened panel.

2 Post-buckling of large stiffened structures: Existing computational strategies

Recent works on analytical or semi-analytical methods for the study of post-buckling of stiffened panels [17, 5, 16] allow efficient preliminary design phases. Nevertheless the needs for accuracy and flexibility in the sizing phases of aircraft structures lead to the use of non-linear finite element models. For this reason this paper focuses on the strategies relating to the solution of finite element discretized problems. The finite element modeling of large stiffened structures results in the following equilibrium equation:

$$\mathbf{K}(\mathbf{q})q = F_{ext} \tag{1}$$

In the case of non-linear behaviour, like local post-buckling, the equilibrium is linearized and solved iteratively according to the Newton-Raphson solution algorithm [8]:

$$\mathbf{K_T}(\mathbf{q})\Delta q = R \tag{2}$$

As the number of d.o.f. increases, the direct solution of the system (2) becomes more expensive. Contrary to the global-local approaches, model reduction techniques and domain decomposition methods are two strategies that enable the computation of an accurate solution of large non-linear finite element problems. A brief description of these strategies is provided in this section.

2.1 Reduced Order Modeling

Model reduction techniques are widely used in the framework of modal analysis [6]. Although they have been introduced in the field of solution of linear and non-linear mechanical problems in the late 70's [14], model reduction techniques are the subject of more recent works [12, 18].

The principle relies on the projection of the unknowns on a reduced subspace. Let $\mathbf{C} = \left[\begin{array}{ccc} C_1 & \dots & C_n \end{array}\right]$ be a reduced basis of n vectors in a solution space of dimension N. The vector of unknowns q is approximated by a linear combination of the $C_{i\in[1:n]}$:

$$\tilde{q} = \mathbf{C}\eta \tag{3}$$

The tangent system (2) is then projected on $\mathbf{C}$:

$$\mathbf{C^T K_T C}\Delta\eta = \mathbf{C^T}R \tag{4}$$

Solving the large system is thus replaced by the projection and the solution of a reduced system which have a smaller computational complexity.

A given reduced basis is adapted to the problem when a linear combination of the vectors reproduces accurately the solution of the discretized equilibrium equation (1). Otherwise the following statement is false despite the convergence of the algorithm:

$$\frac{||R(\tilde{q})||}{||F_{ext}||} < \varepsilon \tag{5}$$

Therefore the choice of the reduced basis is challenging and it has been tackled since the early stage of the model reduction techniques.
In the special case of the post-buckling analysis, the constitutive displacement modes of the reduced basis that can be found in the literature are the following ones:

- the *buckling modes* calculated within an eigenvalue analysis [12, 13].
- the *path derivatives* or higher order derivatives of the solution with respect to a path parameter (applied load or displacement)[15, 12].

Although good agreement to the approximation of post-buckled solutions could be shown in some cases, the reduced basis build in this way does not lead to the fulfillment of the error criterion. That is the reason why, the reduction of non-linear models required the development of strategies for adapting the aforementioned reduced basis to the non-linearities [14, 1, 12].

This is what the "on the fly" completion recently developed by Kerfriden and al. [11] aims at. Contrary to the strategy developed by Kling and al.[12] which performs a full Newton-Raphson increment in the case of an unverified error criterion, the "on the fly" completion makes the most of the projected conjugate gradient algorithm during the increment. The principle of this procedure is recalled for better understanding.

The "on the fly" completion aims at controlling both the full problem error (5) and the reduced problem error (6).

$$\frac{||\mathbf{C^T}R(\tilde{q})||}{||\mathbf{C^T}F_{ext}||} < \varepsilon_{reduced} \tag{6}$$

During the reduced Newton iterations, if the reduced problem error become much smaller (k times) than the full problem error (7), the next prediction step (2) is performed by the mean of a projected conjugate gradient.

$$\frac{||\mathbf{C^T}R(\tilde{q})||}{||\mathbf{C^T}F_{ext}||} < k \times \frac{||R(\tilde{q})||}{||F_{ext}||} \tag{7}$$

