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ON THE INTERAL STRUCTURAL OF SHOCK WAVE IN RELAXING GAS

L.M.B.C Campos *, F.S. Cunha *
***CCTAE, Instituto Superior Técnico, Universidade de Lisboa**

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Abstract

The study of propagation of non-linear sound waves in relaxing media [1-3] shows that two kinds of solutions are possible: (i) if non-linearity predominates over relaxation it cannot prevent the front steepening which leads to shock formation; (ii) if non-linearity is weaker then continuous finite permanent waveforms with finite amplitude are possible. In the present paper it is shown that for a certain range of parameters both 'shock' type (i) and ´wave´ like (ii) solutions can coexist. The implication is that starting from a non-linear wave in a relaxing medium it may or may not develop into a shock. This implies the existence of bifurcations in parameter space, with range for which both ´wave´ and ´shock´ solutions are possible. The bifurcations are extensively described in the literature for non-linear dynamical systems described by ordinary differential equations (o.d.e.s). Waves are described by partial differential equations (p.d.e.s.); in the case of non-linear waves with permanent wave forms the solution of the p.d.e. reduces to an o.d.e., so bifurcations are possible. The bifurcations do not occur for the non-linear sound waves without (with) dissipation specified by Riemann invariants (the Burger's equation), because they have a unique solution associated with a single sound speed. In a relaxing medium there is more than one sound speed, which combined with non-linearity can be expected to lead to bifurcations; this is demonstrated in the present paper.

A prototype problem for the bifurcations of non-linear waves is thus large amplitude acoustics of relaxing media. The latter are important in high-temperature gas dynamics; also in non-linear acoustics in other media subject to combustion, chemical reactions, ionization, radiation and other non-equilibrium conditions. The starting point is the wave equation for weakly non-linear sound waves in a medium with a single relaxation time (§2); in the non-dissipative case non-linear waves with permanent wave form exist (§3). They correspond to four cases of ´shock´ and/or ´wave´ like solutions with three transition conditions (§4). It is shown that relaxation opposes non-linear shock formation by limiting front steepening (§5). For a range of non-linearity and relaxation parameters, ´shock´ and ´wave´ like solutions can coexist, implying the existence of bifurcations (§6). The latter can be expected for non-linear waves of permanent waveform in media with multiple wave speeds (§7). This is demonstrated in Figures 1 to 5 for the non-linear acoustics of relaxing media.

1 Non-linear waves with permanent form

The velocity perturbation of weakly non-linear acoustic waves satisfies the wave equation:

$$\partial V/\partial t + c_0\,\partial V/\partial x + \beta\, V\,\partial V/\partial x + (\xi/2)\,\partial V^2/\partial x^2 = \bar{c}\,\partial \tilde{V}/\partial x\,, \qquad (1)$$

where are assumed: (i) unidirectional propagation in the positive x-direction with equilibrium sound speed c_0 for a linear non-dissipative wave; (ii) weak convective non-linearity factor (2a)=(2b) for an arbitrary substance:

$$\beta = (V_0^3/2c_0^2)(\partial^2 p_0/\partial V_0^2)_s = (c_0^4/2V_0^3)(\partial V_0^2/\partial p_0{}^2)_s = (\gamma+1)/2 \quad \text{(2a-c)}$$

where (3a) is the specific volume; (iii) it simplifies to (2c) for a perfect gas with adiabatic exponent (3b), which is the ratio of the specific heats at constant pressure and temperature; (iv) linear dissipation with diffusivity (3c):

$$V_0 \equiv 1/\rho_0 , \gamma \equiv C_p/C_v : \xi = \left[\eta + k\left(1/C_p + 1/C_v\right)\right]/\rho_0, \quad \text{(3a-c)}$$

due to the shear viscosity η and thermal conductivity k, where ρ_0 is the mean state mass density and V_0 the specific volume (3a); (v) the bulk viscosity absent in (3c) is due to a single relaxation process with frozen sound speed c_∞ appearing in (4a):

$$\bar{c} = c_\infty^2/c_0 - c_0 > 0: (1 - \vartheta\, \partial/\partial t)\tilde{V} \equiv \vartheta\, \partial V/\partial t \quad \text{(4a-b)}$$

and relaxation time ϑ appearing in (4b) together with the relaxation velocity perturbation $\tilde{V}$.

The latter can be eliminated between (4b) and (13) leading to:

$$(1 - \vartheta\, \partial/\partial t)\, [\partial V/\partial t + c_0\, \partial V/\partial x + \beta\, V\, \partial V/\partial x + (\xi/2)\, \partial V^2/\partial x^2] = \bar{c}\, \vartheta\, \partial^2 V/\partial x \partial t \quad \text{(5)}$$

as the wave equation for the velocity perturbation:

$$\partial V/\partial t + c_0\, \partial V/\partial x - (c_0 + \bar{c})\vartheta\, \partial^2 V/\partial x \partial t - \vartheta\, \partial^2 V/\partial t^2 + (\xi/2)\, \partial^2 V/\partial x^2 - (\xi\vartheta/2)\, \partial^3 V/\partial x^2 \partial t + \beta(V - \vartheta\, \partial V/\partial t)\, \partial V/\partial x - \beta\, \vartheta\, V\, \partial^2 V/\partial x \partial t = 0 \quad \text{(6)}$$

