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EULERIAN DERIVATIONS OF NON-INERTIAL NAVIER-STOKES EQUATIONS

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Abstract

The paper presents an Eulerian derivation of the non-inertial Navier-Stokes equations as an alternative to the Lagrangian fluid parcel approach. This work expands on the work of Kageyama and Hyodo [1] who derived the incompressible momentum equation for constant rotation for geophysical applications. In this paper the derivation is done for the Navier-Stokes equations in compressible flow for arbitrary rotation for implementation in aero-ballistic applications.

1 Introduction

Derivation of the non-inertial Navier-Stokes equations (conservation of mass, momentum and energy) is generally done using the fluid parcel approach. Although this method leads to the correct set of equations, it does not clearly indicate the origin of the fictitious forces and can lead to misconceptions.

In deriving the conservation of momentum equation, Newton's second law is modified to include the fictitious forces as body forces in the same manner as which the gravity force is handled:

$$\sum \mathbf{F} + \sum \mathbf{F}_{\text{fictitious}} = m\mathbf{a} \tag{1}$$

The fictitious forces are derived separately using a point mass method to obtain a relation for the inertial acceleration in terms of the non-inertial acceleration components [2]:

$$\mathbf{a} = \frac{d^2\mathbf{X}}{dt^2} + \dot{\boldsymbol{\Omega}}\wedge\mathbf{x} + \boldsymbol{\Omega}\wedge(\boldsymbol{\Omega}\wedge\mathbf{x}) + \dot{\mathbf{V}} + 2\boldsymbol{\Omega}\wedge\mathbf{V} \tag{2}$$

These accelerations are multiplied by the density to obtain the momentum form of the fictitious effects and included on the right hand side of the momentum equation.

This approach, although simple and intuitive, is not rigorous and lends itself to mistakes with regards to the nature of the fictitious forces. It has been observed in literature that these fictitious effects are erroneously added in the conservation of energy equation when this Lagrangian approach is used.

Kageyama and Hyodo [1] proposed an Eulerian method for the derivation of the Coriolis and centrifugal forces in the momentum equation. The derivation was limited to incompressible flow in pure rotation since their application is in the Geophysical field.

In this paper the work of Kageyama and Hyodo [1] is expanded upon to include the full set of Navier-Stokes equations for compressible flow in arbitrary rotation for aero-ballistic applications.

2 Non-inertial Navier-Stokes Equations for Constant, Pure Rotation

This section involves the derivation of the non-inertial Navier-Stokes equations for constant rotation. This section is based on the work of Kageyama & Hyodo [1] and forms the basis for the subsequent section where the work is expanded upon to derive the full set of Navier-Stokes equations for compressible flow in variable rotation.

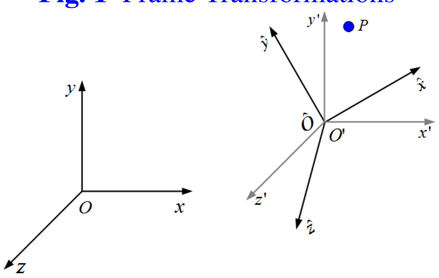
2.1 Frame Transformations

The first step in the derivation is to define the relation of the inertial and non-inertial frames with respect to each other. These relations are mathematically described in terms of transformation operators and is used to change the perspective of the observer.

Assume that three (3) frames exist; O, O' and Ô as indicated in Figure 1. Frame O is the stationary, inertial frame. Frame O' is an orientation preserving frame(**i** and **i**' has the same orientation), which can be either inertial or non-inertial depending on the cases analysed. This frame shares an origin with the rotational frame Ô. Frame Ô is the non-inertial, rotational frame and is therefore not orientation preserving.

Now consider a point P which can be observed from all the frames. Point P is rotating around the origin of frame O, but it is stationary in frames O' and Ô. The set of equations will be developed to describe the motion of point P in the rotational frame Ô.

Fig. 1 Frame Transformations



This point is described in frame O from where a modified Galilean transformation, G^{M}, will be used to describe it in frame O'. The rotational transform, $R^{\Omega t}$, will then be used to transform the resulting equations (as described in frame O') to the rotational frame Ô.

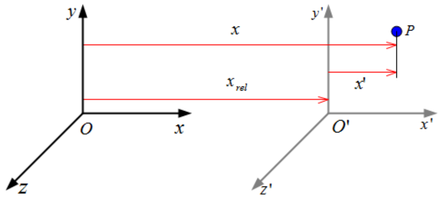
2.1.1 Modified Galilean Transformation

The standard Galilean transform is limited in its application to constant translation in the x-direction. Kageyama & Hyodo [1] modified it to accommodate constant rotational conditions.

The Galilean transform is used to transform between two reference frames that only differ by a constant vector of motion. In Figure 2 such a motion is described between frame O and O'.

Assume that the origins of the two frames intersect at time $t = 0$ and that frame O' is moving at a constant velocity $\mathbf{V}$ in the x-direction. At time $t = \Delta t$, the frames O and O' are then distance x_{rel} from each other.

Fig. 2 Galilean Transformation between Frames

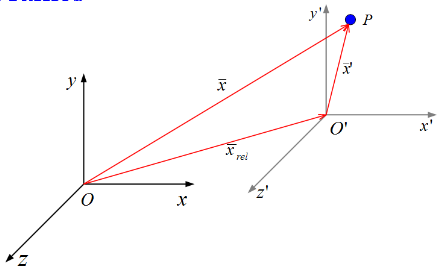


The relationship between the co-ordinates points for this single event between frames O and O' is described by Equation 3. This is known as the standard Galilean transform.

$$\begin{aligned} x' &= x - V\Delta t \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{3}$$

Lets further assume at this point that the constant motion need not be in the x-direction alone and that it can be presented as a vector of motion as shown in Figure 3. Lets further assume that it can be used to described constant motion in rotation as well.

Fig. 3 Modified Galilean Transformation between Frames



In order to simplify this case let all the frames share the same origin and let the point P be stationary in the rotational frame Ô. Therefore point P is rotating with a constant angular velocity around the origin or the inertial frame O. The $\mathbf{x_{rel}}$ component can then be described as:

$$\mathbf{x_{rel}} = \mathbf{V}\Delta t \tag{4}$$

where

$$\mathbf{V} = \mathbf{\Omega} \wedge \mathbf{x} \tag{5}$$

The modified Galilean transform operator is introduced such that any vector observed from the inertial frame O can be related to the vector observed from the orientation preserving frame O' as:

$$\mathbf{u}'(\mathbf{x}', t) = G^{\mathbf{M}}\mathbf{u}(\mathbf{x}, t) \tag{6}$$

This definition will lead to a mathematical description to directly relate the vector fields in the inertial frame O, to the vector fields in the orientation preserving frame O':

$$\begin{aligned} \mathbf{u}'(\mathbf{x}', t) &= G^{\mathbf{M}}\mathbf{u}(\mathbf{x}, t) \\ &= G^{\mathbf{\Omega}\wedge\mathbf{x}}\mathbf{u}(\mathbf{x}, t) \\ &= \mathbf{u}(\mathbf{x}, t) - \mathbf{\Omega} \wedge \mathbf{x} \\ \mathbf{u}'(\mathbf{x}', t) &= \mathbf{u}(\mathbf{x}, t) + \mathbf{x} \wedge \mathbf{\Omega} \end{aligned} \tag{7}$$

2.1.2 Rotational Transformation

Since frame Ô shares an origin with the frame O' the vector components in Ô is related to O' by defining a rotational transform, $R^{\Omega t}$. Equation 7 can be used to describe a vector as seen from frame Ô in relation to a vector in frame O.

$$\begin{aligned} \hat{\mathbf{u}}(\hat{\mathbf{x}}, t) &= R^{\Omega t}\mathbf{u}'(\mathbf{x}', t) \\ &= R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}\mathbf{u}(\mathbf{x}, t) \end{aligned} \tag{8}$$

$R^{\Omega t}$ is therefore the rotational transform that operates on $\mathbf{x}'$ to obtain the $\hat{\mathbf{x}}$ co-ordinates in the rotational frame. From Equation 7 and 8 it can be derived that for the velocity vector the following relation holds:

$$\hat{\mathbf{u}}(\hat{\mathbf{x}}, t) = R^{\Omega t}\{\mathbf{u}(\mathbf{x}, t) + \mathbf{x} \wedge \mathbf{\Omega}\} \tag{9}$$

Lets assume, for convenience sake, that the rotation is around the z-axis of frame O. The vector $\mathbf{\Omega}$ is then described as $\mathbf{\Omega} = (0, 0, \Omega)$. The rotational transform in this case will be described by the following tensor:

$$R^{\Omega t} = \begin{bmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

The first column of this tensor is the projection of the $\mathbf{x}'$ component on $\hat{\mathbf{x}}$,$\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. In the same manner is the second and third columns the projection of $\mathbf{y}'$ and $\mathbf{z}'$ respectively on the rotational axes. The quantitative values of the rotation tensor will be different for each case.