Equation (8) defines a projector onto $Im(C)^{\perp K_T}$. The idea is to separate the search space into two subspaces $Im(C)$ and $Im(C)^{\perp K_T}$ (9).

$$\mathbf{P} = \mathbf{I} - \mathbf{C}\left(\mathbf{C}\mathbf{K_T}\mathbf{C}^T\right)^{-1}\mathbf{C}^T\mathbf{K_T} \tag{8}$$

$$\begin{cases} \Delta q = \mathbf{C}\Delta\eta + \Delta q_K \\ \Delta q_K \in Im(C)^{\perp K_T} \end{cases} \tag{9}$$

The prediction step is thus replaced by the uncoupled equations (4) and (10).

$$(\mathbf{K_T}\mathbf{P})\,\Delta q_K = R - \mathbf{K_T}\mathbf{C}\Delta\eta \tag{10}$$

The solution of (10) is orthogonal to the reduced basis and can be added to it after normalization. This completion can occur several times until convergence of both the reduced and the full problem errors.

In the case of local non-linearities, it is worth noting that the entire reduced basis is completed while only local modifications would be sufficient. This seems to be one of the of relevant development path for improving the performances of the adaptive reduced basis strategies.

2.2 Domain decomposition methods

Since the basic idea of domain decomposition methods is to spread computational costs over several processor units, their development is linked to the advent of parallel computers. The structure is decomposed into non-overlapping sub-domains and kinematic (resp. static) continuity is ensured by conditions on displacement (resp. force) unknowns over the boundaries [10]. This decomposition should be done such that calculation cost is balanced between processors in order to optimize parallel computation.

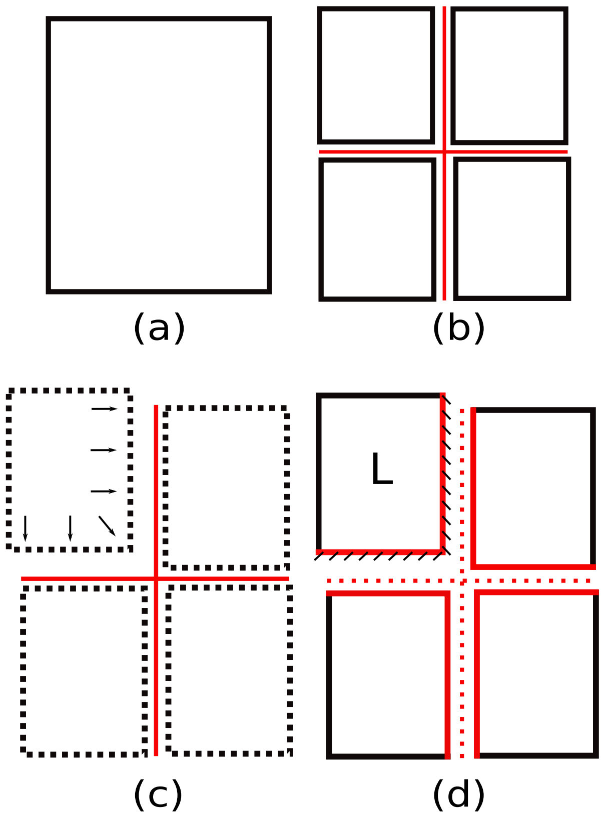

Fig. 1 Steps of the domain decomposition method (in the case of a primal approach). (a) Initial structure, (b) decomposition of the structure in 4 sub-domains and boundary, (c) condensation of the sub-domain and solution of the boundary unknowns, (d) linear localization of the boundary displacements on each sub-domain

The main steps for the resolution of the linear system of each Newton iteration (2) are presented in the case of a primal approach, when displacement continuity over the boundaries is verified a priori. Fig.1 represents it schematically while details are given in the following enumeration (The superscript s refers to a sub-domain identifier and the subscript b (resp. i) indicates the boundary (resp. internal) d.o.f. of a subdomain).

