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METHOD DISTURBANCES IN THE LIMIT DEFORMATIONS IN THE CROSS-SECTION WING IS

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Abstract

The three-dimensional arrangement represents a solution of a elastic problem concerning the determination of a deflected mode of an arbitrary cross-section solid restrained bar deformation, loaded with concentrated torque at the free end.

1 General Introduction

The movement limits available in a torque bar cross-section recall a local stress burst and are known as Saint Venant's effect. As this area viewing the structural strength as the more dangerous one so the stress level determination needs to raise bearing strength of such details. There are a considerable number of such areas in the aircraft structure, but the area of the torque box-to-wing connection is the most dangerous and high-loaded in the centre-section. Because there is a considerable change of the deflection rate in this area, so a singular stress burst is observed. The precise acknowledgement of the stresses is needed to choose a constructive decision correctly. The existing calculation methods can give only an approximated valuation of it. The numerical methods just give only a fluctuating solution for this area. Quite a simple reliable solution shall be available to choose an optimal decision which could permit to evaluate the stresses and deformations picture.

As a three-dimensional deflected mode (DM) arises explicitly in this area, so in the first approximation let us consider the model as a solid elastic bar and evaluate DM near the restricted cross-section during the torsion by the concentrated moment.

2 Problem definition

Let the bar have an arbitrary cross-section. The lateral surface is free of external stresses. One bar butt is completely unchangeable. Another butt applied an arbitrary force system which is statically equivalent to the torque action M_z. The commencement of Cartesian coordinate system $x0y$ is in the centre of gravity of the unchangeable section while z axis is rectilinear and passes throughout the centres of gravity of the cross-sections.

In the absence of body forces the movements of an arbitrary point of the body shall satisfy Lame's equations

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} = 0, \quad i, j = 1,2,3, \quad (2.1)$$

were u_i is an arbitrary bar point movements induced by its deformation;

$\lambda = E\nu/(1+\nu)(1-2\nu)$, $\mu = E/2(1+\nu)$ are Lame's ratios;

E, ν are a modulus of elasticity and Poisson's ratio.

Boundary conditions shall be satisfied both on the lateral surface

$$\sigma_{ij} n_j = 0, i, \quad j = 1,2,3 \quad (2.2)$$

and the bar butts:

$$z = 0; \quad u_i = 0 \quad ,i, j = 1,2,3$$

$$z = l; \quad \iint_\Omega x\sigma_{33} d\Omega = M_y, \quad \iint_\Omega (x\sigma_{23} - y\sigma_{13}) d\Omega = 0, \quad (2.3)$$

$$\iint_\Omega y\sigma_{33} d\Omega = \iint_\Omega \sigma_{13} d\Omega = \iint_\Omega \sigma_{23} d\Omega = \iint_\Omega \sigma_{33} d\Omega = 0,$$

where σ_{ij} are stress tensor components, n_j are normal directional cosines to the surface; Ω is the cross-section area, and l is the bar length.

The strained and deformed bar condition is to be determined.

3 Method of solution

The tension burst having a damping exponential character along the longitudinal coordinate z is known to surge near the restricted section. There is a homogeneous stress faraway from the restricted section. Therefore, using the boundary layer conception, an asymptotic solution called an internal one can be achieved near the restricted section and be jointed to the external solution by van Dyke's method for a homogeneous stress. As an external solution any known one can be used for a free bending of the given geometry bar.

Let us enter an internal variable to construct an internal resolution

$$\chi = \frac{1}{k}\left(1 - e^{-nz}\right) \qquad (3.1)$$

where k is the small parameter characterizing the bar contraction ratio; n is a constant.

Let us write the internal resolution of the movements as a series designating $u = u_1, v = u_2, w = u_3$:

$$\begin{aligned} u^i &= ku_1^i(x,y,\chi) + k^2u_2^i(x,y,\chi), \\ v^i &= kv_1^i(x,y,\chi) + k^2v_2^i(x,y,\chi), \\ w^i &= kw_1^i(x,y,\chi) + k^2w_2^i(x,y,\chi). \end{aligned} \qquad (3.2)$$

Thereupon, having substituted the variables in equations (2.1) and resolutions (2.2) and equating like powers terms we will have a system of six equations:

$$\frac{\partial^2 u_1}{\partial \chi^2} = \frac{\partial^2 v_1}{\partial \chi^2} = \frac{\partial^2 w_1}{\partial \chi^2} = 0; \qquad (3.3)$$

$$\mu n \frac{\partial^2 u_2}{\partial \chi^2} - 2\mu n \frac{\partial^2 u_1}{\partial \chi^2}\chi + (\lambda+\mu)\frac{\partial^2 w_1}{\partial x \partial \chi} - \mu n \frac{\partial u_1}{\partial \chi} = 0;$$

$$\mu n \frac{\partial^2 v_2}{\partial \chi^2} - 2\mu n \frac{\partial^2 v_1}{\partial \chi^2}\chi + (\lambda+\mu)\frac{\partial^2 w_1}{\partial y \partial \chi} - \mu n \frac{\partial v_1}{\partial \chi} = 0;$$

$$(\lambda+2\mu)n\frac{\partial^2 w_2}{\partial \chi^2} - 2(\lambda+2\mu)n\frac{\partial^2 w_1}{\partial \chi^2} - (\lambda+2\mu)n\frac{\partial w_1}{\partial \chi} + $$

$$+ (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = 0.$$

The solution of the first three equations (3.3) satisfying the boundary conditions (2.3) at z=0 will be of the following form

$$\begin{aligned} &u_1 = a_1(x,y)\chi;\ v_1 = b_1(x,y)\chi; \\ &w_1 = c_1(x,y)\chi \end{aligned} \qquad (3.4)$$