Now that the modified Galilean invariance and the rotational transform has been described for constant rotational conditions, both can be used in the derivation of the non-inertial Navier-Stokes equations for constant rotation.

2.2 Incompressible Flow Conditions

In this section the non-inertial Navier-Stokes equations for conservation of mass, momentum and energy for constant rotation in incompressible flow will be derived using an Eulerian approach.

2.2.1 Continuity Equation

The conservation of mass, known as the continuity equation, in the inertial frames takes the form:

$$\frac{\partial\rho}{\partial t} + (\nabla \cdot \rho\mathbf{u}) = 0 \tag{11}$$

The first term represents the temporal change in density due to compressibility of the flow. Since this case involves incompressible flow this term can be neglected, but for the purposes of the derivation it will remain in the equation until the last step. The second term is the divergence of density and velocity which represents the residual mass flux of a given control volume.

Scalars, such as density and mass flux, are invariant under Galilean transformation. The first term of the continuity equation in the inertial frame can therefore be directly equated to the term in the non-inertial frame:

$$\frac{\partial\hat{\rho}}{\partial t} = R^{\Omega t}\frac{\partial\rho}{\partial t} \tag{12}$$

The second term of the continuity equation will be affected by both frame transformations since it contains the velocity vector:

$$\begin{aligned} (\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}}) &= R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}(\nabla \cdot \rho\mathbf{u}) \\ &= R^{\Omega t}[\nabla \cdot \rho(G^{\mathbf{\Omega}\wedge\mathbf{x}}\mathbf{u})] \end{aligned} \tag{13}$$

Equation 9 is used to complete the modified Galilean transformation, and the equation becomes:

$$(\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}}) = R^{\Omega t}\{\nabla \cdot \rho(\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\Omega})\} \\ = R^{\Omega t}\{\nabla \cdot (\rho\mathbf{u}) + \nabla \cdot \rho(\mathbf{x} \wedge \boldsymbol{\Omega})\} \tag{14}$$

The second term of the equation above falls away since the divergence of the cross product of distance and rotation is zero:

$$\nabla \cdot (\mathbf{x} \wedge \boldsymbol{\Omega}) = 0 \tag{15}$$

The equation thus becomes:

$$\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = R^{\Omega t}(\nabla \cdot \rho\mathbf{u}) \tag{16}$$

The addition of Equation 12 and Equation 16 leads to a relation between the continuity equation in the inertial and rotational frames:

$$\frac{\partial\hat{\rho}}{\partial t} + \hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = R^{\Omega t}(\frac{\partial\rho}{\partial t} + \nabla \cdot \rho\mathbf{u}) \tag{17}$$

The right hand side of the equation is equal to zero since this represents the continuity equation in the inertial frame (Equation 11):

$$\frac{\partial\hat{\rho}}{\partial t} + \hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = 0 \tag{18}$$

Since this is the incompressible case, the temporal term is equal to zero. The continuity equation for the rotational frame therefore takes the form:

$$\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = 0 \tag{19}$$

The physical meaning of this equation describes the very nature of incompressible flow assumption; the residual mass flux in a specific control volume is zero. This means that there are no compressible effects in the flow because the same amount of mass flux that enters a domain will exit it.

The transient density term causes a change in the residual mass flux in the domain that manifests itself in the form of compressibility.

2.2.2 Momentum Equation

The inertial equation for incompressible momentum conservation is describe by the equation below:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} \tag{20}$$

The first term that will be transformed to obtain an expression that relates the inertial to the rotational frame is the time dependant term. It will be done by finding an expression for the time derivative in the limit:

$$\frac{\partial\hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) = \lim_{\Delta t \to 0} \frac{\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t+\Delta t) - \hat{\mathbf{u}}(\hat{\mathbf{x}}, t)}{\Delta t} \tag{21}$$

An expression for $\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t + \Delta t)$ must be found. The form that the expression must take, will directly relate the frames to each other:

$$\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t+\Delta t) = R^{\Omega(t+\Delta t)}G^{\boldsymbol{\Omega}\wedge\mathbf{x}_{t+\Delta t}} \\ [\mathbf{u}(\mathbf{x}_{t+\Delta t}, t+\Delta t)] \tag{22}$$

The tools that is required to obtain an expression for the relation above is described in the derivation below.

Perform a Taylor series expansion for $\mathbf{x}_{t+\Delta t}$:

$$\mathbf{x}_{t+\Delta t} = \mathbf{x}_t + \Delta t\mathbf{V} + O(\Delta t^2) \tag{23}$$

The resulting series is truncated at the second order term and the derivative term is substituted through Equation 5. Re-arrangement of the terms will lead to an expression for displacement over the specific time interval:

$$\mathbf{x}_{t+\Delta t} - \mathbf{x}_t = \mathbf{x}_{\Delta t} = \Delta t(\boldsymbol{\Omega} \wedge \mathbf{x}) \tag{24}$$

A Fourier series expansion is done for $\mathbf{u}(\mathbf{x}_{t+\Delta t}, t + \Delta t)$, and with substitution of Equation 24 it results in:

$$\mathbf{u}(\mathbf{x}_{t+\Delta t}, t+\Delta t) \\ = \mathbf{u}(\mathbf{x}_t, t) + [\Delta t(\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla]\mathbf{u}(\mathbf{x}_t, t) \\ + (\Delta t\frac{\partial}{\partial t})\mathbf{u}(\mathbf{x}_t, t) \tag{25}$$

Equation 25 is substituted into Equation 22 to get the expression:

$$\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t+\Delta t) \\ = R^{\Omega(t+\Delta t)}G^{\boldsymbol{\Omega}\wedge\mathbf{x}_{t+\Delta t}}\{\mathbf{u}(\mathbf{x}_t, t) \\ + [\Delta t(\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla]\mathbf{u}(\mathbf{x}_t, t) + (\Delta t\frac{\partial}{\partial t})\mathbf{u}(\mathbf{x}_t, t)\} \tag{26}$$

$G^{\boldsymbol{\Omega}\wedge\mathbf{x}_{t+\Delta t}}$ can be simplified as shown below if

Equation 23 is substituted in the operator and truncated at the first order:

$$\begin{aligned} G^{\mathbf{\Omega}\wedge\mathbf{x}_{t+\Delta t}} &= G^{\mathbf{\Omega}\wedge\{\mathbf{x}_t+\Delta t(\Omega\wedge\mathbf{x}_t+O[\Delta t^2])\}} \\ &= G^{\mathbf{\Omega}\wedge\{\mathbf{x}_t(1+O[\Delta t])\}} \\ &\approx G^{\mathbf{\Omega}\wedge\mathbf{x}_t} \end{aligned} \tag{27}$$

The expression for $\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t+\Delta t)$, then becomes:

$$\begin{aligned} &\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t+\Delta t) \\ &= R^{\Omega(t+\Delta t)}G^{\mathbf{\Omega}\wedge\mathbf{x}}\{\mathbf{u}(\mathbf{x}_t, t) \\ &+ [\Delta t(\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla]\mathbf{u}(\mathbf{x}_t, t) \\ &+ (\Delta t\frac{\partial}{\partial t})\mathbf{u}(\mathbf{x}_t, t)\} \end{aligned} \tag{28}$$

A final set of tools is required before the expressions for Equation 21 can be completed.