In the derivation of (1) are assumed weak non-linearity and dissipation, that is small β and ξ. The relaxation time in (4b) need not be small, and this $\beta\vartheta$, $\xi\vartheta$ are not negligible though $\beta\xi$ would be. The solution is sought as a waveform with permanent shape (7a) convected at a velocity w_0; the latter is not known 'a priori' since several wave speeds exist in a relaxing medium. Substitution of (7a) in (6) yields (7b):

$$V(x,t) = f(x - w_0 t): \quad (c_0 - w_0)f' + [\xi/2 - w_0\vartheta(w_0 - c_0 - \bar{c})]f'' + (\xi/2)w_0\vartheta f''' + \beta f' f + \beta\, w_0\, \vartheta\left(f'^2 + f'' f\right) = 0. \quad \text{(7a,b)}$$

The first integration is immediate:

$$(c_0 - w_0)f + [\xi/2 - w_0\vartheta(w_0 - c_0 - \bar{c})] f' + (\xi/2)w_0\vartheta f'' + (\beta/2)f^2 + \beta\, w_0\, \vartheta f' f = A \quad \text{(8)}$$

where A is a constant of integration.

Neglecting dissipation (9a) but not relaxation the order of the equation drops from two to one (9b):

$$\xi = 0: (c_0 - w_0)f - w_0\vartheta(w_0 - c_0 - \bar{c})f' + (\beta/2)f^2 + \beta\, w_0\, \vartheta f' f = A. \quad \text{(9a,b)}$$

Choosing the convection velocity (10a) to be the sound speed plus the relaxation velocity simplifies (9b) to (10b):

$$w_0 = c_0 + \bar{c} = c_\infty^2/c_0 > c_0, c_\infty: \quad \beta f^2 - 2\bar{c} f + 2\, \beta\, \vartheta(c_0 + \bar{c})f' f = 2A \quad \text{(10a,b)}$$

The convection velocity (10a) always exceeds both the equilibrium c_0 and frozen c_∞ sound speeds appearing in (4a). The arbitrary constant in (10b) can be determined (11c) from an asymptotic velocity (11a) assuming it is uniform (11b):

$$f(\infty) = V_- , f'(\infty) = 0: 2A = \beta V_-^2 - 2\, \bar{c}\, V_- \quad \text{(11a-c)}$$

Substituting (11c) in (10b) yields (12b):

$$y \equiv x - w_0 t = x - c_\infty^2\, t/c_0 :$$

$$-2(\vartheta\, c_\infty^2/c_0) f\, df/dy = f^2 - V_-^2 - 2(\bar{c}/\beta)(f - V_-) = (f - V_-)(f + V_- - 2\,\bar{c}/\beta), \quad (12a,b)$$

which is a separable equation in the: (i) convected coordinate (12a); (10a); (ii) the wave form (7a).

2 Two regimes of propagation of relaxation waves

Separating the variables in (12b) leads to (13b):

$$2\varepsilon \equiv V_-/(V_- - \bar{c}/\beta): \quad (13a,b)$$

$$(2\vartheta)^{-1} d(t - c_0\, x/c_\infty^2) = [(f - V_-)(f + V_- - 2\,\bar{c}/\beta)]^{-1} f df = [\varepsilon/(f - V_-) + (1-\varepsilon)/(f + V_- - 2\,\bar{c}/\beta)$$

in (13b) appears the parameter (13a). Integrating (13b) leads to:

$$\log V_* + (t - c_0\, x/c_\infty^2)/(2\vartheta) = \varepsilon \log(f - V_-) + (1-\varepsilon) \times \log(f + V_- - 2\,\bar{c}/\beta)\ ; \quad (14)$$

the constant of integration V_* is the amplitude of the velocity perturbation (7a) of the relaxation wave:

$$(V - V_-)^{\varepsilon}\,(V + V_- - 2\,\bar{c}/\beta)^{1-\varepsilon} = V_* \exp[(t - c_0\, x/c_\infty^2)/(2\vartheta)]. \quad (15)$$

This specifies the velocity perturbation of a weakly non-linear acoustic wave in a relaxing medium, for which there are four cases.

The first case (I) is $\varepsilon > 1$ in (13a) which (16a) implies (16b):

$$\varepsilon > 1:\ V_1 \equiv 2\,\bar{c}/\beta > V_- > \bar{c}/\beta \equiv V_2 = V_1/2\,; \quad (16a\text{-}c)$$

the condition (16c) is required so that $\varepsilon > 0$, since for $0 < V_- < \bar{c}/\beta$ then (13a) is negative; also $V_- < 0$ would lead to $\varepsilon < 1$ so it is excluded from this case I. The condition $\varepsilon > 1$ thus requires V_- to lie between the critical velocities V_1 and V_2. In this case $\varepsilon > 0$ and $1 - \varepsilon < 0$ in (15), so far downstream x→+∞ (upstream x→-∞) as the r.h.s vanishes (diverges), then V must tend to (17a) [(17b)]:

$$\varepsilon > 1:$$

$$\begin{cases} V_I(+\infty) = \lim_{x\to+\infty} V(x,t) = V_- > \bar{c}/\beta > 0 & (17a) \\ V_I(-\infty) \equiv \lim_{x\to-\infty} V(x,t) = 2\,\bar{c}/\beta - V_- \equiv V_+ > 0 \ or\ \infty. & (17b) \end{cases}$$

The downstream velocity (17a) confirms the asymptotic boundary condition (11a). From (16b,c) it follows that the velocity perturbation is a compression both upstream and downstream $V_\pm > 0$. Also:

$$\varepsilon > 1: V_I(-\infty) - V_I(+\infty) = V_+ - V_- = 2(\bar{c}/\beta - V_-) < 0\,, \quad (18)$$

shows that the upstream velocity is smaller; thus this case I corresponds to a compression wave $V_I(+\infty) > V_I(-\infty) > 0$.