Substituting (3.4) into the remained system equations (3.3) and when satisfying the boundary conditions (1.3) on the butt z=0 we shall have

$$u_2 = \frac{1}{2}\left(a_1(x,y) - \frac{\lambda+\mu}{n\mu}\frac{\partial c_1(x,y)}{\partial x}\right)\chi^2 + a_2(x,y)\chi \quad ;$$

$$v_2 = \frac{1}{2}\left(b_1(x,y) - \frac{\lambda+\mu}{n\mu}\frac{\partial c_1(x,y)}{\partial y}\right)\chi^2 + b_2(x,y)\chi \quad ; (3.5)$$

$$w_2 = \frac{1}{2}\left(c_1(x,y) - \frac{\lambda+\mu}{n(\lambda+2\mu)}\left(\frac{\partial a_1(x,y)}{\partial x} + \frac{\partial b_1(x,y)}{\partial y}\right)\right)\chi^2 +$$

$$+ c_2(x,y)\chi \quad .$$

Unknown transverse coordinates functions a_1, a_2 ,b_1, b_2, c_1, c_2 can be determined according to the conditions of the internal and external (2.2)resolutions jointing:

$$\left(u^0\right)^i = \left(u^i\right)^0; \left(v^0\right)^i = \left(v^i\right)^0; \left(w^0\right)^i = \left(w^i\right)^0, \qquad (3.6)$$

where "i" index means the resolution on the internal coordinate (3.1), while "0" on the external coordinate z.

Having written the joining condition (3.6) and equating the terms of the same order k, we receive six equations relative to six unknown functions the solution of which will be the internal resolution.

Then, a compound, evenly suitable for all bar solution can be achieved by the joining procedure [1]. However, the compound solution cannot be the single one. The solutions can differ in the joining area. A specified choice of n constant input into the internal variable (3.1) enables to determine the compound resolution in just only way.

It is logical to use for it the principle of the bar strain potential energy minimum [2]

$$\partial \mathrm{E}/\partial n = 0, \qquad (3.7)$$

where $\mathrm{E} = \int_0^l \iint_\Omega \frac{1}{2}\sum_{i,j=1}^{3} \sigma_{i,j}\varepsilon_{i,j} d\Omega dl$ is the bar strain potential energy; ε_{ij} are components of the tensor of strain.

Equation (3.7) determines explicitly the n constant which characterizes the constraint effect.

4 Solution

The three-dimensional problem of the free torsion of solid slightly conic bars with an arbitrary cross-section was solved by D.Yu. Panov [3] within the method of small parameter. So, let this solution be used as an external resolution.

As well as in the works [3] a bar which lateral surface is determined by equation

$$f\left(x(1-kz), y(1-kz)\right)=0$$

will be considered.

Otherwise, $f(\xi,\eta)=0$ if auxiliary variables $\xi=x(1-kz)$ and $\eta=y(1-kz)$ are entered. The third auxiliary variable ζ coincides with axis z.

Either a value of cross-sectional dimensions relative change per a unit of bar length or relative bar k tapering is used as the small parameter. Further let the value k be considered so small that value k^2 can be neglected relative to k in the first power.

Let us write an internal resolution of movements like (3.7), considering second-order of smallness of k in it.

$$\begin{aligned} u^i &= ku_1^i(x,y,\chi)+k^2u_2^i(x,y,\chi),\\ v^i &= kv_1^i(x,y,\chi)+k^2v_2^i(x,y,\chi),\\ w^i &= kw_1^i(x,y,\chi)+k^2w_2^i(x,y,\chi). \end{aligned} \tag{4.1}$$

Putting components of movements (3.4) and (3.5) into (4.1), an internal resolution result. The joining conditions (3.6) shall be used to determine six unknown functions a_1, b_1, c_1, a_2, b_2, c_2. To the effect let the internal resolution (4.1) expressed through the external variable z be equal to the external resolution [3] with internal variable χ (3.5).

$$\begin{aligned} &a_1(x,y)\left(1-e^{-nz}\right)+\frac{1}{2}a_1(x,y)\left(1-e^{-nz}\right)^2-\frac{1}{2}\frac{\lambda+\mu}{n\mu}\frac{\partial c_1(x,y)}{\partial x}\left(1-e^{-nz}\right)^2+ka_2(x,y)\left(1-e^{-nz}\right)=\frac{1}{n}\tau y\left(1-e^{-nz}\right)+k\tau P_1(x,y),\\ &b_1(x,y)\left(1-e^{-nz}\right)+\frac{1}{2}b_1(x,y)\left(1-e^{-nz}\right)^2-\frac{1}{2}\frac{\lambda+\mu}{n\mu}\frac{\partial c_1(x,y)}{\partial y}\left(1-e^{-nz}\right)^2++kb_2(x,y)\left(1-e^{-nz}\right)=\\ &=\frac{1}{n}\tau x\left(1-e^{-nz}\right)+k\tau P_2(x,y), \end{aligned} \tag{4.2}$$

where τ is relative bar torsion per a unit of length;

$$P_1(x,y)=-\frac{1}{2}y\left(x^2+y^2\right)+x\Phi(x,y)-(1-2\nu)\left(\int^x\Phi dx+h_1(y)\right);$$

$$P_2(x,y)=-\frac{1}{2}x\left(x^2+y^2\right)+y\Phi(x,y)-(1-2\nu)\left(\int^y\Phi dy+h_2(x)\right);$$

Φ(x,y) is the torsion function.