The assumption was made that point P is fixed in the rotating frame and the rotation is around the shared origin or the frames, then an expression can be derived for $\mathbf{x}_t$:

$$\begin{aligned} \hat{\mathbf{x}} &= R^{\Omega(t+\Delta t)}\mathbf{x}_{t+\Delta t} = R^{\Omega t}\mathbf{x}_t \\ \mathbf{x}_t &= R^{\Omega\Delta t}\mathbf{x}_{t+\Delta t} \end{aligned} \tag{29}$$

This relation is substituted in the Taylor series expansion for $\mathbf{x}_{t+\Delta t}$:

$$\begin{aligned} \mathbf{x}_{t+\Delta t} &= \mathbf{x}_t + \Delta t\mathbf{V} + O[\Delta t^2] \\ &= R^{\Omega\Delta t}\mathbf{x}_{t+\Delta t} + \Delta t(\mathbf{\Omega}\wedge\mathbf{x_t}) + O[\Delta t^2] \end{aligned} \tag{30}$$

Re-arrange this equation and consider in the limit as Δt approaches 0:

$$\lim_{\Delta t\to 0}\frac{R^{\Omega\Delta t}\mathbf{x}_{t+\Delta t} - \mathbf{x}_{t+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}(\mathbf{x}_t\wedge\mathbf{\Omega}) \tag{31}$$

Considering this relation for any vector $\mathbf{b}$, and take into account that $\mathbf{x}_{t+\Delta t}\to\mathbf{x}_t$ as $\Delta t\to 0$, the following equation is arrived at:

$$\lim_{\Delta t\to 0}\frac{R^{\Omega\Delta t}\mathbf{b} - \mathbf{b}}{\Delta t} = \mathbf{b}\wedge\mathbf{\Omega} \tag{32}$$

Substitute Equation 28 into Equation 21 to obtain the equation:

$$\begin{aligned} &\frac{\partial\hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) \\ &= \lim_{\Delta t\to 0}\frac{R^{\Omega(t+\Delta t)}G^{\mathbf{\Omega}\wedge\mathbf{x}}\{[1-\frac{1}{R^{\Omega\Delta t}}}{\Delta t} \\ &\frac{+(\Delta t(\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla)]\mathbf{u}(\mathbf{x}_t, t) + (\Delta t\frac{\partial}{\partial t})\mathbf{u}(\mathbf{x}_t, t)\}}{\Delta t} \end{aligned} \tag{33}$$

By using Equation 32, and after re-arrangement of the terms the following expression is arrived at:

$$\begin{aligned} \frac{\partial\hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) = R^{\Omega t}[\frac{\partial}{\partial t} + (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla - \mathbf{\Omega}\wedge] \\ [G^{\mathbf{\Omega}\wedge\mathbf{x}}\mathbf{u}(\mathbf{x}_t, t)] \end{aligned} \tag{34}$$

Substitute Equation 9 into the equation above will result in:

$$\begin{aligned} &\frac{\partial\hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) \\ &= R^{\Omega t}[\frac{\partial}{\partial t} + (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla - \mathbf{\Omega}\wedge](\mathbf{u}(\mathbf{x}_t, t)) \\ &+ R^{\Omega t}[\frac{\partial}{\partial t} + (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla - \mathbf{\Omega}\wedge](\mathbf{x}\wedge\mathbf{\Omega}) \end{aligned} \tag{35}$$

In the equation above the transient component of $[\frac{\partial}{\partial t} + (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla - \mathbf{\Omega}\wedge](\mathbf{x}\wedge\mathbf{\Omega})$ is equal to zero:

$$\frac{\partial}{\partial t}(\mathbf{x}\wedge\mathbf{\Omega}) = \frac{\partial\mathbf{x}}{\partial t}\wedge\mathbf{\Omega} + (\mathbf{x}\wedge\frac{\partial\mathbf{\Omega}}{\partial t}) = 0 \tag{36}$$

The first term is zero because the magnitude of $\mathbf{x}$ is constant over the time domain; its magnitude does not change with respect to the origin since this case involves pure rotation. The second term is zero due to constant rotation of the point P. In the case where the rotation is not constant, this term will play a role as seen in this next section.

By introduction of the identity below, the terms $[(\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla - \mathbf{\Omega}\wedge](\mathbf{x}\wedge\mathbf{\Omega})$ can be simplified:

$$(\mathbf{a}\cdot\nabla)(\mathbf{x}\wedge\mathbf{\Omega}) = \mathbf{a}\wedge\mathbf{\Omega} \tag{37}$$

The entire term is hence cancelled out:

$$\begin{aligned} &[(\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla - \mathbf{\Omega}\wedge](\mathbf{x}\wedge\mathbf{\Omega}) \\ &= [(\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla](\mathbf{x}\wedge\mathbf{\Omega}) - \mathbf{\Omega}\wedge(\mathbf{x}\wedge\mathbf{\Omega}) \\ &= \mathbf{\Omega}\wedge(\mathbf{x}\wedge\mathbf{\Omega}) - \mathbf{\Omega}\wedge(\mathbf{x}\wedge\mathbf{\Omega}) \\ &= 0 \end{aligned} \tag{38}$$

This leads to the final description of the unsteady terms in the momentum equation. Note the appearance of one part of the Coriolis effect manifesting in the relation below.

$$\begin{aligned} &\frac{\partial\hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) \\ &= R^{\Omega t}[\frac{\partial}{\partial t} + (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla - \mathbf{\Omega}\wedge](\mathbf{u}(\mathbf{x}_t, t)) \end{aligned} \tag{39}$$

The relation of the inertial to the rotational advection term is described in the following manner:

$$\begin{aligned}&(\hat{\mathbf{u}}\cdot\hat{\nabla})\hat{\mathbf{u}}\\&=R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}(\mathbf{u}\cdot\nabla)\mathbf{u}\\&=R^{\Omega t}(G^{\mathbf{\Omega}\wedge\mathbf{x}}\mathbf{u}\cdot\nabla)G^{\mathbf{\Omega}\wedge\mathbf{x}}\mathbf{u}\end{aligned}\tag{40}$$

Substitution of Equation 9 into the equation above results in:

$$\begin{aligned}&(\hat{\mathbf{u}}\cdot\hat{\nabla})\hat{\mathbf{u}}\\&=R^{\Omega t}[(\mathbf{u}+\mathbf{x}\wedge\mathbf{\Omega})\cdot\nabla](\mathbf{u}+\mathbf{x}\wedge\mathbf{\Omega})\\&=R^{\Omega t}[(\mathbf{u}+\mathbf{x}\wedge\mathbf{\Omega})\cdot\nabla]\mathbf{u}\\&+R^{\Omega t}[(\mathbf{u}+\mathbf{x}\wedge\mathbf{\Omega})\cdot\nabla](\mathbf{x}\wedge\mathbf{\Omega})\end{aligned}\tag{41}$$

Dividing out all the terms gives the final relation of the advection term between the frames. Note the appearance of the centrifugal effect and the other part of the Coriolis effect from the transformation of the advection term.

$$\begin{aligned}&(\hat{\mathbf{u}}\cdot\hat{\nabla})\hat{\mathbf{u}}\\&=R^{\Omega t}[(\mathbf{u}\cdot\nabla)\mathbf{u}+((\mathbf{x}\wedge\mathbf{\Omega})\cdot\nabla)\mathbf{u}\\&+(\mathbf{u}\wedge\mathbf{\Omega})+(\mathbf{x}\wedge\mathbf{\Omega})\wedge\mathbf{\Omega}]\end{aligned}\tag{42}$$

The gradient of pressure term in the momentum equation is described between the frames in the following manner:

$$\hat{\nabla}\hat{p}=R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}\nabla p\tag{43}$$