1. For each sub-domain (local operation), condensation of the internal d.o.f. for the tangent operator and the residual:

$$\begin{aligned} \mathbf{S}^{\mathbf{s}}_{\mathbf{T}} &= \mathbf{K}^{\mathbf{s}}_{\mathbf{T}bb} - \mathbf{K}^{\mathbf{s}}_{\mathbf{T}bi}{\mathbf{K}^{\mathbf{s}}_{\mathbf{T}ii}}^{-1}\mathbf{K}^{\mathbf{s}}_{\mathbf{T}ib} \\ r^s &= R^s_b - \mathbf{K}^{\mathbf{s}}_{\mathbf{T}bi}{\mathbf{K}^{\mathbf{s}}_{\mathbf{T}ii}}^{-1}R^s_i \end{aligned} \qquad (11)$$

2. Resolution of the condensed linearized problem over the boundary unknowns (global operation):

$$\Delta q_b = {\mathbf{S}_{\mathbf{T}}}^{-1} r \qquad (12)$$

3. Localization of the boundary displacement in each sub-domain (local operation):

$$\Delta q^s_i = {\mathbf{K}^{\mathbf{s}}_{\mathbf{T}ii}}^{-1}(R^s_i - \mathbf{K}^{\mathbf{s}}_{\mathbf{T}ib}\Delta q^s_b) \qquad (13)$$

The local operations are independent per sub-domain. The global system is usually solved by iterative parallel solvers of Krylov type which do not require assembling condensed local operators. Details on Krylov solver for domain decomposition methods can be found in the literature [9].

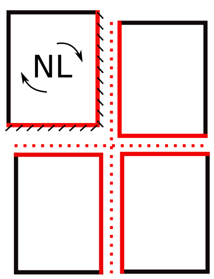

Fig. 2 Non-linear localization

However Cresta and al. [7] showed that the classical domain decomposition methods had poor convergence properties in presence of local nonlinearity. Indeed the most critical local phenomenon controls the global convergence. Then they proposed a strategy that differentiates the treatment of the local nonlinearities. An iterative solver is introduced in the place of the localization solver (see fig.2). The idea is to localize the increment of boundary displacement of a sub-domain as if it were the boundary conditions of a particular non-linear problem.

$$\Delta q^s_i = {\mathbf{K}^{\mathbf{s}}_{\mathbf{T}ii}}^{-1}(q)(R^s_i(q) - \mathbf{K}^{\mathbf{s}}_{\mathbf{T}ib}(q)\Delta q^s_b) \qquad (14)$$

A local convergence criterion ε_L is defined as parameter for the method. A parametric study [7] showed that a compromise should be found between a coarse local convergence criterion (leading to high number of global iteration) and a fine one (leading to a high number of local iterations with a stagnation of the number of global iterations).

The non-linear localization algorithm was evaluated for the analysis of a frame structure under bending load [7]. Local buckling of constitutive beams occurred. The table 1 summarizes the convergence results of both classical domain decomposition and non-linear localization algorithms. It shows an important gain in terms of number of global and local computations. As a consequence, the computation time is reduced.

Strategy	loading steps	Number of global iterations	Number of local computations
a	8	63	630
b	6	35	470

Table 1 Convergence results: comparison between (a) conventional domain decomposition algorithm and (b) domain decomposition algorithm with non-linear localization [7]

3 Domain decomposition with non-linear reduced localization: Coupling domain decomposition and model reduction

Non-linear localization improves performance of domain decomposition in the case of local nonlinearities by the transfer of computation cost from global to local level. Thus the local operations become overriding and further improvements may come from local model reduction. Especially in the case of post-buckling analysis of stiffened structures where refined model is necessary for taking into account the structural details and physical phenomena.

3.1 Principles

The proposed strategy relies on the coupling of domain decomposition with non-linear localization and a model reduction technique. It aims at reducing the costs of all the local operations in a domain decomposition procedure such as condensation and localization.