Functions $h_1(y)$ and $h_2(x)$ are selected from the condition

$$\frac{\partial}{\partial y}\left(\int^x\Phi(x,y)dx+h_1(y)\right)+\frac{\partial}{\partial x}\left(\int^y\Phi(x,y)dy+h_2(x)\right)=0 \tag{4.3}$$

Let us consider that in transitional area where occurs joining $\left(1+e^{-nz}\right)\cong 1$. In such a case the equations (4.2) cold be re-written as

$$\frac{3}{2}a_1(x,y)-\frac{1}{2}\frac{\lambda+\mu}{n\mu}\frac{\partial c_1(x,y)}{\partial x}+ka_2(x,y)=-\frac{1}{n}\tau y+k\tau P_1(x,y)$$

$$\frac{3}{2}b_1(x,y)-\frac{1}{2}\frac{\lambda+\mu}{n\mu}\frac{\partial c_1(x,y)}{\partial y}+kb_2(x,y)=-\frac{1}{n}\tau x+k\tau P_2(x,y)$$

$$\frac{3}{2}c_1(x,y)-\frac{1}{2}\frac{\lambda+\mu}{n(\lambda+2\mu)}\left(\frac{\partial a_1(x,y)}{\partial x}+\frac{\partial b_1(x,y)}{\partial y}\right)+$$
$$+kc_2(x,y)=\tau\Phi(x,y)$$

Equating terms with like powers k, the unknown functions will be found:

$$a_1(x,y)=-\frac{2}{3}\frac{1}{n}\tau y+\frac{1}{3}\frac{\lambda+\mu}{n\mu}\frac{\partial c_1(x,y)}{\partial x};$$

$$a_2(x,y)=\tau P_1(x,y);$$

$$b_1(x,y)=-\frac{2}{3}\frac{1}{n}\tau x+\frac{1}{3}\frac{\lambda+\mu}{n\mu}\frac{\partial c_1(x,y)}{\partial y};$$

$$b_2(x,y)=\tau P_2(x,y); \qquad c_2(x,y)=0$$

To determine $c_1(x,y)$ there is an equation

$$\frac{3}{2}c_1-\frac{1}{6}\frac{(\lambda+\mu)^2}{n^2\mu(\lambda+2\mu)}\left(\frac{\partial^2c_1}{\partial x^2}+\frac{\partial^2c_1}{\partial y^2}\right)=\tau\Phi(x,y)$$

Considering that $\frac{\partial^2\Phi}{dx^2}+\frac{\partial^2\Phi}{dy^2}$, there will be found that $c_1=\frac{2}{3}\tau\Phi$ and accordingly

$$a_1=-\frac{2}{3}\frac{1}{n}\tau y+\frac{2}{9}\tau\frac{\lambda+\mu}{n\mu}\frac{\partial\Phi}{\partial x};$$
$$b_1=-\frac{2}{3}\frac{1}{n}\tau x+\frac{2}{9}\tau\frac{\lambda+\mu}{n\mu}\frac{\partial\Phi}{\partial y}.$$

The compound solution satisfying the problem alongside all the segment $[0,l]$ will be determined as following [4].

$$u=u^0+u^i-\left(u^i\right)^0,$$
$$v=v^0+v^i-\left(v^i\right)^0,$$
$$w=w^0+w^i-\left(w^i\right)^0.$$

Putting the resolutions of movements into (4.4) an evenly suitable solution of the problem will be achieved accurate within k^2

$$\begin{aligned}
u&=-\tau yz+\frac{\tau}{3n}\left(y+\frac{2}{3}\frac{\lambda+\mu}{\mu}\frac{\partial\Phi}{\partial x}\right)\left(1-e^{-nz}\right)e^{-nz}+\\
&\quad+2k\tau yz^2+k\tau P_1\left(1-e^{-nz}\right)\\
v&=-\tau xz+\frac{\tau}{3n}\left(x-\frac{2}{3}\frac{\lambda+\mu}{\mu}\frac{\partial\Phi}{\partial y}\right)\left(1-e^{-nz}\right)e^{-nz}\\
&\quad-2k\tau xz^2+k\tau P_2\left(1-e^{-nz}\right)\\
w&=\frac{2}{3}\tau\Phi\left(1-e^{-nz}\right)+\frac{1}{3}\tau\Phi\left(1-e^{-nz}\right)^2+k\tau zP_3(x,y)
\end{aligned}\qquad(4.5)$$

where $P_3(x,y)=-2\Phi(x,y)-x\frac{\partial\Phi}{\partial x}-y\frac{\partial\Phi}{\partial y}$.