The second case II is $\varepsilon < 0$ in (19a) implying (19b) by (13a):

$$\varepsilon < 0:\ 0 < V_- < \bar{c}/\beta = V_2 = V_1/2\,. \quad (19a,b)$$

In this case $\varepsilon < 0$ and $1 - \varepsilon > 0$ so that far downstream x→+∞ (upstream x→-∞) the r.h.s. of (15) vanishes (diverges) implying that V tends to (20a) [(20b)] :

$$\varepsilon < 0:$$

$$\begin{cases} V_{II}(+\infty) = \lim_{x\to+\infty} V(x,t) = V_+ = 2\,\bar{c}/\beta - V_- > \bar{c}/\beta = V_2 > 0, & (20a) \\ V_{II}(-\infty) = \lim_{x\to-\infty} V(x,t) = V_- < \bar{c}/\beta = V_2\,. & (20b) \end{cases}$$

In this case II the asymptotic limits (20a,b) interchange those (17a,b) of case I: (i) the asymptotic velocity (11a) is the upstream velocity (20b); (ii) the upstream velocity (20a) is smaller than the downstream velocity in this case III:

$$\varepsilon < 0: V_{II}(-\infty) - V_{II}(+\infty) = V_- - V_+ = 2\left(V_- - \bar{c}/\beta\right) < 0\,; \quad (21)$$

again this is an compression wave $V_{II}(+\infty) > V_{II}(-\infty)$, as in case I.

The remaining cases satisfy (22a) which is met (13a) by two distinct conditions (22b) and (22c):

$$0 < \varepsilon < 1: \quad V_- > 2\,\bar{c}/\beta = V_1 \ \ or\ \ V_- < 0. \quad (22a\text{-}c)$$

Thus case III involves a compression (22b) and case IV a rarefaction (22c). In both cases $\varepsilon > 0$ and $1-\varepsilon > 0$, so far downstream x→+∞ (upstream x→-∞) as the r.h.s. of (15) vanishes (diverges), then V must tend to (23a) [(23b)]:

$$0 < \varepsilon < 1:$$

$$\begin{cases} V_{III,IV}(+\infty) = \lim_{x\to+\infty} V(x,t) = V_- \; ou \; V_+, & (23a) \\ V_{III,IV}(-\infty) = \lim_{x\to-\infty} V(x,t) = \infty\,. & (23b) \end{cases}$$

In the preceding cases I (II) in (16b)[(19b)] the non-linearity parameter was small $\beta < 2\,\bar{c}/V_-$ ($\beta < \bar{c}/V_-$), so relaxation dominates and limits the amplitude of the wave form. In the case III the non-linearity parameter (22b) is large $\beta > 2\,\bar{c}/V_-$, and relaxation is insufficient to prevent shock formation, so a continuous solution does not exist for all space, viz. there is a upstream divergence (23b). The same conclusion applies to case IV.

3 Regimes of propagation and transition velocities

In the cases III and IV there are in each two sub-cases for the downstream condition (23a) combined with (22b,c):

$$0 < \varepsilon < 1, \; V_- < 0,$$

$$V_{IIIA}(+\infty) = \lim_{x\to+\infty} V(x,t) = V_- < 0, \quad (24a,b)$$

$$0 < \varepsilon < 1, \; V_- > 2\,\bar{c}/\beta,$$

$$V_{IVA}(+\infty) = V_- > 2\,\bar{c}/\beta > 0, \quad (25a,b)$$

$$0 < \varepsilon < 1, \; V_- < 0,$$

$$V_{IIIB}(+\infty) = V_+ = 2\,\bar{c}/\beta - V_- > 0, \quad (26a,b)$$

$$0 < \varepsilon < 1, V_- > 2\,\bar{c}/\beta,$$

$$V_{IVB}(+\infty) = V_+ = 2\,\bar{c}/\beta - V_- < 0. \quad (27a,b)$$

Thus two sub-cases include an initial compression [(25a,b);(26a,b)] and rarefaction [(24a,b);(27a,b)]. From the point-of-view of non-linearity parameter there are two regimes:

$$regime: \begin{cases} \beta \geq 2\,\bar{c}/V_- \equiv \bar{\beta} : shock, & (28a) \\ \beta < 2\,\bar{c}/V_- \equiv \bar{\beta} : wave. & (29b) \end{cases}$$

From the point-of-view of the parameter (13a) there are three cases:

$$regime: \begin{cases} \varepsilon > 1: \quad wave\ I, & (30a) \\ 0 \leq \varepsilon \leq 1: shock\ III\ or\ IV, & (29b) \\ \varepsilon < 0: \quad wave\ II. & (29c) \end{cases}$$

From the point-of-view of the velocity V_- there are four ranges:

$$regime: \begin{cases} V_- \leq 0: \; shock\ IIIA\ or\ IIIB, & (31a) \\ 0 < V_- \leq \bar{c}/\beta: \; wave\ II, & (32b) \\ \bar{c}/\beta \leq V_- < 2\,\bar{c}/\beta : wave\ I, & (30c) \\ V_- \geq 2\,\bar{c}/\beta\ shock\ IVA\ or\ IVB, & (30d) \end{cases}$$

this leads to three transition velocities, namely $V_- = 0, V_1, V_2$ with (V_1, V_2) appearing in (16b,c).