At the local level, both condensation and localization are performed in a reduced subspace. Equations (11) and (14) become respectively (15) and (16).

$$\begin{aligned} \mathbf{S^s_T} &= \mathbf{K^s_{T}}_{bb} - \mathbf{K^s_{T}}_{b\eta}\mathbf{K^s_{T}}_{\eta\eta}^{-1}\mathbf{K^s_{T}}_{\eta b} \\ r^s &= R^s_b - \mathbf{K^s_{T}}_{b\eta}\mathbf{K^s_{T}}_{\eta\eta}^{-1}R^s_\eta \end{aligned} \qquad (15)$$

$$\Delta\eta^s = \mathbf{K^s_{T}}_{\eta\eta}^{-1}(R^s_\eta(q) - \mathbf{K^s_{T}}_{\eta b}(q)\Delta q^s_b) \qquad (16)$$

with

$$\begin{aligned} \mathbf{K^s_{T}}_{\eta\eta} &= \mathbf{C^T}\mathbf{K^s_{T}}_{ii}\mathbf{C} \\ \mathbf{K^s_{T}}_{\eta b} &= \mathbf{C^T}\mathbf{K^s_{T}}_{ib} \\ \mathbf{K^s_{T}}_{b\eta} &= \mathbf{K^s_{T}}_{bi}\mathbf{C} \end{aligned} \qquad (17)$$

and

$$R^s_\eta = \mathbf{C^T}R^s_i \qquad (18)$$

In this manner, the interior part of the local tangential operators is not factorized. From the standpoint of parallel computation, the improvement of load balancing between sub-domains is derived from the local models reduction to approximately the same number of unknowns (subspace coordinates).

3.2 Simple local reduced basis for post-buckling analysis and interactions between sub-domains

A local reduced basis is proposed for post-buckling analysis from the following statements:

- no experience on the model has been acquired from previous calculation.
- although good accuracy is required, the cost of the initial reduced basis constitution must not minimize excessively the gains resulting from the model reduction.
- an adaptive procedure may ensure the accuracy of the model reduction and compensate the lack of a priori knowledges.

On this basis, the proposed local reduced basis is composed of the following easily computed displacement modes:

- the unbuckled solution of the fundamental equilibrium path which is either calculated by a linear analysis or extracted from the solution of the first increment.
- the firsts buckling modes which are calculated within an eigenvalue analysis after the first increment converged. Only the modes whose critical load may be reached in the range of loading ($Fc < F_{ext}$) are kept in the reduced basis.

The idea is that these displacement modes are the main working displacement modes and can be completed by some low-energy displacement modes to reach the buckled equilibrium. Indeed in the case of a plate in compression, because of stress redistribution the out-of-plane displacements and the in-plane displacements are not accurately approximated by a combination of buckling modes and the solution of in-plane compression . Therefore a completion procedure is added to the strategy at a local level.

A previous work submitted for publication [2] demonstrated the relevance of this simple adaptive reduced subspace on a simple plate under shear and compression loading without domain decomposition. The gain in CPU time reaches 90% for refined meshes.

3.3 “On the fly” completion of the local subspaces

As described in the previous subsection, a completion of the local reduced subspaces may be necessary. For this reason, “on the fly” completion is introduced at the local level, which is consistent with the prospects of [11]. It is indeed worth noting that the reduced subspace of the whole model is intrinsically split on the subdomains and can be independently completed, thus saving computational resources.

The completion occurs during the non-linear localization step. The local convergence test in the reduced subspace (19) determines whether the next prediction step of the Newton-Raphson procedure is solved in the reduced subspace only or also in the complementary subspace [11] so that the local reduced basis is enriched (see section 2.1)

$$\frac{||R^s_\eta(\tilde{q})||}{||\mathbf{C}^\mathbf{T} F^s_{ext}||} < k \times \frac{||R^s(\tilde{q})||}{||F^s_{ext}||} \quad (19)$$

3.4 The entire procedure

This subsection summarizes the procedure of domain decomposition with non-linear reduced localization for post-buckling analysis into a flowchart fig.3.

It is interesting to note that other strategies for the computation of different local nonlinearities derive by the modification of the local subspaces initialization.