Using the known formulae of the linear theory of elasticity, the stresses corresponding to the received system of movements (4.5)will be found.

$$\begin{aligned}
\sigma_{11}&=\frac{4}{9}\frac{\lambda+\mu}{n}\tau\frac{\partial^2\Phi}{\partial x^2}\left(1-e^{-nz}\right)e^{-nz}+\frac{2}{3}\lambda\tau n\Phi\left(2-e^{-nz}\right)e^{-nz}+2k\mu\tau x\left(\frac{\partial\Phi}{\partial x}-y\right)\left(1-e^{-nz}\right)-k\lambda\tau\left(x\frac{\partial\Phi}{\partial x}+y\frac{\partial\Phi}{\partial y}\right)e^{-nz}-2k\lambda\tau\Phi e^{-nz},\\
\sigma_{12}&=\frac{4}{9}\frac{\lambda+\mu}{n}\tau\frac{\partial^2\Phi}{\partial x\partial y}\left(1-e^{-nz}\right)e^{-nz}+k\mu\tau\left(\left(\frac{\partial\Phi}{\partial y}+x\right)x+\left(\frac{\partial\Phi}{\partial x}-y\right)y\right)\left(1-e^{-nz}\right),\qquad(4.6)\\
\sigma_{22}&=\frac{4}{9}\frac{\lambda+\mu}{n}\tau\frac{\partial^2\Phi}{\partial y^2}\left(1-e^{-nz}\right)e^{-nz}+\frac{2}{3}\lambda\tau n\Phi\left(2-e^{-nz}\right)e^{-nz}+2k\mu\tau y\left(\frac{\partial\Phi}{\partial y}+x\right)\left(1-e^{-nz}\right)-k\lambda\tau\left(x\frac{\partial\Phi}{\partial x}+y\frac{\partial\Phi}{\partial y}\right)e^{-nz}-2k\lambda\tau\Phi e^{-nz},\\
\sigma_{23}&=\mu\tau x\left(1-e^{-nz}\right)+\frac{2}{3}\mu\tau\frac{\partial\Phi}{\partial y}\left(1-e^{-nz}\right)+\frac{1}{3}\mu\tau\frac{\partial\Phi}{\partial y}\left(1-e^{-nz}\right)^2+\frac{2}{3}\mu\tau x\left(2-e^{-nz}\right)e^{-nz}+\\
&\quad+\frac{2}{9}(\lambda+\mu)\tau\frac{\partial\Phi}{\partial y}\left(2e^{-nz}-1\right)e^{-nz}+k\mu\tau nP_2e^{-nz}-3k\mu\tau z\left(\frac{\partial\Phi}{\partial y}+x\right)-k\mu\tau z\left(x\frac{\partial^2\Phi}{\partial x\partial y}+y\frac{\partial^2\Phi}{\partial y^2}\right)-k\mu\tau xz,\\
\sigma_{33}&=\frac{2}{3}(\lambda+2\mu)\tau n\Phi\left(2-e^{-nz}\right)e^{-nz}-2k\mu\tau\left(x\frac{\partial\Phi}{\partial x}+y\frac{\partial\Phi}{\partial y}\right)-2k\mu\tau\Phi+2k\lambda\tau\Phi2\nu\left(1-e^{-nz}\right)-k\lambda\tau\left(x\frac{\partial\Phi}{\partial x}+y\frac{\partial\Phi}{\partial y}\right)e^{-nz}.
\end{aligned}$$

Condition (3.7) enables to determine n constant explicitly.

Thereby, the solution of the given geometry bar bending torsion problem is determined by expressions (4.5) and (4.6) completely.

PRISMATIC BAR

The received solutions (4.5) and (4.6) show that at the bending torsion of small tapering bars all stresses and movements are nonzero ones. Let us consider now an extreme case of the pass of a slightly tapering bar into a prismatic one. I.e., parameter k in solutions (4.5) and (4.6) is vanishing.

So, the stresses and movements for prismatic bars are as following

$$\begin{aligned}
u&=-\tau yz+\frac{1}{3n}\tau\left(y-\frac{2}{3}\frac{\lambda+\mu}{\mu}\frac{\partial\Phi}{\partial x}\right)\left(1-e^{-nz}\right)e^{-nz},\\
v&=-\tau xz-\frac{1}{3n}\tau\left(x-\frac{2}{3}\frac{\lambda+\mu}{\mu}\frac{\partial\Phi}{\partial y}\right)\left(1-e^{-nz}\right)e^{-nz},\qquad(4.7)\\
w&=\frac{2}{3}\tau\Phi\left(1-e^{-nz}\right)+\frac{1}{3}\tau\Phi\left(1-e^{-nz}\right)^2
\end{aligned}$$