It was discussed earlier that scalars are invariant under the modified Galilean transformation. Scalars will not be invariant under the rotational transform if spatial operations is performed on it since the axis along which the discretization is performed, changes between frames. The relation between the gradient of pressure in the inertial and rotational frames is therefore described by:

$$\hat{\nabla}\hat{p}=R^{\Omega t}\nabla p\tag{44}$$

The diffusion term in the inertial frame can be related to the rotational frame in the following manner:

$$\begin{aligned}&\nu\hat{\nabla}^2\hat{\mathbf{u}}\\&=R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}\nu\nabla^2\mathbf{u}\\&=R^{\Omega t}\nu\nabla^2G^{\mathbf{\Omega}\wedge\mathbf{x}}\mathbf{u}\\&=R^{\Omega t}\nu\nabla^2(\mathbf{u}+\mathbf{x}\wedge\mathbf{\Omega})\\&=R^{\Omega t}\nu[\nabla^2\mathbf{u}+\nabla^2(\mathbf{x}\wedge\mathbf{\Omega})]\end{aligned}\tag{45}$$

If it is considered that:

$$\nabla^2(\mathbf{x}\wedge\mathbf{\Omega})=0\tag{46}$$

It can be shown that the diffusion term is invariant under constant transformation:

$$\nu\hat{\nabla}^2\hat{\mathbf{u}}=R^{\Omega t}\nu\nabla^2\mathbf{u}\tag{47}$$

Note that the pressure and viscous terms are Galilean invariant in this instance and combine the two components in a vector $\mathbf{f}(\mathbf{x},t)$:

$$\mathbf{f}(\mathbf{x},t)=-\nabla p+\nu\nabla^2\mathbf{u}\tag{48}$$

The new, combined parameter in the inertial and rotational frames is related in the following manner due to the invariance:

$$\hat{\mathbf{f}}(\hat{\mathbf{x}},t)=R^{\Omega t}\mathbf{f}(\mathbf{x},t)\tag{49}$$

The transformation of the momentum is completed through the summation of the unsteady and advection terms in the rotational and inertial frames as determined in Equation 39 and Equation 42:

$$\begin{aligned}&\frac{\partial\hat{\mathbf{u}}}{\partial t}+(\hat{\mathbf{u}}\cdot\hat{\nabla})\hat{\mathbf{u}}\\&=R^{\Omega t}[\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}+2\mathbf{u}\wedge\mathbf{\Omega}\\&+\mathbf{x}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}]\\&=R^{\Omega t}[\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}]\\&+R^{\Omega t}[2\mathbf{u}\wedge\mathbf{\Omega}+\mathbf{x}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}]\end{aligned}\tag{50}$$

The first term grouping of the equation above is simplified as shown in the equations below. This was done using Equation 20, Equation 48 and Equation 49.

$$\begin{aligned}&R^{\Omega t}[\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}]\\&=R^{\Omega t}[-\Delta p+\nu\nabla^2\mathbf{u}]\\&=R^{\Omega t}\mathbf{f}(\mathbf{x},t)\\&=\hat{\mathbf{f}}(\hat{\mathbf{x}},t)\end{aligned}\tag{51}$$

The second term grouping, with the insertion of Equation 9, becomes:

$$\begin{aligned}&R^{\Omega t}[2\mathbf{u}\wedge\mathbf{\Omega}+\mathbf{x}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}]\\&=2(R^{\Omega t}\mathbf{u})\wedge\mathbf{\Omega}+(R^{\Omega t}\mathbf{x})\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}\\&=2[\hat{\mathbf{u}}-R^{\Omega t}(\mathbf{x}\wedge\mathbf{\Omega})]\wedge\mathbf{\Omega}\\&+(R^{\Omega t}\mathbf{x})\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}\\&=2\hat{\mathbf{u}}\wedge\mathbf{\Omega}-\hat{\mathbf{x}}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}\end{aligned}\tag{52}$$

The two simplifications above is filled back into Equation 50 and results in the non-inertial momentum equation for constant rotation.

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} = -\hat{\nabla}\hat{p} + \nu\hat{\nabla}^2\hat{\mathbf{u}} + 2\hat{\mathbf{u}} \wedge \mathbf{\Omega} - \hat{\mathbf{x}} \wedge \mathbf{\Omega} \wedge \mathbf{\Omega} \tag{53}$$

It can be seen from the equation above that the fictitious forces associated with constant rotation is the centrifugal and the Coriolis effects. The centrifugal effect originates from the transformation of the advection terms while the Coriolis effect is form both the transient and advection terms.

2.2.3 Energy Equation

The general energy equation in the inertial frame takes the following form:

$$\frac{\partial \rho\varepsilon}{\partial t} + (\nabla \cdot \rho\varepsilon\mathbf{u}) = -p(\nabla \cdot \mathbf{u}) + \nabla \cdot (k\nabla T) \tag{54}$$

The time dependant term is transformed in a similar manner to the pressure term in the momentum equation since internal energy is a scalar. It was already discussed that scalars are invariant under transformation. The first term is therefore transformed though the following operation:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} = R^{\Omega t}\frac{\partial \rho\varepsilon}{\partial t} \tag{55}$$

The convective term is transformed between the frames with the use of the rotational transform, modified Galilean transform and by substitution of Equation 9

$$\begin{aligned}
&(\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) \\
&= R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}(\nabla \cdot \rho\varepsilon\mathbf{u}) \\
&= R^{\Omega t}[\nabla \cdot \rho\varepsilon(\mathbf{u} + \mathbf{x} \wedge \mathbf{\Omega})] \\
&= R^{\Omega t}[\nabla \cdot \rho\varepsilon\mathbf{u} + \nabla \cdot \rho\varepsilon(\mathbf{x} \wedge \mathbf{\Omega})]
\end{aligned} \tag{56}$$

It was already shown in Equation 15 that the second term on the right hand side is equal to zero. Therefore the transformed equation becomes:

$$(\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = R^{\Omega t}(\nabla \cdot \rho\varepsilon\mathbf{u}) \tag{57}$$

The terms that represents the rate of work done by the normal pressure forces is transform between the frames and Equation 9 is inserted:

$$\begin{aligned}
&-\hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) \\
&= R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}[-p(\nabla \cdot \mathbf{u})] \\
&= R^{\Omega t}[-p\nabla \cdot (\mathbf{u} + \mathbf{x} \wedge \mathbf{\Omega})] \\
&= R^{\Omega t}[-p\nabla \cdot \mathbf{u} + -p\nabla \cdot (\mathbf{x} \wedge \mathbf{\Omega})]
\end{aligned} \tag{58}$$

It can be shown that this terms is also invariant under transformation by the insertion of Equation 15:

$$-\hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) = R^{\Omega t}[-p\nabla \cdot \mathbf{u}] \tag{59}$$

The diffusion is invariant under transformation since the heat transfer coefficient (k) and temperature (T) are scalars. The transformation between the frames then becomes:

$$\begin{aligned}
\hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) &= R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}[\nabla \cdot (k\nabla T)] \\
&= R^{\Omega t}[\nabla \cdot (k\nabla T)]
\end{aligned} \tag{60}$$

All the transformed terms of the energy equation is summed to obtain the equation below.

$$\begin{aligned}
&\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) + \hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) - \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) \\
&= R^{\Omega t}[\frac{\partial \rho\varepsilon}{\partial t} + (\nabla \cdot \rho\varepsilon\mathbf{u}) + p(\nabla \cdot \mathbf{u}) \\
&- \nabla \cdot (k\nabla T)]
\end{aligned} \tag{61}$$

The right hand side of the equation is equal to zero, as shown in Equation 54. The energy equation in the non-inertial frame for constant rotation is invariant under transformation in this specific case:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = -\hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) + \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) \tag{62}$$

This equation can be further simplified with the assumption of incompressibility and using Equation 19:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) \tag{63}$$

Note that the energy equation is invariant under transformation.

2.3 Compressible Flow Conditions

In this section the non-inertial Navier-Stokes equations for conservation of mass, momentum and energy for constant rotation in compressible flow will be derived using an Eulerian approach.