The first transition (31a) is between a shock (30a) and a wave (30b) and simplifies (13b) to (31b):

$$V_- = 0 \; (2\vartheta)^{-1} d(t - c_0 x/c_\infty^2) = (f - 2\,\bar{c}/\beta)^{-1} df. \quad (33a,b)$$

This corresponds to:

$$\varepsilon = 0:$$

$$V(x,t) = 2\,\bar{c}/\beta + V_* \exp[(t - c_0 x/c_\infty^2)/(2\vartheta)], \quad (34a,b)$$

implying a finite velocity far downstream (33a) and divergence (33b) upstream:

$$\varepsilon = 0:$$

$$V(+\infty) = \lim_{x\to+\infty} V(x,t) = 2\,\bar{c}/\beta\,, \quad (35a)$$

$$V(-\infty) = \lim_{x\to-\infty} V(x,t) = \infty\,. \quad (33b)$$

The third transition (34a) is also between wave (30c) and shock (30d) and simplifies (13b) to (34b):

$$V_- = 2\,\bar{c}/\beta : (2\vartheta)^{-1} d(t - c_0 x/c_\infty^2) = (f - 2\,\bar{c}/\beta)^{-1} df; \quad (36a,b)$$

this leads to:

$$\varepsilon = 1:$$

$$V(x,t) = 2\,\bar{c}/\beta + V_* \exp[(t - c_0 x/c_\infty^2)/(2\vartheta)]; \qquad (37a,b)$$

thus the wave form is the same (35a,b)≡(32a,b) in the two cases ε=0 and ε=1 of transition between wave and shock. The result (32b)[(35b)] could be obtained from (15) with ε=0 (ε=1) using (31a) [(34a)]; both cases imply (33a,b). The second transition velocity $V_- = \bar{c}/\beta$ corresponds to ε=∞ in (13a) and limit is less obvious from (15).

The second transition (36a) is between waves (30b,c) and simplifies (13b) to (36b):

$$V_- = \bar{c}/\beta : (2\vartheta)^{-1} d(t - c_0 x/c_\infty^2) = (f - \bar{c}/\beta)^{-2} f df = [(f - \bar{c}/\beta)^{-1} + (\bar{c}/\beta)(f - \bar{c}/\beta)^{-2}] df. \qquad (38a,b)$$

This integrates:

$$\log V_* + (t - c_0 x/c_\infty^2)/(2\vartheta) = \log(f - \bar{c}/\beta) - (\bar{c}/\beta)/(f - \bar{c}/\beta), \qquad (39)$$

and leads to:

$$\varepsilon = \infty:$$

$$V_* \exp[(t - c_0 x/c_\infty^2)/(2\vartheta)] = (V - \bar{c}/\beta) \exp[(1 - \beta V/\bar{c})^{-1}] \qquad (40a,b)$$

far downstream x⟶+∞ (upstream x⟶-∞) the l.h.s. of (38b) vanishes (diverges) and the r.h.s. implies that $1 - \beta V/\bar{c} < 0$ $(1 - \beta V/\bar{c} > 0)$ so that it tends to (39a) [(39b)]:

$$\varepsilon = \infty:$$

$$\begin{cases} V(+\infty) = \lim_{x\to+\infty} V(x,t) = \bar{c}/\beta + O\,, & (41a) \\ V(-\infty) = \lim_{x\to-\infty} V(x,t) = \bar{c}/\beta - O\,. & (39b) \end{cases}$$

In this case the upstream and downstream velocities are close on opposite sides of (36a).

4 Centre velocity and slope of wavefront

All the preceding solutions, e.g. (15) involve an arbitrary amplitude V_*. The latter can be replaced by the velocity at a given point; choosing V_0 at the ′centre′ of the waveform (40a), the velocity perturbation (15) satisfies (40b):

$$c_\infty^2 t = c_0 x: (V_0 - V_-)^\epsilon (V_0 - V_+)^{1-\epsilon} = V_*. \qquad (42a,b)$$

It reduces to (41c) [(42c)] in the case (31a)≡(32a)≡(41a,b)[(34a)≡(35a)≡(42a,b)] of transition between wave and shock:

$$\varepsilon = 0, V_- = \bar{c}/\beta:$$

$$V_0 = V_* + V_+ = V_* + 2\,\bar{c}/\beta - V_- = V_* + \bar{c}/\beta, \qquad (43a\text{-}c)$$

$$\varepsilon = 1, V_- = 2\,\bar{c}/\beta:\; V_0 = V_* + V_- = V_* + 2\,\bar{c}/\beta. \qquad (44a\text{-}c)$$

In the case (38a)≡(36a)≡(43a,b) of transition between waves (38b) leads to (43c):

$$\varepsilon = \infty, V_- = \bar{c}/\beta:$$

$$V_* = (V_0 - \bar{c}/\beta) \exp[(1 - \beta V_0/\bar{c})^{-1}]. \qquad (45a\text{-}c)$$

The relation (43c) between V_0 and V_* can be put in the form:

$$X \equiv \beta V_0/\bar{c} - 1: \qquad (46a,b)$$

$$\beta V_*/\bar{c} = X \exp[-1/X],$$

which involves the two velocities (V_0, V_*) made dimensionless dividing by (43b).