4 Implementation and performance of the strategy

4.1 Preliminaries

The domain decomposition strategy with non-linear reduced localization for post-buckling analysis was implemented in a research finite element code developed by the authors of this paper. The code is written in python which is a scientific object oriented programming language and makes the most of a well known numerical linear algebra library (LAPACK). This work was performed using High Performance Computing (HPC) resources from CALMIP (http://www.calmip.cict.fr).

In the performance study, the parameters were chosen empirically. Further work should demonstrate the influence of each local parameter (k,ε_L). Furthermore, as Noor and al. [14] recommended for the reduced model techniques, the global error threshold ε_G is set to 5.10^{-2}.

4.2 Local post-buckling of a stiffened panel

In order to give some insight into the performances of the strategy for industrial applications, the case study takes inspiration from the works led by Bertolini and al. [4] on stiffeners debonding of an aircraft panel due to local post-buckling. Fig.4 describes the case study. The stiffening of the plate is realized by constraining out-of-plane and rotational d.o.f.. The structure is divided into four sub-domains. It can be seen that the stiffeners are not symmetrically positioned so that the sub-domain number 1 buckles first and no global buckling mode appears in the range of loading.

The structure is discretized by quadratic quadrangular finite elements and transverse shear is taken into account in the model. The computation of the model is carried out with two levels of mesh refinement in order to emphasize a possible dependence to the number of d.o.f..

4.3 Analysis of the results

Since the first increment of the proposed strategy is not reduced, the result analysis focuses on the second increment in which the sub-domain 1 buckles. The table 2 summarizes the convergence results of the compared strategies.

Strategy	Global iterations	Local iterations	Subspace completions
Mesh 1: 20 by 20 elements, 2000 d.o.f.			
a	3	39	-
b	5	44	17
Mesh 2: 40 by 40 elements, 8000 d.o.f.			
a	6	53	-
b	5	51	23

Table 2 Convergence results. (a) Domain decomposition with non-linear localization, (b) Domain decomposition with non-linear reduced localization for post-buckling.

The convergence properties are not significantly affected by the reduction of the non-