$$\sigma_{11}=\frac{4}{9}\frac{\lambda+\mu}{n}\tau\frac{\partial^2\Phi}{\partial x^2}\left(1-e^{-nz}\right)e^{-nz}+\frac{2}{3}\lambda\tau n\Phi\left(2-e^{-nz}\right)e^{-nz},\ \sigma_{12}=\frac{4}{9}\frac{\lambda+\mu}{n}\tau\frac{\partial^2\Phi}{\partial x\partial y}\left(1-e^{-nz}\right)e^{-nz},$$
$$\sigma_{22}=\frac{4}{9}\frac{\lambda+\mu}{n}\tau\frac{\partial^2\Phi}{\partial y^2}\left(1-e^{-nz}\right)e^{-nz}+\frac{2}{3}\lambda\tau n\Phi\left(2-e^{-nz}\right)e^{-nz},$$
$$\sigma_{13}=-\mu\tau y\left(1-e^{-nz}\right)+\frac{2}{3}\mu\tau\frac{\partial\Phi}{\partial x}\left(1-e^{-nz}\right)+\frac{1}{3}\mu\tau\frac{\partial\Phi}{\partial x}\left(1-e^{-nz}\right)^2-\frac{2}{3}\mu\tau y\left(2-e^{-nz}\right)e^{-nz}+ \qquad (4.8)$$
$$+\frac{2}{9}(\lambda+\mu)\tau\frac{\partial\Phi}{\partial x}\left(2e^{-nz}-1\right)e^{-nz},$$
$$\sigma_{23}=\mu\tau x\left(1-e^{-nz}\right)+\frac{2}{3}\mu\tau\frac{\partial\Phi}{\partial y}\left(1-e^{-nz}\right)+\frac{1}{3}\mu\tau\frac{\partial\Phi}{\partial y}\left(1-e^{-nz}\right)^2+\frac{2}{3}\mu\tau x\left(2-e^{-nz}\right)e^{-nz}+$$
$$+\frac{2}{9}(\lambda+\mu)\tau\frac{\partial\Phi}{\partial y}\left(2e^{-nz}-1\right)e^{-nz},$$
$$\sigma_{33}=\frac{2}{3}(\lambda+2\mu)\tau n\Phi\left(2-e^{-nz}\right)e^{-nz}.$$

It is obtained that the prismatic bar all the same has a compound stress. Faraway from this area e^{-nz} is vanishing and expressions (4.7) and (4.8) give the Saint-Venant's solution.

When considering the fixed section as z=0 it turns out that the normal stresses can even exceed tangents here, and the tangent stress σ_{12}, as supposed, equals to zero. And the biggest of the normal stresses will be the longitudinal stress σ_{33} because

$$\frac{\sigma_{33}}{\sigma_{11}}=\frac{\sigma_{33}}{\sigma_{22}}=1+2\frac{\mu}{\lambda}$$

For example, at Poisson's ratio **ν**=1/3 the longitudinal normal stress σ_{33} will be twice more than transversal ones.

BARS OF SOME KINDS OF CROSS-SECTIONS

Let us consider a solution of the problem for some cross-section Ω specific areas in detail.

Let area Ω represent an ellipse with semi axes *a* and *в* in section Z=0 and be determined by the equation

$$(\xi,\eta)=\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1$$

where $\xi=X(1-KZ),\ \eta=Y(1-KZ)$

The torsion function for this area is known to appear as

$$\Phi(\xi,\eta)=C\eta\xi,$$

where $C=-\left(a^2-b^2\right)/\left(a^2+b^2\right)$.

And the relative angle of bar torsion per a unit of length equals to

$$\tau=\frac{M_Z\left(a^2+b^2\right)}{\pi\mu a^3b^3}.$$

Functions $h_1(\eta)$ and $h_2(\xi)$are taken from condition (4.3), when opening, it results

$$\frac{1}{2}C(\xi^2+\eta^2)+\frac{\partial h_1}{\partial\eta}+\frac{\partial h_2}{\partial\xi}=0.$$

Hence $\quad h_1=-\frac{1}{6}C\eta^3,\quad h_2=-\frac{1}{6}C\xi^3,$

otherwise discarding terms which have *K* squared or in higher powers, it results

$$u=-\tau yZ(1-2KZ)+\frac{1}{3n}\tau y\left(1+\frac{2}{3}\frac{\lambda+\mu}{\mu}C\right)\left(1-e^{-n*Z}\right)e^{-nZ}+$$
$$+K\tau Cy\left[x^2-\left(\frac{1}{2}-\nu\right)\left(x^2-\frac{1}{3}y^2\right)\right]\left(1-e^{-nZ}\right)-$$
$$-\frac{1}{2}K\tau y\left(x^2+y^2\right)\cdot\left(1-e^{-nZ}\right), \qquad (4.9)$$
$$\upsilon=\tau xZ(1-2KZ)-\frac{1}{3n}\tau x\left(1-\frac{2}{3}\frac{\lambda+\mu}{\mu}C\right)\left(1-e^{-nZ}\right)e^{-nZ}+$$
$$+K\tau Cx\left[y^2-\left(\frac{1}{2}-\nu\right)\left(y^2-\frac{1}{3}x^2\right)\right]\left(1-e^{-nZ}\right)+$$
$$+\frac{1}{2}K\tau x\left(x^2+y^2\right)\cdot\left(1-e^{-nZ}\right),$$
$$\omega=\frac{2}{3}\tau Cxy\left(1-e^{-nZ}\right)+\frac{1}{3}\tau Cxy\left(1-e^{-nZ}\right)^2-4K\tau Cxyz.$$