The slope of the waveform is given (12b) in all space by:

$$f' V = -(c_0/c_\infty^2)(V - V_-)(V - V_+)/(2\vartheta). \qquad (47)$$

In particular it takes the value (46b) at the centre (46a):

$$c_\infty^2 t = c_0 x: f_0{}' = -(c_0/c_\infty^2)(V_0 - V_- - V_+ + V_- V_+/V_0)/(2\vartheta) = -(c_0/c_\infty^2)[V_0\beta - 2\bar{c} - V_-(2\bar{c} - V_-\beta)/V_0]/(2\vartheta\,\beta) \qquad (48a,b)$$

where was used (17b). The slope of the wave form varies inversely with the relaxation time, i.e. the wave front is steep (shallow) for short (long) relaxation time, corresponding to an almost discontinuous (mostly blurred) shock wave. This demonstrates that a long relaxation time is the dominant time scale for a relaxation wave with a shallow front like a "blurred shock". The negative sign conforms that the amplitude decreases across the front. The present problem has a solution with permanent wave form (15) which involves or leads to ten velocities: (i) the velocity (10a) of convection of the wave form; (ii) the asymptotic velocity (11a) as a boundary (or initial) condition; (iii) the other boundary velocity (17b) related to (ii); (iv/v) the upper and lower transition velocities (16b,c) (vi) the wave amplitude V_* in (15) which is arbitrary; (viii) the middle velocity V_0 in (40a,b) and (43c); (viii-x) the preceding involve the equilibrium c_0 and frozen c_∞ sound speed appearing in the relaxation speed $\bar{c}$ in (4a)

The velocity perturbation satisfies (15) in dimensionless form:

$$\varepsilon \neq 0, \infty: (W-1)^{\varepsilon}(W-1+1/\varepsilon)^{1-\varepsilon} = W_* e^{\tau}, \quad \text{(49a,b)}$$

where: (i) the velocity is made dimensionless dividing by V_- in (48a,b); (ii) the convected coordinate (48b) combines position and time:

$$W = V/V_- \ , \tau \equiv (t - c_0 x/c_\infty^2)/(2\vartheta). \quad \text{(50a,b)}$$

For fixed time $z \to -\infty$ downstream and $z \to +\infty$ upstream, and $z = 0$ corresponds to the mid-point (40a). The expression (47b) does not hold (47a) for $\varepsilon = 0$, which corresponds (13a) to $V_- = 0$, invalidating the choice (48a) of dimensionless velocity. In the case $\varepsilon = 0$, the velocity perturbation (32b) takes the dimensionless form (49c):

$$\varepsilon = 0: W \equiv \beta V/2\bar{c}, W = 1 + W_* e^{\tau}, \quad \text{(51a-c)}$$

using (49b) instead of (48a). The expression (47b) also does not hold for $\varepsilon = \infty$ corresponding (13c) to $V_- = \bar{c}/\beta$ in (38b):

$$V_- = \bar{c}/\beta:$$

$$V(x,t) = \bar{c}/\beta + V_* \exp[-(t - c_0 x/c_\infty^2)/(2\vartheta)] \exp[-(1 - \beta V/c)^{-1}]; \quad \text{(52a,b)}$$

the corresponding dimensionless (49b) form is:

$$\varepsilon = \infty: W \equiv \beta V/\bar{c}, \quad \text{(53a-c)}$$

$$W = 1 + W_* e^{\tau} \exp[-(1-W)^{-1}].$$

In all three cases [(47a,b);(48a,b)], [(49a-c);(48b)] and [(51a-c);(48b)] there are two dimensionless variables: (i) the amplitude V normalized W in different ways (48a), (49b) and (50b); (ii) the same (48b) convected coordinate z. There are two dimensionless parameters: (i) the normalized amplitude W_*; (ii) the dimensionless combination (13a) of non-linearity and relaxation effects

5 Bifurcations due to the combination of non-linearity and relaxation

Each of the seven cases is illustrated in figures 1 to 5, for unit amplitude (52a) and seven values (52b) of the parameter (13a):

$$W_* = 1, \varepsilon = -1, 0, 1/4, 1/2, 3/4, 1, 2. \quad \text{(54a,b)}$$

These correspond to: (i) the initial velocities relative to V_2 in (16c):

$$V_-/V_2 = \beta V_-/\bar{c} = \varepsilon/(\varepsilon - 1/2) = \{2/3, 0, -1, \infty, 3, 2, 4/3\}; \quad \text{(55)}$$