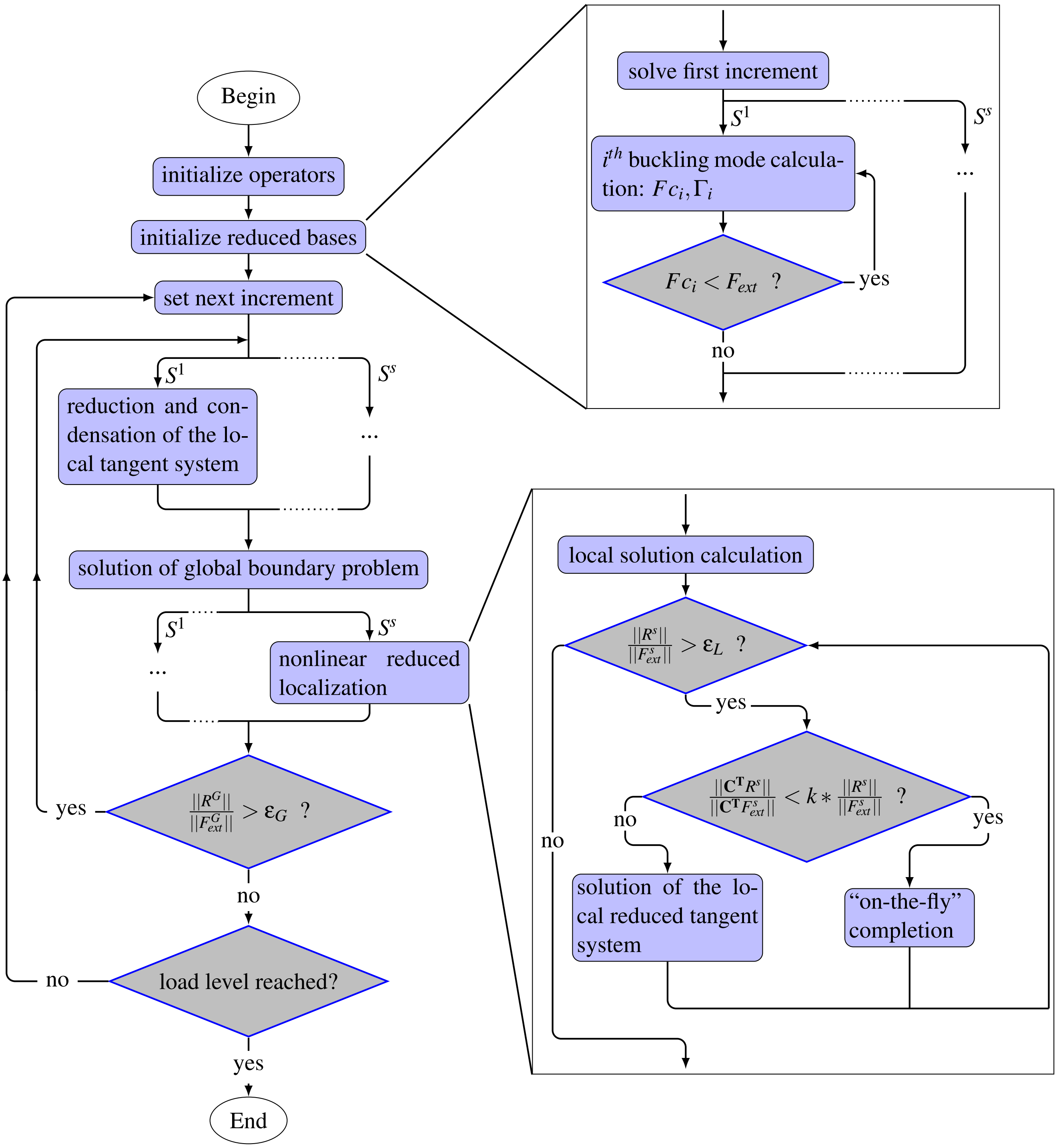

Fig. 3 Flowchart of the procedure. S^i refers to the i^{th} subdomain.