2.3.1 Continuity Equation

The general continuity equation in the rotational frame was derived in Equation 18. This has shown that the equation is invariant under transformation. The compressible, non-inertial equation thus remains:

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = 0 \tag{64}$$

2.3.2 Momentum Equation

The incompressible form of the momentum equation as shown in Equation 20, made the assumption that the change in density is negligible. Therefore, the equation could be simplified by dividing density into all the terms as there are no temporal or spatial gradients in density. The diffusion term in particular could be simplified in a manner that would facilitate easy transformation where the divergence of the gradient of velocity yields the same result as taking the laplacian of the velocity. This is not the case when compressibility has to be accounted for. The compressible Navier-Stokes Equation in the inertial frame will take the form:

$$\begin{aligned}&\frac{\partial}{\partial t}\rho\mathbf{u} + \nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u}) \\ &= -\nabla P + \nabla \cdot [\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}]\end{aligned} \tag{65}$$

The non-inertial form of the separate terms of the equation, must be derived from this form to obtain the compressible transformation.

First consider the unsteady term in the rotational frame and apply the product rule for partial derivatives. This operation will result in two terms that was not considered during the incompressible case:

$$\frac{\partial}{\partial t}(\hat{\rho}\hat{\mathbf{u}}) = \hat{\rho}\frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}}\frac{\partial \hat{\rho}}{\partial t} \tag{66}$$

With the aid of Equations 8 (which defined the transformation between the frames) and Equation 34 (which defined the relation between the incompressible transient term in different frames) the above becomes:

$$\begin{aligned}\frac{\partial}{\partial t}(\hat{\rho}\hat{\mathbf{u}}) = R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}[\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{\Omega} \wedge \mathbf{x}) \cdot \nabla\mathbf{u} \\ - \rho\mathbf{\Omega} \wedge \mathbf{u} + \mathbf{u}\frac{\partial \rho}{\partial t}]\end{aligned} \tag{67}$$

The product rule is then used to combine the terms $\rho\frac{\partial \mathbf{u}}{\partial t}$ and $\mathbf{u}\frac{\partial \rho}{\partial t}$ so that the equation simplifies to:

$$\begin{aligned}\frac{\partial}{\partial t}(\hat{\rho}\hat{\mathbf{u}}) = R^{\Omega t}[\frac{\partial}{\partial t}(\rho) + \rho(\mathbf{\Omega} \wedge \mathbf{x}) \cdot \nabla \\ - \rho\mathbf{\Omega}\wedge]G^{\mathbf{\Omega}\wedge\mathbf{x}}\mathbf{u}\end{aligned} \tag{68}$$

The equation is of the same form as Equation 34, it can therefore be shown that the final form of the equation will be similar to Equation 39, but with the inclusion of the density scalar:

$$\begin{aligned}\frac{\partial}{\partial t}(\hat{\rho}\hat{\mathbf{u}}) = R^{\Omega t}[\frac{\partial}{\partial t}(\rho) + \rho(\mathbf{\Omega} \wedge \mathbf{x}) \cdot \nabla \\ - \rho\mathbf{\Omega}\wedge]\mathbf{u}\end{aligned} \tag{69}$$

The relation between the frames for the advection term in the compressible Navier-Stokes momentum equation is:

$$\hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) = R^{\Omega t}G^{\mathbf{\Omega}\wedge\mathbf{x}}[\nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u})] \tag{70}$$

By using Equation 7 the equation above is expanded into:

$$\begin{aligned}&\hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) \\ &= R^{\Omega t}\{\nabla \cdot \rho[(\mathbf{u} + \mathbf{x} \wedge \mathbf{\Omega}) \otimes (\mathbf{u} + \mathbf{x} \wedge \mathbf{\Omega})]\} \\ &= R^{\Omega t}\{(\nabla \cdot \rho\mathbf{u}) \otimes \mathbf{u} + (\nabla \cdot \rho\mathbf{u}) \otimes (\mathbf{x} \wedge \mathbf{\Omega}) \\ &+ [\nabla \cdot \rho(\mathbf{x} \wedge \mathbf{\Omega})] \otimes \mathbf{u} \\ &+ [\nabla \cdot \rho(\mathbf{x} \wedge \mathbf{\Omega})] \otimes (\mathbf{x} \wedge \mathbf{\Omega})\}\end{aligned} \tag{71}$$

As shown in the previous section, the identity below can be used to simplify the equation.

$$(\nabla \cdot \mathbf{a}) \otimes (\mathbf{x} \wedge \mathbf{\Omega}) = \mathbf{a} \wedge \mathbf{\Omega} \tag{72}$$

This will lead to the following expression for relating the diffusion term in the rotational frame to the terms in the inertial frame:

$$\begin{aligned}&\hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) \\ &= R^{\Omega t}[\nabla \cdot \rho\mathbf{u} \otimes \mathbf{u} + \rho\mathbf{u} \wedge \mathbf{\Omega} \\ &+ (\nabla \cdot \rho(\mathbf{x} \wedge \mathbf{\Omega}) \otimes \mathbf{u} + (\rho\mathbf{x} \wedge \mathbf{\Omega}) \wedge \mathbf{\Omega}]\end{aligned} \tag{73}$$

The pressure gradient term in the momentum equation is transformed in a similar manner than shown in the previous section. This part of the equation remain invariant since it is a scalar.

$$\begin{aligned} \hat{\nabla}\hat{P} &= R^{\Omega t} G^{\mathbf{\Omega}\wedge\mathbf{x}} \nabla P \\ \hat{\nabla}\hat{P} &= R^{\Omega t} \nabla P \end{aligned} \tag{74}$$

In the transformation of the diffusion term the difference between the compressible and incompressible cases must be noted. The divergence of the velocity vector is not equal to zero, therefore the completed diffusion term must be accounted for. The expression for relating the diffusion term between the frames hence becomes:

$$\begin{aligned} &\hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla}\cdot\hat{\mathbf{u}})\hat{\mathbf{I}}] \\ &= R^{\Omega t} G^{\mathbf{\Omega}\wedge\mathbf{x}} \nabla \cdot [\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \\ &+ \lambda(\nabla\cdot\mathbf{u})\mathbf{I}] \end{aligned} \tag{75}$$

With the implementation of Equation 9, the right hand side of the equations becomes:

$$\begin{aligned} &R^{\Omega t}\nabla \cdot \{\mu[\nabla(\mathbf{u} + \mathbf{x}\wedge\mathbf{\Omega}) + \nabla(\mathbf{u} + \mathbf{x}\wedge\mathbf{\Omega})^T] \\ &+ \lambda(\nabla\cdot(\mathbf{u} + \mathbf{x}\wedge\mathbf{\Omega}))\mathbf{I}\} \end{aligned} \tag{76}$$

If it is considered that,

$$\nabla(\mathbf{x}\wedge\mathbf{\Omega}) + \nabla(\mathbf{x}\wedge\mathbf{\Omega})^T = 0 \tag{77}$$

and

$$\nabla\cdot(\mathbf{x}\wedge\mathbf{\Omega}) = 0 \tag{78}$$

It can be shown that, as in the case of incompressible flow, the diffusion component of the momentum equation is invariant for constant rotation conditions:

$$\begin{aligned} &\hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla}\cdot\hat{\mathbf{u}})\hat{\mathbf{I}}] \\ &= R^{\Omega t}\nabla \cdot [\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \lambda(\nabla\cdot\mathbf{u})\mathbf{I}] \end{aligned} \tag{79}$$

The same principle as in the previous section is used to derive the final equation:

$$\begin{aligned} &\frac{\partial\hat{\rho}\hat{\mathbf{u}}}{\partial t} + \hat{\nabla}\cdot(\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}}) \\ &= -\hat{\nabla}\hat{P} \\ &+ \hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla}\cdot\hat{\mathbf{u}})\hat{\mathbf{I}}] \\ &+ 2\rho\hat{\mathbf{u}}\wedge\mathbf{\Omega} - \rho\hat{\mathbf{x}}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega} \end{aligned} \tag{80}$$

It must be noted that the incompressible momentum equation is a special case of the compressible momentum equation where it is assumed that the velocity of the flow is low enough (generally incompressibility assumptions is assumed below Mach 0.3) that the compressible effects does not have a significant effect on the flow properties. No truly incompressible conditions exist.