(ii) the ratio of non-linearity parameter to the critical value (28a,b):

$$\psi = \beta/\bar{\beta} = \beta V_-/2\bar{c} = V_-/V_1 = \varepsilon/(2\varepsilon - 1) = \{1/3, 0, -1/2, \infty, 3/2, 1, 2/3\}, \quad \text{(56)}$$

coincides with the ratio V_- to V_2 in (16b), i.e. one-half of (53). In all cases the same dimensionless similarity variable (48b) is used. Since it is a combined space-time variable corresponding to a waveform convected at a constant velocity, it has two interpretations: (i) before (§3-5), it was considered at a fixed time, specifying the waveform as a function of position, so that $\tau \to \infty$ upstream $x \to -\infty$ and $\tau \to -\infty$ downstream $x \to \infty$; (ii) in the sequel (§6 and Figures 1 to 5) it is considered at a fixed position, specifying the waveform as a function of time so that $\tau \to \infty$ for late time $t \to \infty$ and $\tau \to -\infty$ for early time $t \to -\infty$. Thus the correspondence (i) and (ii) is: (i) upstream $x \to -\infty$ and late time $t \to \infty$ for $\tau \to \infty$; (ii) downstream $x \to \infty$ and early time $t \to -\infty$ for $\tau \to -\infty$. The solutions bounded for all time $\tau \to \pm\infty$ are waves. The solutions unbounded for $\tau \to +\infty$ or $\tau \to -\infty$ show that a bounded continuous solution is not possible; a bounded solution is possible if it is discontinuous, corresponding to a shock. The two transition cases (55a) both correspond a single solution (55c):

$$\varepsilon = 0, 1; V_- = 0, 2\bar{c}/\beta: W = 1 + W_* e^{\tau}, \tag{57a-c}$$

for the initial velocities (55b), with distinct normalizations: (i) for $\varepsilon = 1$ in [(47b);(48a)]; (ii) for $\varepsilon = 0$ in (49a-c). The plots in Figure 1, for distinct amplitudes $W_* = -1, 0, +1$ show: (i) constant velocity for zero amplitude; (ii) divergence $W_\pm = \pm\infty$ at late time $\tau \to \infty$ starting from unity $W_\pm = 1$ at early time $\tau \to -\infty$. In the remaining figures 2 to 5, of non-transition cases, the unit amplitude is used (56a), so that only the dimensionless ratio of non-linearity and relaxation is varied.

The value (56b) below the lower transition leads (47b) to (56c):

$$W_* = 1; \quad \varepsilon = -1: (W - 2)^2 = e^{\tau}(W - 1); \tag{58a-c}$$

this corresponds to a quadratic equation (57b) in the case of relative non-linearity parameter (57a):

$$\psi = 1/3: \; W^2 - (4 + e^{\tau})W + 4 + e^{\tau} = 0. \tag{59a,b}$$

The roots lead to two solutions (58b) for the same initial velocity (54)≡(58a):

$$V_- = 2\bar{c}/3\beta: 4 + e^{\tau} \pm \sqrt{e^{\tau}(e^{\tau} + 4)} = 2W_\pm = 3\beta V/V_-\bar{c}. \tag{60a,b}$$

Both solutions (Figure 2) start at early time $\tau \to -\infty$ with finite amplitude $W_\pm(-\infty) = 2$; the solution above $W_+(-\infty) = 2 + O$ diverges $W_+(+\infty) = \infty$ at late time as $\tau \to \infty$, i.e forms a shock. The solution below $W_-(-\infty) = 2 - O$ tends at late time $\tau \to \infty$ to one half of the initial value $W_-(+\infty) = 1$, as follows from:

$$\lim_{\tau\to\infty} W_- = \frac{1}{2}\lim_{\tau\to\infty}\left[4 + e^{\tau}\left(1 - \sqrt{1 + 4e^{-\tau}}\right)\right] = \frac{1}{2}\lim_{\tau\to\infty}\{4 + e^{\tau}[1 - (1 + 2e^{-\tau} + O(e^{-2\tau}))]\} = 1. \tag{61}$$

This is confirmed by (56c): (i) at early time $\tau \to -\infty$ then $W_\pm \to 2$; (ii) at late time $\tau \to \infty$ either $W_- \to 1$ or $W_+ \to \infty$. Thus a small difference in downstream velocity $V_- = 2\bar{c}/3\beta \pm O$ leads to a shock or a wave.

The value (60b) above the upper transition leads (47b) to (60c):

$$W_* = 1; \; \varepsilon = 2 \quad (W - 1)^2 = e^{\tau}(W - 1/2); \tag{62a-c}$$

this is again a quadratic equation (61b) for the case of relative non-linearity parameter (61a):

$$\psi = 2/3: \; W^2 - (2 + e^{\tau})W + 1 + \frac{1}{2} e^{\tau} = 0. \tag{63a,b}$$

The roots (61b) lead to two solutions (62b) for the initial velocity (62a):

$$V_- = 4\bar{c}/3\beta: \tag{64a,b}$$

$$2W_\pm = 2 + e^{\tau} \pm \sqrt{e^{\tau}(e^{\tau} + 2)} = 3\beta V/(2V_- c).$$

The asymptotic value (Figure 3) at early time $\tau \to -\infty$ is the same in both cases $W_\pm(-\infty)$=1. The upper value $W_+(+\infty) = \infty$ forms a shock at late time $\tau \to +\infty$. The lower value remains a wave of finite amplitude, with at late time $\tau \to \infty$ value $W_-(+\infty) = 1/2$, as follows from:

$$\lim_{\tau\to\infty} W_- = \frac{1}{2}\lim_{\tau\to\infty}\left[2 + e^{\tau}\left(1 \pm \sqrt{1 + 2e^{-\tau}}\right)\right] = \frac{1}{2}\lim_{\tau\to\infty}\{2 + e^{\tau}[1 - (1 + e^{-\tau} + O(e^{-2\tau}))]\} = 1/2. \tag{65}$$

This is confirmed from (60c) since: (i) at early time $\tau \to -\infty$ then $W_\pm \to 1$; (ii) as late time

$\tau \to \infty$ then either $W_+ \to \infty$ or $W_- \to 1/2$. Again a small difference in the downstream velocity $V_- = 4\bar{c}/3\beta \pm O$ leads to a shock or a wave.