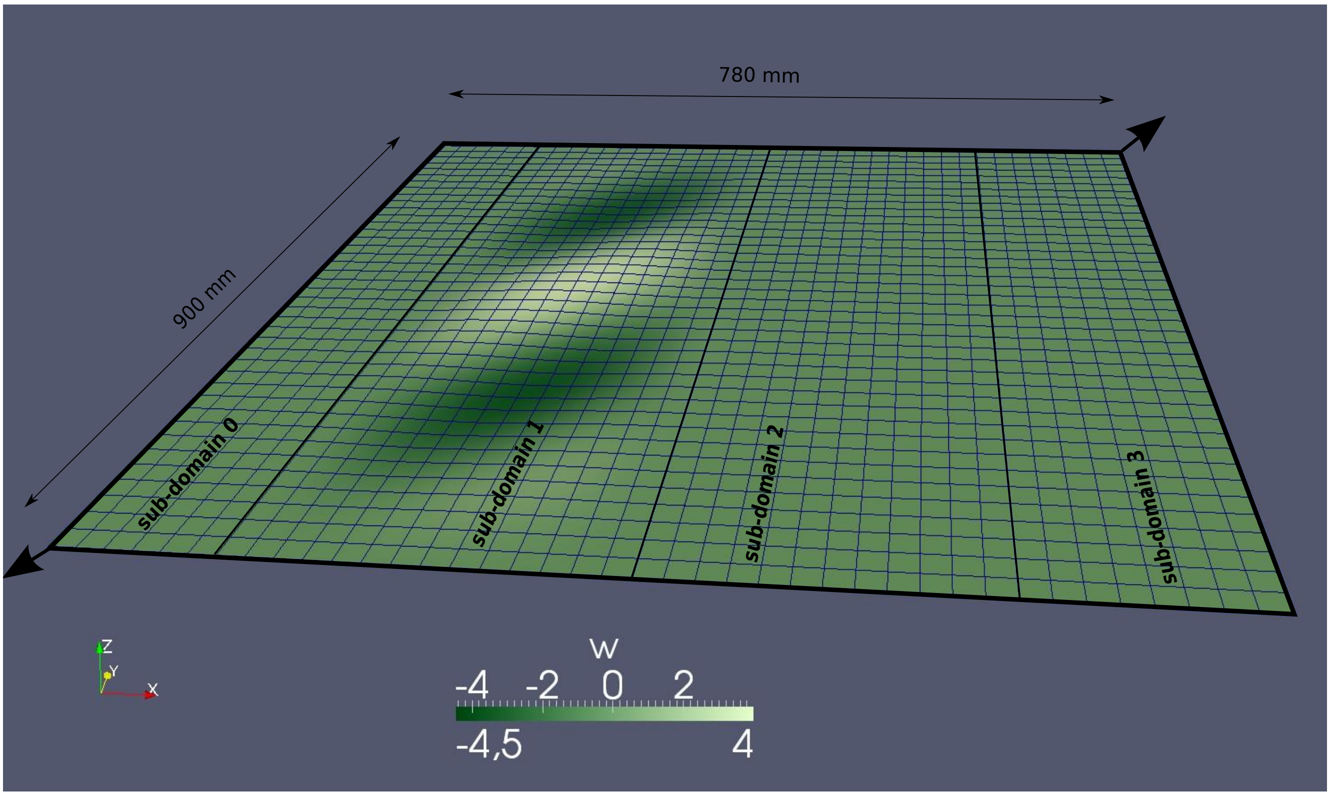

Fig. 4 Stiffened panel: Local post-buckling in the case of a shear test. Out-of-plane displacement in millimeter.

linear localization neither by the mesh refinement. However, tab.3 shows that the CPU times of the condensation is reduced by up to a factor 10.

Strategy	Localization	Localization with completion	Condensation
Mesh 1: 20 by 20 elements, 2000 d.o.f.			
a	0.44	-	1.6
b	0.012	0.6	0.32
a/b	36	0.73*	5
Mesh 2: 40 by 40 elements, 8000 d.o.f.			
a	6.9	-	34.4
b	0.23	11.7	3.28
a/b	30	0.6*	10.5

Table 3 CPU time (in seconds) per local operations of sub-domain 1.(a) Domain decomposition with non-linear localization, (b) Domain decomposition with non-linear reduced localization for post-buckling. *comparison between localization of strategy a and localization with completion of strategy b.

In the case of localization without completion, the CPU time is also strictly minimized but these gains may be canceled if too many completions are required. On the first hand, the ratios of localization CPU times are stable while the mesh refinement increases. On the other hand, the ratio of CPU times of condensations increases with the mesh refinement.

Tab.4 confirms the relevance of the proposed simple subspace. While almost half of the localization iterations achieves a completion of the subspace, the gain remains about 50 percent.

Strategy	Mesh 1	Mesh 2
a	89.6	1066
b	50.1	504
Gain	44%	53%

Table 4 Total CPU time (in seconds) and gain (in percent) for local computations of sub-domain 1.(a) Domain decomposition with non-linear localization, (b) Domain decomposition with non-linear reduced localization for post-buckling.

Lastly, an increase in the gain with the number of d.o.f. is outlined. This is consistent with the assumption that the proposed strategy is designed to deal with refined models.

5 Conclusion and perspectives

In this paper, a strategy coupling domain decomposition and model reduction for post-buckling analysis was proposed. The CPU time is reduced by around 50 percent in comparison to an existing computational strategy. The case study is chosen to demonstrate the relevance of the proposed strategy in an industrial framework, although the modeling is simplified. Nevertheless some aspects have not been addressed and require further developments.

Whereas the interactions between sub-domains are taken into account, for reasons of simplicity, no interactions between buckled sub-domains occurred in the case study. The buckling of a sub-domain lead indeed to the change of the stiffness seen by its neighboring sub-domains. This may change their buckling modes.