For stresses in formulae (3.9) it results

$$\sigma_{11}=\frac{2}{3}\lambda n\tau Cxy\left(2-e^{-nZ}\right)e^{-nZ}+2K\mu\tau xy(C-1)\left(1-e^{-nZ}\right)-4K\lambda\tau Cxye^{-nZ},$$

$$
\begin{aligned}
\sigma_{12} &= \frac{4}{9}\frac{\lambda+\mu}{n}\tau C\left(1-e^{-nZ}\right)e^{-nZ} + K\mu\tau\left[(C+1)x^2+(C-1)y^2\right]\left(1-e^{-nZ}\right),\\
\sigma_{13} &= -\mu\tau y\left(1-e^{-nz}\right)+\frac{2}{3}\mu\tau Cy\left(1-e^{-nz}\right)+\frac{1}{3}\mu\tau Cy\left(1-e^{-nz}\right)^2-\frac{2}{3}\mu\tau y\left(2-e^{-nz}\right)e^{-nZ}+\frac{2}{9}(\lambda+\mu)\tau yC\left(2e^{-nz}-1\right)e^{-nz}-\\
&\quad -4k\mu\tau yz(C-1)+k\mu\tau nCy\left[x^2-\left(\frac{1}{2}-\nu\right)\left(x^2-\frac{1}{3}y^2\right)\right]e^{-nz}-\frac{1}{2}k\mu\tau ny\left(x^2+y^2\right)e^{-nz},\\
\sigma_{22} &= \frac{2}{3}\lambda n\tau Cxy\left(2-e^{-nz}\right)e^{-nz}+2k\mu\tau xy(C+1)\left(1-e^{-nZ}\right)-4k\lambda\tau Cxye^{-nz},\\
\sigma_{23} &= \mu\tau x\left(1-e^{-nz}\right)+\frac{2}{3}\mu\tau Cx\left(1-e^{-nz}\right)+\frac{1}{3}\mu\tau Cx\left(1-e^{-nz}\right)^2+\frac{2}{3}\mu\tau x\left(2-e^{-nZ}\right)e^{-nz}+\frac{2}{9}(\lambda+\mu)\tau Cx\left(2e^{-nz}-1\right)e^{-nz}-\\
&\quad -4k\mu\tau xz\cdot(C+1)+k\mu\tau nCx\left[y^2-\left(\frac{1}{2}-\nu\right)\left(y^2-\frac{1}{3}x^2\right)\right]e^{-nz}+\frac{1}{2}k\mu\tau nx\left(x^2+y^2\right)e^{-nz},\\
\sigma_{33} &= \frac{2}{3}(\lambda+2\mu)\tau nCxy\left(2-e^{-nz}\right)e^{-nz}-2K\tau\lambda Cxye^{-nz}+2k\tau\lambda Cxy\left[2\nu\left(1-e^{-nz}\right)-1\right]-8k\tau\mu Cxy.
\end{aligned}
\qquad (4.10)
$$

Let us write equation (2.14) to find out the n ratio.

$$
\begin{aligned}
&n^4\frac{44}{81}a^2b^2C^2\left(\frac{\lambda}{\mu}+2\right)+n^2\left\{a^2\left[\frac{2}{3}(C+1)(2B_1-B_2)-\frac{1}{81}B_1^2-\frac{1}{18}B_2^2\right]-4b^2\left[(C-1)(2A_1+A_3)+\frac{2}{3}A_1A_3+\frac{1}{2}A_1^2+\frac{1}{4}A_3^2\right]\right\}+\\
&+nK\left\{16b^2(C-1)(4A_1+A_2)-\frac{64}{27}C\left(\frac{\lambda}{\mu}+1\right)\left[a^2(C+1)+b^2(C-1)\right]\right\}-\frac{16}{27}\left(\frac{\lambda}{\mu}+1\right)^2C^2=0,
\end{aligned}
\qquad (4.11)
$$

where

$$A_1=-\frac{14}{9}C-\frac{1}{3}-\frac{2}{9}\frac{\lambda}{\mu}C;\; A_2=\frac{1}{2}(C-1)+\nu C;$$

$$A_3=\frac{7}{9}C+\frac{2}{3}+\frac{4}{9}\frac{\lambda}{\mu}C;$$

$$B_1=4C+\frac{2}{3}\left(\frac{\lambda}{\mu}+1\right)C-1;\; B_2=C+\frac{4}{3}\left(\frac{\lambda}{\mu}+1\right)C-2.$$

It is not a difficult task to solve equation (4.11) relative to n. Only a real positive root satisfies the problem solution

Parameter n is an important characteristic of the constraint effect which specifies not only magnitude of the boundary effect area resulted from constraint, but both intensity and tension change character in the area as well. Besides, n parameter enables to make a qualitative analysis of deflected mode of the material in the area influenced by the constraint effect linked with geometric dimensions bar and material characteristics. So, on the assumption of equation (4.11) one can conclude that n, and consequently the constraint effect, in case of a prismatic bar does not depend on its length that reaffirms ipso facto the Saint-Venant's hypothesis. The tapering has some, but not too strong, influence over n parameter. It is obvious from the graph of dependence of dimensionless factor $n^*=n^*a$ on the dimensionless factor $k^*=k^*l$, at $b/a=0.5$ (Fig. 1). This dependence has a practically linear appearance at any Poisson's ratio value. A small growth of n with the bar tapering increase causes a constraint area reduction. The normal stress σ_{33} reduction in a constraint section Z=0 with the tapering increase reaffirms the Ye. P. Grossman's conclusion [5] about the constraint effect and the tapering effect creating normal stresses with different signs at torsion.