2.3.3 Energy Equation

The general energy equation remains the same as described in the previous section:

$$\frac{\partial\rho\varepsilon}{\partial t} + (\nabla\cdot\rho\varepsilon\mathbf{u}) = -p(\nabla\cdot\mathbf{u}) + \nabla\cdot(k\nabla T) \tag{81}$$

This equation remains invariant in the non-inertial frame as shown in Equation 62

$$\frac{\partial\hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla}\cdot\hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = -\hat{p}(\hat{\nabla}\cdot\hat{\mathbf{u}}) + \hat{\nabla}\cdot(\hat{k}\hat{\nabla}\hat{T}) \tag{82}$$

Take again note that there are not fictitious effects present in the energy equation.

2.4 Incompressible Equations as a Special Case of the Compressible Equations

The assumption of incompressibility can be made when the flow velocity is substantially small (below Mach 0.3) so that the mass flux is close to zero. This is a special case of compressible flow that assumes that no temporal changes in density occurs. As such, if the compressible Navier-Stokes equations where derived correctly, applying the incompressible assumptions to it, should lead to the derived, incompressible Navier-Stokes equations.

The compressible continuity equation in the rotational frame was determined to be:

$$\frac{\partial\hat{\rho}}{\partial t} + \hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}} = 0 \tag{83}$$

The applied assumption of incompressibility will result in the transient change in density being zero:

$$\frac{\partial\hat{\rho}}{\partial t} = 0 \tag{84}$$

The equation therefore becomes:

$$\hat{\nabla}\cdot\hat{\rho}\hat{\mathbf{u}} = 0 \tag{85}$$

This provides the same result Equation 19 which is the derived, incompressible continuity equation in the rotational frame.

The derived, compressible momentum equation in the rotational frame is:

$$\begin{aligned}&\frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t}+\hat{\nabla}\cdot(\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}})\\&=-\hat{\nabla}\hat{P}\\&+\hat{\nabla}\cdot[\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}}+\hat{\nabla}\hat{\mathbf{u}}^T)+\hat{\lambda}(\hat{\nabla}\cdot\hat{\mathbf{u}})\hat{\mathbf{I}}]\\&+2\rho\hat{\mathbf{u}}\wedge\mathbf{\Omega}-\rho\hat{\mathbf{x}}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}\end{aligned} \tag{86}$$

The first term that must be simplified to account from incompressibility is the diffusion term:

$$\hat{\nabla}\cdot[\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}}+\hat{\nabla}\hat{\mathbf{u}}^T)+\hat{\lambda}(\hat{\nabla}\cdot\hat{\mathbf{u}})\hat{\mathbf{I}}] \tag{87}$$

Lets consider the x-momentum components of the divergence of the deviatoric stress tensor. Assume in this instance that the dynamic viscosity μ is a constant. Simplify the relation below to obtain the form as shown below:

$$\begin{aligned}&\frac{\partial}{\partial x}(2\mu\frac{\partial u}{\partial t}+\lambda\nabla\cdot\mathbf{u})+\frac{\partial}{\partial y}(\mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}))\\&+\frac{\partial}{\partial z}(\mu(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}))\\=&\mu(2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x})\\&+\frac{\partial}{\partial x}(\lambda\nabla\cdot\mathbf{u})\\&=\mu\nabla^2 u+\mu\frac{\partial}{\partial x}(\nabla\cdot\mathbf{u})+\frac{\partial}{\partial x}(\lambda\nabla\cdot\mathbf{u})\end{aligned} \tag{88}$$

This relation can be written in the vector form to account for all the components of the diffusive momentum if it is assumed that the second viscosity, λ, is constant (Stokes Hypothesis):

$$\hat{\nabla}\cdot\hat{\mu}\hat{\nabla}\hat{\mathbf{u}}+\hat{\nabla}\cdot[(\hat{\mu}+\hat{\lambda})(\hat{\nabla}\cdot\hat{\mathbf{u}})] \tag{89}$$

The second term in the relation above will be equal to zero if the incompressible continuity equation is substituted into the relation. This result in the following equation:

$$\begin{aligned}\frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t}&+\hat{\nabla}\cdot(\hat{\rho}\hat{\mathbf{u}}\otimes\hat{\mathbf{u}})=-\hat{\nabla}\hat{P}+\hat{\nabla}\cdot\hat{\mu}\hat{\nabla}\hat{\mathbf{u}}\\&+2\rho\hat{\mathbf{u}}\wedge\mathbf{\Omega}-\rho\hat{\mathbf{x}}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}\end{aligned} \tag{90}$$

Since density is constant in the equation above, it can be divided into the equation, which will lead to the non-inertial momentum equation:

$$\begin{aligned}\frac{\partial \hat{\mathbf{u}}}{\partial t}+(\hat{\mathbf{u}}\cdot\hat{\nabla})\hat{\mathbf{u}}&=-\hat{\nabla}\hat{p}+\nu\hat{\nabla}^2\hat{\mathbf{u}}\\&+2\hat{\mathbf{u}}\wedge\mathbf{\Omega}-\hat{\mathbf{x}}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}\end{aligned} \tag{91}$$

This equation above it the same as Equation 53 which was derived from first principles.

The conservation of energy equation in the rotational frame for compressible flow is described by:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t}+(\hat{\nabla}\cdot\hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}})=-\hat{p}(\hat{\nabla}\cdot\hat{\mathbf{u}})+\hat{\nabla}\cdot(\hat{k}\hat{\nabla}\hat{T}) \tag{92}$$

If the continuity equation is applied to this equation it will result in:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t}+(\hat{\nabla}\cdot\hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}})=\hat{\nabla}\cdot(\hat{k}\hat{\nabla}\hat{T}) \tag{93}$$

This is the same as Equation 63 where the incompressible energy equation in the rotational frame was derived.

This section therefore indicates that there are no observed discrepancies between the derived equations for the compressible and incompressible cases in the rotational frame.

3 Non-inertial Navier-Stokes Equations for Variable, Pure Rotation

In this section the non-inertial Navier-Stokes equation for variable rotation around the axis of the inertial frame will be derived.

It will observed that the equations for mass and energy conservation for constant and variable rotation remains the same. The equation for conservation of momentum has an additional term to account for the variability in rotation. It will be shown that the equations for constant and variable rotational acceleration are equivalent and that all additional terms due to the changes in acceleration is negligible in the limit as Δt approaches zero.

3.1 Frame Transformations

Assume that the same three frame exist as in the previous derivation: O, O' and Ô. O is again the stationary frame, O' is the orientation preserving frame and Ô is the rotational frame. In the previous derivation Ô was rotating at a constant velocity around the shared origin. In this instance Ô are rotating around the shared origin with a constant rotational acceleration.

3.1.1 Modified Galilean Transformation

In this section the modified Galilean transform obtained from the previous section will be augmented to account for rotational acceleration.