The use of a mixed domain decomposition approach [7] would enable taking into account these stiffness changes. A strategy for updating the local buckling modes would also become necessary in addition to the management of the size of the reduced basis that must not grow unreasonably.

Finally this paper opens perspectives of development for the computational strategies for local post-buckling analysis of large stiffened structures by the use of both physical knowledges and parallel computing.

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References

[1] B. O. Almroth, F. A. Brogan, and P. Stern. Automatic choice of global shape functions in structural analysis. *AIAA Journal*, 16(5):525–528, 1978.

[2] L. Barrière, B. Castanié, S. Marguet, and P. Cresta. A simple adaptive model order reduction strategy for post-buckling analysis. *Thin-Walled Structures*, 2012. Submitted.

[3] J. Bertolini, B. Castanié, J.J. Barrau, and J.-P. Navarro. Multi-level experimental and numerical analysis of composite stiffener debonding. part 1: Non-specific specimen level. *Composite Structures*, 90(4):381 – 391, 2009.

[4] J. Bertolini, B. Castanié, J.J. Barrau, J.P. Navarro, and C. Petiot. Multi-level experimental and numerical analysis of composite stiffener debonding. part 2: Element and panel level. *Composite Structures*, 90(4):392 – 403, 2009.

[5] E Byklum, E. Steen, and J. Amdahl. A semi-analytical model for global buckling and post-buckling analysis of stiffened panels. *Thin Walled Structures*, 42:701–717, 2003.

[6] R. R Craig and M. C. C Bampton. Coupling of substructures for dynamic analyses. *AIAA JOURNAL*, Vol. 6:pages 1313–1319, 1968.

[7] P. Cresta, O. Allix, C. Rey, and S. Guinard. Nonlinear localization strategies for domain decomposition methods: Application to post-buckling analyses. *Computer Methods in Applied Mechanics and Engineering*, 196(8):1436–1446, January 2007.

[8] M.A. Crisfield. *Non-linear Finite Element Analysis of Solids and Structures*. Wiley, 1991.

[9] P. Gosselet. *Méthodes de décompostion de domaine et méthodes d'accélération pour les problémes multichamps en mécanique non-linéaire*. PhD thesis, Université Paris VI, 2003.

[10] P Gosselet and C. Rey. Non-overlapping domain decomposition methods in structural mechanics. *Archive of Computational Methods in Engineering*, 13(4):515–572, 2006.

[11] P. Kerfriden, P. Gosselet, S. Adhikari, and S.P.A. Bordas. Bridging proper orthogonal decomposition methods and augmented Newton-Krylov algorithms: An adaptive model order reduction for highly nonlinear mechanical prob-

lems. *Computer Methods in Applied Mechanics and Engineering*, 200(5-8):850–866, January 2011.

[12] A. Kling, R. Degenhardt, and R. Zimmermann. A hybrid subspace analysis procedure for nonlinear postbuckling calculation. *Composite Structures*, 73:162–170, 2006.

[13] D. Nagy. Modal representation of geometrically nonlinear behavior by the finite element method. *Computers & Structures*, 10(4):683–688, August 1979.

[14] A. K. Noor, C. M. Andersen, and J. M. Peters. Reduced basis technique for collapse analysis of shells. *AIAA Journal*, 19(3):393–397, March 1981.

[15] A. K. Noor and J. M. Peters. Reduced basis technique for nonlinear analysis of structures. *AIAA Journal*, 18(4):455–462, April 1980.

[16] D.G. Stamatelos, G.N. Labeas, and K.I. Tserpes. Analytical calculation of local buckling and post-buckling behavior of isotropic and orthotropic stiffened panels. *Thin Walled Structures*, 49:422–430, 2011.

[17] Bisagni C. Vescovini, R. Single-mode solution for post-buckling analysis of composite panels with elastic restraints loaded in compression. *Composites: Part B*, 2011.

[18] K. Willcox and J. Peraire. Balanced model reduction via the proper orthogonal de- composition. *AIAA*, 40:2323–2330, 2002.

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