Dimensionless ratio n^* dependences on semi axes ellipse ratio at the prismatic bar torsion enable to make some qualitative conclusions: firstly, ratio n^* has the minimum value at b/a=0.85-0.9, i.e. the bars with such semi axes ratio has the biggest extension area of the boundary effect caused by constraint; secondly, on increasing the Poisson's effect the boundary effect area reduces because of the n^* increasing, and normal stresses slightly grow up; thirdly, the constrain effect vanishes for a round bar ($b/a\rightarrow 1$) and a very thin plate ($b/a\rightarrow 0$), as it was expected because ratio $n^*\rightarrow\infty$.

Comparing normal dimensionless stress distribution graphs

$\sigma_{33}{}^{*}=\sigma_{33}{}^{*}a^3/M_z$

along line AB of the elliptic section prismatic bar at the following calculated data: ν=0.32, b/a=0.5, a/b=0.25, x/a=0.75, y/a=-0.33

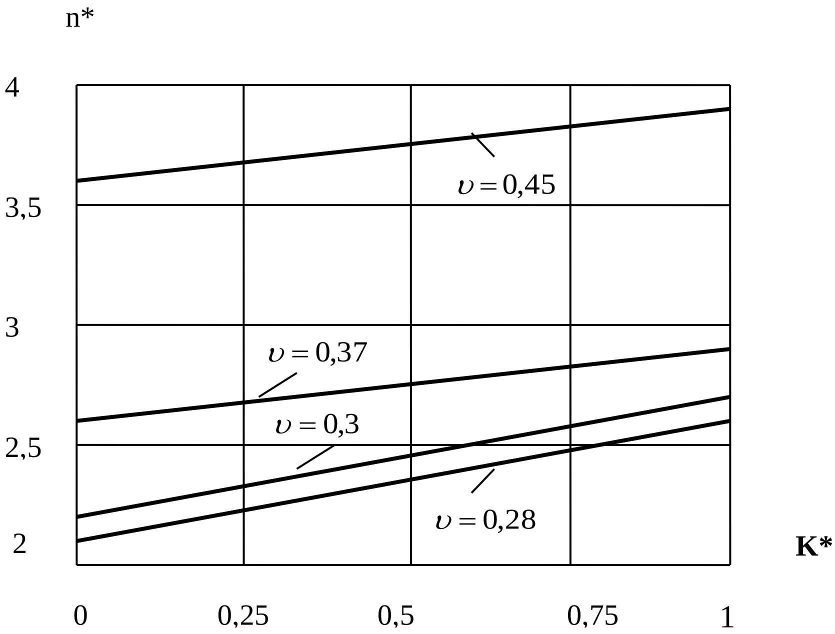

Fig. 1 Dependence of dimensionless parameter $n^* = na$ on the bar tapering

(Fig.2) shows that σ_{33}* has the same character of behavior in all the three comparable solutions. The divergence is observed in numerical values and reaches difference of 25% between the solution presented in this work and the N.V. Zvolinsky's solution [6] in the constrained section. The Foeppl's results differ essentially less.

The N.V. Zvolinsky's, Foeppl's and S.P. Timoshenko's [7] solutions do not satisfy all elastostatics equations. In the first solution do not satisfy precisely the equilibrium equations, two other solutions do not satisfy Saint Venant identity, i.e. those are approximate solutions. The solution given in this work has a more common character because it is true for arbitrary cross-section bars bothprismatic ones and those having deflection from prismaticity. The longitudinal normal stress overstating as compared with other solutions is explained, perhaps, by non-complete satisfaction of the boundary conditions on the lateral surface of the bar near the constrained section z=0,

The comparative graphs for the distribution of dimensionless normal and tangent stresses $\sigma^*_{ij} = \sigma_{ij} \cdot a^3 / M_z$ along axis z at prismatic elliptic bar torsion are given in fig.3. The calculated data are the same. The normal lateral stress σ^*_{22} will be equal to σ^*_{11} because the bar has a constant section alongside.

The extreme case of the ellipse degeneracy into a circle is important practically. The solution for the round section can be obtained supposing C=0 in (4.9) and (4.10).