Assume that the frame origins intersect at time $t = 0$ and that frame O' is moving at acceleration $\mathbf{a}$ in three dimensional space. At time $t = \Delta t$ frame O and O' are distance $\mathbf{x_{rel}}$ from each other. In Equation 4 there was not an accelerating component, but in this case it must be incorporated in the expression to account for the distance travelled by the particle:

$$\mathbf{x_{rel}} = \mathbf{V}\Delta t + \frac{1}{2}\mathbf{a}\Delta t^2 \tag{94}$$

In the equation above the velocity is again described as in Equation 5:

$$\mathbf{V} = \mathbf{\Omega} \wedge \mathbf{x} \tag{95}$$

The acceleration is the time derivative of the velocity:

$$\begin{aligned}\frac{\partial \mathbf{V}}{\partial t} &= \frac{\partial}{\partial t}\mathbf{\Omega} \wedge \mathbf{x} \\ &= \frac{\partial \mathbf{\Omega}}{\partial t} \wedge \mathbf{x} + \mathbf{\Omega} \wedge \frac{\partial \mathbf{x}}{\partial t}\end{aligned} \tag{96}$$

The second term is equal to zero since this case involves pure rotation. The accelerating component for the rotational case is therefore expressed as:

$$\mathbf{a} = \dot{\mathbf{\Omega}} \wedge \mathbf{x} \tag{97}$$

Equation 94 is a Taylor series expansion that was truncated after the second order term since constant acceleration was assumed. Had the acceleration not been constant, the additional terms will be accounted for by the inclusion of further derivative terms:

$$\mathbf{x_{rel}} = \mathbf{V}\Delta t + \frac{1}{2!}\mathbf{a}\Delta t^2 + \frac{1}{3!}\dot{\mathbf{a}}\Delta t^3 + ... \tag{98}$$

In this equation it can already be seen that the effect of further derivatives on $\mathbf{x_{rel}}$ becomes smaller and smaller. Further on in this section it will be shown that only one additional term are introduced in the non-inertial Navier-Stokes momentum equation due to variable acceleration. The continuity and energy conservation equations remains invariant.

In the same manner as in the previous section, Equation 7, the relation between the order preserving frame and the inertial frame is defined with the inclusion of the accelerating components:

$$\begin{aligned}&\mathbf{u}'(\mathbf{x}', t) \\ &= G^{\mathbf{M}}\mathbf{u}(\mathbf{x}, t) \\ &= G^{\mathbf{\Omega}\wedge\mathbf{x}+(\dot{\mathbf{\Omega}}\wedge\mathbf{x})\Delta t}\mathbf{u}(\mathbf{x}, t) \\ &= \mathbf{u}(\mathbf{x}, t) + \mathbf{x} \wedge \mathbf{\Omega} + (\mathbf{x} \wedge \dot{\mathbf{\Omega}})\Delta t\end{aligned} \tag{99}$$

3.1.2 Rotational Transformation

The rotational transform for this case can be defined in the same manner as Equation 8. The vector components in Ô is related to O' by defining a rotational transform and substituting Equation 99 to relate Ô to O:

$$\begin{aligned}\hat{\mathbf{u}}(\hat{\mathbf{x}}, t) &= R^{\Omega t+\dot{\Omega}t^2}\mathbf{u}'(\mathbf{x}', t) \\ &= R^{\Omega t+\dot{\Omega}t^2}G^{\mathbf{\Omega}\wedge\mathbf{x}+(\dot{\mathbf{\Omega}}\wedge\mathbf{x})\Delta t}\mathbf{u}(\mathbf{x}, t)\end{aligned} \tag{100}$$

$R^{\Omega t+\dot{\Omega}t^2}$ is the rotational transform that operates on $\mathbf{x}'$ to obtain the $\hat{\mathbf{x}}$ co-ordinates in the accelerating, rotational frame.

Lets assume, for convenience sake, that the rotation is around the z-axis, then the vector $\mathbf{\Omega}$ is described as $\mathbf{\Omega} = (0, 0, \Omega)$ and vector $\dot{\mathbf{\Omega}}$ is described as $\dot{\mathbf{\Omega}} = (0, 0, \dot{\Omega})$. The rotational transform in this case will be described by:

$$R^{\Omega t+\dot{\Omega}t^2} = \begin{bmatrix} \cos(\Omega t + \dot{\Omega}t^2) & \sin(\Omega t + \dot{\Omega}t^2) & 0 \\ -\sin(\Omega t + \dot{\Omega}t^2) & \cos(\Omega t + \dot{\Omega}t^2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{101}$$

From Equation 99 and 100 it can be derived that for the velocity vector the following relation holds:

$$\begin{aligned}&\hat{\mathbf{u}}(\hat{\mathbf{x}}, t) \\ &= R^{\Omega t+\dot{\Omega}t^2}[\mathbf{u}(\mathbf{x}, t) + \mathbf{x} \wedge \mathbf{\Omega} + (\mathbf{x} \wedge \dot{\mathbf{\Omega}})\Delta t]\end{aligned} \tag{102}$$

3.2 Incompressible Flow Conditions

In this section the non-inertial Navier-Stokes equations for conservation of mass, momentum and energy for variable rotation in incompressible flow will be derived using an Eulerian approach.

3.2.1 Continuity Equation

Consider the continuity equation in the inertial reference frame:

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \mathbf{u}) = 0 \tag{103}$$

As was discussed in the previous section, scalars are invariant under transformation and therefore the time dependant term in the inertial and accelerating frame is related by:

$$\frac{\partial \hat{\rho}}{\partial t} = R^{\Omega t + \dot{\Omega} t^2} \frac{\partial \rho}{\partial t} \tag{104}$$

The relation of the second term in the continuity equation becomes:

$$\begin{aligned} &(\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}}) \\ &= R^{\Omega t + \dot{\Omega} t^2} G^{\mathbf{\Omega} \wedge \mathbf{x} + (\dot{\mathbf{\Omega}} \wedge \mathbf{x})\Delta t} (\nabla \cdot \rho \mathbf{u}) \\ &= R^{\Omega t + \dot{\Omega} t^2} \nabla \cdot \rho (G^{\mathbf{\Omega} \wedge \mathbf{x} + (\dot{\mathbf{\Omega}} \wedge \mathbf{x})\Delta t} \mathbf{u})\} \end{aligned} \tag{105}$$

With the implementation of Equation 102 the relation is simplified to:

$$\begin{aligned} &(\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}}) \\ &= R^{\Omega t} \nabla \cdot \rho [\mathbf{u} + \mathbf{x} \wedge \mathbf{\Omega} + (\mathbf{x} \wedge \dot{\mathbf{\Omega}})\Delta t] \end{aligned} \tag{106}$$

The second and third terms in the relation above is equal to zero:

$$\begin{aligned} \nabla \cdot (\mathbf{x} \wedge \mathbf{\Omega}) = 0 \\ \nabla \cdot (\mathbf{x} \wedge \dot{\mathbf{\Omega}}) = 0 \end{aligned} \tag{107}$$

The relation is hence simplified to an invariant relation:

$$\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = R^{\Omega t + \dot{\Omega} t^2} (\nabla \cdot \rho \mathbf{u}) \tag{108}$$

The addition of Equation 104 and Equation 108 gives a relation for continuity in the rotational frame:

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = R^{\Omega t + \dot{\Omega} t^2} (\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u}) \tag{109}$$

With the help of Equation 103 and the assumption that the flow is incompressible, the final equation for mass conservation in the accelerating frame is obtained:

$$\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = 0 \tag{110}$$

Consider the term $(\mathbf{x} \wedge \dot{\mathbf{\Omega}})\Delta t$ in Equation 106. This term originates from the Taylor series expansion in Equation 98 and contains a Δt component. Any further expansions, due to changes in acceleration, will also contain a Δt^n component. These acceleration terms will become negligible in the limit:

$$\lim_{\Delta t \to 0} (\mathbf{x} \wedge \dot{\mathbf{\Omega}})\Delta t = 0 \tag{111}$$

In the light of this, the derivations for the continuity equation in the rotational frame have been proven to be invariant to rotation whether the acceleration it is zero, constant or variable. In any rotational frame Equation 110 holds for incompressible conditions.

3.2.2 Momentum Equation

The conservation of momentum equation in the inertial frame can be expressed by:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \tag{112}$$

The terms will again, as in the previous section, be treated separately and then combined to obtain the final transformed equation.