In the intermediate range $0 \le \varepsilon \le 1$ only shock solutions exists. The cases $\varepsilon = 0$ and $\varepsilon = 1$ have been considered (55a-c) and the other transition case:

$$W_* = 1;\ \varepsilon = 1/2 \qquad \psi = \infty = V_-, \qquad \text{(66a-e)}$$
$$e^{2\tau} = (W-1)(W+1) = W^2 - 1,$$

is (64b), and leads to a large initial velocity (64d); it corresponds (47b) to a solution (64e) like (49c) in Figure 1, replacing τ by 2τ and W by W^2. The 'unphysical' case (64a-e) separates the remaining two cases with positive or negative initial velocity, e.g.:

$$W_* = 1;\ \varepsilon = 1/4,\ \psi = -1/2, V_- = -\bar{c}/\beta, W = -\beta V/\bar{c}: \qquad \text{(67a-f)}$$

$$(W-1)(W-3)^3 = e^{4\tau},$$

$$W_* = 1;\ \varepsilon = 3/4 \qquad \psi = 3/2, V_- = 3\bar{c}/\beta, W = -\beta V/3\bar{c}: \qquad \text{(68a-f)}$$

$$(W-1)^3(W+1/3) = e^{4\tau},$$

using (47b). In both cases (65a-f) in Figure 4 and (66a-f) in Figure 5, there is divergence at late time $\tau \to +\infty$ and a shock forms. The shock arises from distinct early time $\tau \to -\infty$ conditions $W(-\infty) = 1, 3$ in (65a-f) and $W(-\infty) = 1, -1/3$ in (66a,f).

6 Discussion

The plotting of waveforms of the figures 1 to 5 was made as a function of the convected space-time coordinate (48b) in two ways. At fixed time as a function of position, finite amplitude downstream $x \to +\infty$ corresponds upstream $x \to -\infty$ to: (i) a finite amplitude for a wave; (ii) a finite amplitude for a 'shock', meaning that a continuous bounded solution is not possible, and a shock must form to have a bounded discontinuous solution. Bearing in mind the similarity variable (48b) in space-time, and entirely equivalent interpretation can be made at fixed position as a function of time. For the same amplitude: downstream as $x \to +\infty$ corresponds far into the past $t \to -\infty$, and the upstream condition $x \to -\infty$ corresponds to late time $t \to +\infty$, with two cases: (i) finite amplitude for a wave; (ii) divergent solution or 'blow-up' for a shock, i.e. a discontinuity is needed to prevent divergence.

The two parameters affecting the propagation of weakly non-linear sound waves in a relaxing medium are: (i) the ratio (67a) of the non-linearity parameter β to the value (28a) beyond which only shocks exist and below which waves (28b) are possible:

$$\psi = \beta/\bar{\beta} = V_-\beta/(2\bar{c})\,; \qquad \text{(69a,b)}$$
$$2\varepsilon = 1/[1 - \bar{c}/(V_-\beta)],$$

(ii) the exponent (13a)≡(67b) in the wave form (15). The two are related by (68a,b):

$$\psi = \varepsilon/(2\varepsilon - 1), \qquad \text{(70a,b)}$$
$$\varepsilon = 1/[2 - 1/(\psi)] = \psi/(2\psi - 1),$$

(i) starting with a rarefaction wave downstream $\psi<0$ always leads (case III) to a shock upstream; (ii) starting with a strong compression downstream $\psi>1$ also leads always to a shock upstream (case IV). A wave is possible as well as a shock in the bifurcation range $0<\psi<1$, which includes two sub-ranges separated by $\psi=1/2$; for $\psi<1/2$ case II with $\varepsilon<0$ and for $\psi>1/2$ case I with $\varepsilon>1$.

There is substantial literature [4-9] on bifurcations for non-linear dynamical systems described by ordinary differential equations (o.d.e). This raises the ´mathematical question´: (i) how can bifurcations occur for waves which are the solution of partial differential equations? The waves considered are permanent wave forms, and thus cases in which the p.d.e. has a solution depending on a single combination of space-time variables, corresponding to a constant propagation speed. In this case the solution of the non-linear p.d.e. reduces to a non-linear o.d.e., so bifurcations could occur.

The latter remark raises a physical question: (ii) the non-linear non-dissipative sound waves described by Riemann invariants have no bifurcations [10-14]; (iii) nor do the weakly non-linear dissipative sound waves described by the Burger's equation [15-18]. Why then do bifurcations occur for waves in relaxing media? The answer is that in the case (ii) the non-linear sound wave is determined uniquely at all events in space-time by the two Riemann invariants associated with propagation in opposite directions at a group velocities $U_{\pm} = V \pm c_0$ associated with a single sound speed c_0. Thus there can be no bifurcations. The inclusion of dissipation does not change the unicity of solution as shown [15] by the Cole-Hopf transformation [19-20] of the Burgers [21] equation into a heat equation.