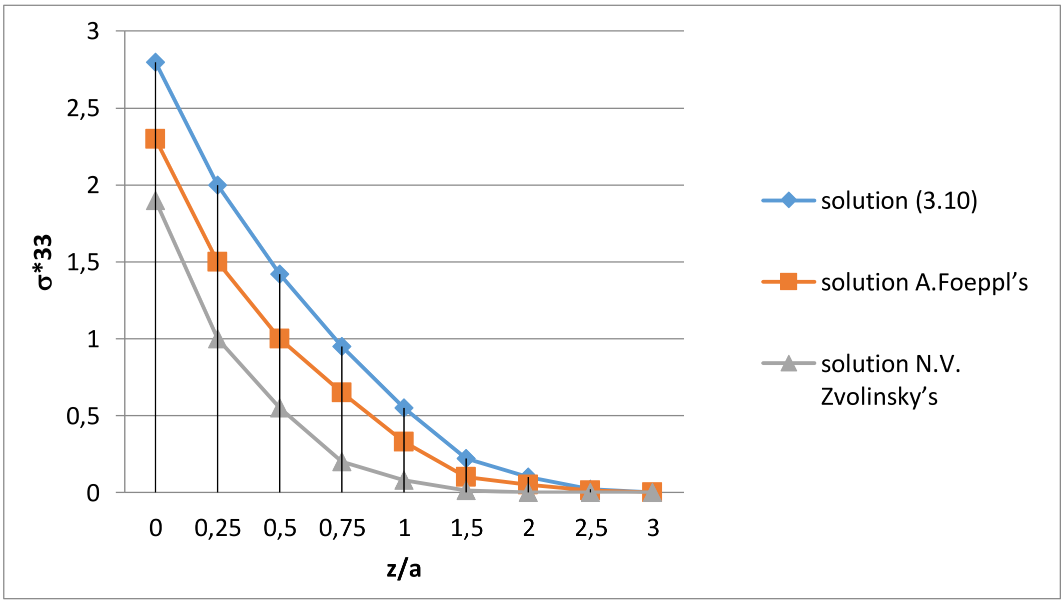

Fig.2 Comparative graphs for the distribution of dimensionless longitudinal normal stress σ_{33}* at the elliptic bar torsion

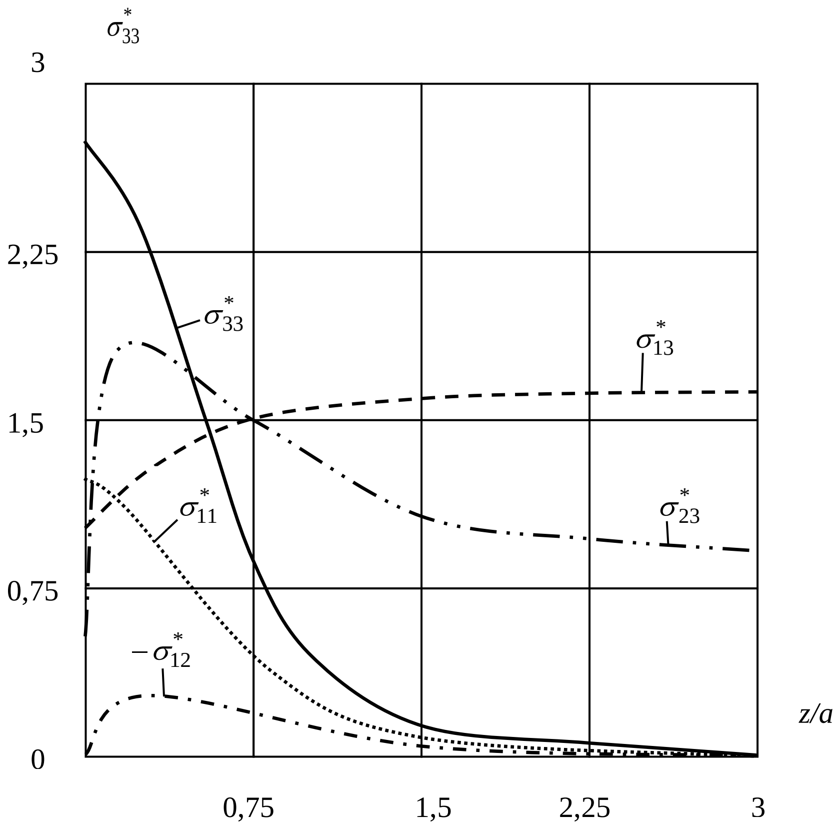

Fig.3 Comparative graphs for the distribution of normal and tangent stresses along axis z at the elliptic bar torsion

$$u = -\tau yz + \frac{1}{3n}\tau y(1-e^{-nz})e^{-nz},$$
$$v = \tau xz - \frac{1}{3n}\tau x(1-e^{-nz})e^{-nz}, \qquad (4.12)$$

$$\omega = 0,$$
$$\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{33} = 0,$$
$$\sigma_{13} = -\mu\tau y + \mu\tau y e^{-nz} - \frac{2}{3}\mu\tau y(2-e^{-nz})e^{-nz},$$
$$\sigma_{23} = \mu\tau x - \mu\tau x e^{-nz} + \frac{2}{3}\mu\tau x(2-e^{-nz})e^{-nz}. \qquad (4.13)$$

Expressions (4.12) and (4.13) depend on the constraint. However, as it was mentioned previously for the round section case $n \to \infty$, therefore expressions (4.12) and (4.13) are reduced to a free torsion solution. It is affirmed completely in practice.

Thereby, as torsion function $\Phi(\xi,\eta)$ is known for many kinds of cross-sections, therefore expressions for movements and stresses an constrained torsion can be written down easily.

The given solution permits to assess the level of all stresses and deformations depending on section form and material elastic responses. As the elastic problem solution has been received, the solution for a real stress case will be obtained by writing an analogous solution for other stresses and applying the principle of superposition.

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