In the case of a relaxing medium there [1-3,12] is more than one sound speed, namely: (i) the frozen sound speed, or instantaneous wave speed before the medium has relaxed; (ii) the equilibrium sound speed after the medium has relaxed. The existence of more than one wave speed can lead to multiple solutions of the equations of fluid dynamics, as is well known in high-temperature gas dynamics [22]. In conclusion the bifurcations can occur for non-linear waves in media with more than one wave speed

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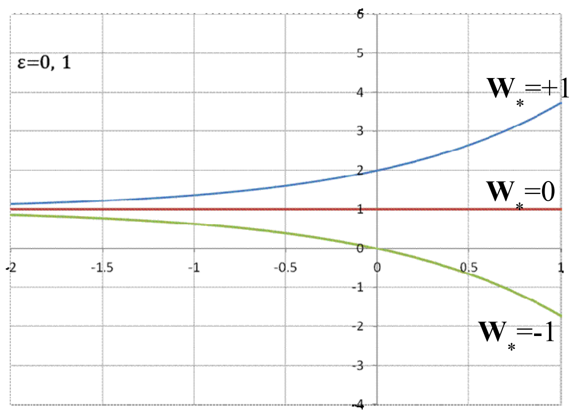

Figure 1- Normalized velocity perturbation (48a) versus dimensionless similarity parameter (48a) for three values of amplitude $W_*=0,\pm1$. The non-zero values lead to divergent waves, unless a shock forms.

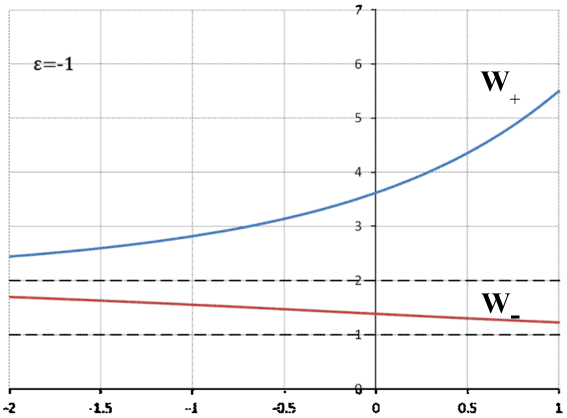

Figure 2- As figure 1 for unit amplitude and parameter range for which there exists a continuous solution W_- with finite amplitude everywhere for all time. The other solution W_+ diverges or forms a shock.

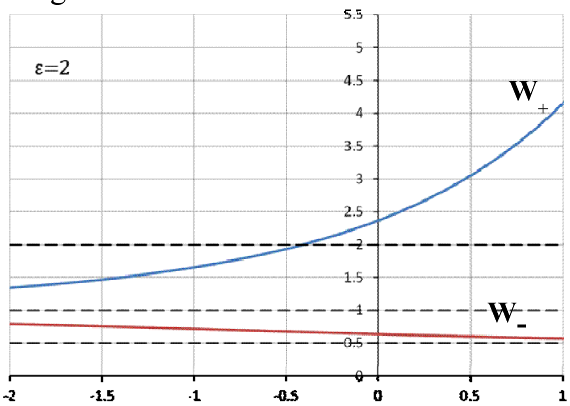

Figure 3- As figure 2 in a different parameter range for which both shock and wave solutions exist.

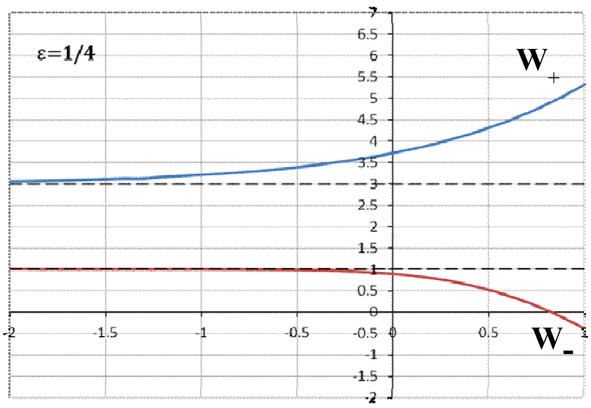

Figure 4 As Figures 1-3 for a parameter range with two divergent or shock type solutions.

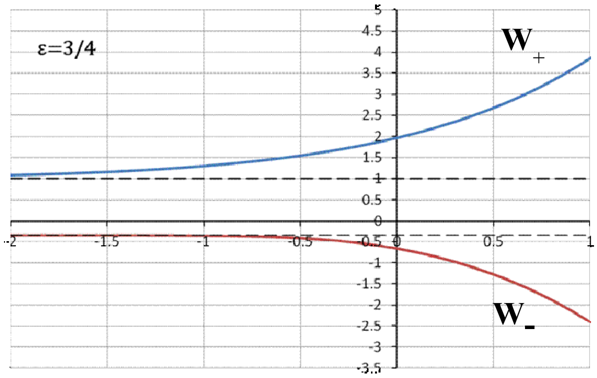

Figure 5 As figure 4 with a different parameter value also leading to two divergent or shock type solutions.

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