The first transformation will concern the unsteady term where an expression must be found for:

$$\begin{aligned} &\frac{\partial \hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) \\ &= \lim_{\Delta t \to 0} \frac{\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t + \Delta t) - \hat{\mathbf{u}}(\hat{\mathbf{x}}, t)}{\Delta t} \end{aligned} \tag{113}$$

The first task is to find an expression for the term $\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t + \Delta t)$. The expression must take the form:

$$\begin{aligned} &\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t + \Delta t) \\ &= R^{\Omega(t+\Delta t) + \dot{\Omega}(t^2 + \Delta t^2)} \\ &[G^{\mathbf{\Omega} \wedge \mathbf{x}_{t+\Delta t} + (\dot{\mathbf{\Omega}} \wedge \mathbf{x}_{t+\Delta t})\Delta t} \mathbf{u}(\mathbf{x}_{t+\Delta t}, t + \Delta t)] \end{aligned} \tag{114}$$

For simplification the rotational and modified Galilean transforms will be shown in the following manner:

$$R^{\Omega(t+\Delta t)+\dot{\Omega}(t^2+\Delta t^2)} = R^{M^{t+\Delta t}}$$
$$G^{\mathbf{\Omega}\wedge\mathbf{x}_{t+\Delta t}+(\dot{\mathbf{\Omega}}\wedge\mathbf{x}_{t+\Delta t})\Delta t} = G^{M^{t+\Delta t}} \tag{115}$$

The Taylor series expansion for $\mathbf{x}_{t+\Delta t}$ is expressed as:

$$\mathbf{x}_{t+\Delta t} = \mathbf{x}_t + \Delta t\mathbf{V} + \frac{1}{2}\Delta t^2\mathbf{a} + O(\Delta t^3) \tag{116}$$

The equation above is truncate at the second order. With the substitution of Equation 96 and Equation 97 and further re-arrangement the equation becomes:

$$\begin{aligned}&\mathbf{x}_{t+\Delta t} - \mathbf{x}_t\\&= \mathbf{x}_{\Delta t} = \Delta t(\mathbf{\Omega}\wedge\mathbf{x}) + \frac{1}{2}\Delta t^2(\dot{\mathbf{\Omega}}\wedge\mathbf{x})\end{aligned} \tag{117}$$

The Fourier series expansion is obtained for $\mathbf{u}(\mathbf{x}_{t+\Delta t}, t+\Delta t)$. Substitute Equation 117 into the expression to obtain:

$$\begin{aligned}&\mathbf{u}(\mathbf{x}_{t+\Delta t}, t+\Delta t)\\&= \mathbf{u}(\mathbf{x}_t, t) + \{[\Delta t(\mathbf{\Omega}\wedge\mathbf{x})\\&+ \frac{1}{2}\Delta t^2(\dot{\mathbf{\Omega}}\wedge\mathbf{x})]\cdot\nabla\}\mathbf{u}(\mathbf{x}_t, t)\\&+ (\Delta t\frac{\partial}{\partial t})\mathbf{u}(\mathbf{x}_t, t)\end{aligned} \tag{118}$$

The equation above is substituted into Equation 114 to get the expression:

$$\begin{aligned}&\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t+\Delta t)\\&= R^{M^{t+\Delta t}}G^{M^{t+\Delta t}}\{\mathbf{u}(\mathbf{x}_t, t) + [\Delta t(\mathbf{\Omega}\wedge\mathbf{x})\\&+ \frac{1}{2}\Delta t^2(\dot{\mathbf{\Omega}}\wedge\mathbf{x})]\cdot\nabla]\mathbf{u}(\mathbf{x}_t, t)\\&+ (\Delta t\frac{\partial}{\partial t})\mathbf{u}(\mathbf{x}_t, t)\}\end{aligned} \tag{119}$$

In the previous section is was shown that the following simplification can be made:

$$G^{M^{t+\Delta t}} \approx G^{M^t} \tag{120}$$

Using the simplification for the modified Galilean transformation above leads to the expression:

$$\begin{aligned}&\hat{\mathbf{u}}(\hat{\mathbf{x}}_{t+\Delta t}, t+\Delta t)\\&= R^{M^{t+\Delta t}}G^{M^t}\{\mathbf{u}(\mathbf{x}_t, t)\\&+ [\Delta t(\mathbf{\Omega}\wedge\mathbf{x}) + \frac{1}{2}\Delta t^2(\dot{\mathbf{\Omega}}\wedge\mathbf{x})]\cdot\nabla]\mathbf{u}(\mathbf{x}_t, t)\\&+ (\Delta t\frac{\partial}{\partial t})\mathbf{u}(\mathbf{x}_t, t)\}\end{aligned} \tag{121}$$

In order to complete the expression in Equation 113 further expressions must be defined. Assume that the point P is fixed in the rotating frame, and the rotation is around the origin (meaning that L and L' share an origin), then:

$$\begin{aligned}\hat{\mathbf{x}} &= R^{M^{t+\Delta t}}\mathbf{x}_{t+\Delta t} = R^{M^t}\mathbf{x}_t\\\mathbf{x}_t &= R^{M^{\Delta t}}\mathbf{x}_{t+\Delta t}\end{aligned} \tag{122}$$

Use the expression above and conduct a Taylor series expansion for $\mathbf{x}_{t+\Delta t}$:

$$\begin{aligned}\mathbf{x}_{t+\Delta t} &= \mathbf{x}_t + \Delta t\mathbf{V} + \frac{1}{2}\Delta t^2\mathbf{a} + O[\Delta t^3]\\&= R^{M^{\Delta t}}\mathbf{x}_{t+\Delta t} + \Delta t(\mathbf{\Omega}\wedge\mathbf{x_t})\\&+ \frac{1}{2}\Delta t^2(\dot{\mathbf{\Omega}}\wedge\mathbf{x_t}) + O[\Delta t^3]\end{aligned} \tag{123}$$

Re-arrange the expression above and consider it in the limit:

$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{R^{M^{\Delta t}}\mathbf{x}_{t+\Delta t} - \mathbf{x}_{t+\Delta t}}{\Delta t}\\&= \lim_{\Delta t\to 0}[(\mathbf{x_t}\wedge\mathbf{\Omega})\\&- \frac{1}{2}\Delta t(\dot{\mathbf{\Omega}}\wedge\mathbf{x_t}) - O[\Delta t^3]]\end{aligned} \tag{124}$$

If the above is considered for any vector $\mathbf{b}$, and if it is taken into account that $\mathbf{x}_{t+\Delta t} \to \mathbf{x}_t$ as $\Delta t \to 0$, the following equation related to rotation is obtained:

$$\lim_{\Delta t\to 0}\frac{R^{M^{\Delta t}}\mathbf{b} - \mathbf{b}}{\Delta t} = \mathbf{b}\wedge\mathbf{\Omega} \tag{125}$$

With all the required expressions in place Equation 113 can now be completed:

$$\begin{aligned}\frac{\partial\hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) = \lim_{\Delta t\to 0}&\frac{R^{M^{t+\Delta t}}G^{M^t}\{[1 - \frac{1}{R^{\Omega\Delta t}}}{\Delta t}\\&\frac{+\Delta t(\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla}{\Delta t}\\&\frac{+\frac{1}{2}\Delta t^2(\dot{\mathbf{\Omega}}\wedge\mathbf{x})\cdot\nabla]\mathbf{u}(\mathbf{x}_t, t)}{\Delta t}\\&\frac{+(\Delta t\frac{\partial}{\partial t})\mathbf{u}(\mathbf{x}_t, t)\}}{\Delta t}\end{aligned} \tag{126}$$

Equation 125 is used to simplify the expression above and with some re-arrangement of terms the following expression is obtained:

$$\begin{aligned}\frac{\partial\hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) = R^{M^t}[&\frac{\partial}{\partial t} + (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla - \mathbf{\Omega}\wedge]\\&[G^{M^t}\mathbf{u}(\mathbf{x}_t, t)]\end{aligned} \tag{127}$$

This equation above will retain its current form irrespective of any further changes in acceleration. All other terms that in inserted to account for variation in acceleration will become negligible when the expression is considered in the limit.

Equation 102 is substituted in the equation above to remove the modified Galilean operator from the equation:

$$\begin{aligned}&\frac{\partial \hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t)\\&= R^{M^t}[\frac{\partial}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{u}(\mathbf{x}_t, t))\\&+ R^{M^t}[\frac{\partial}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \boldsymbol{\Omega})\\&+ R^{M^t}[\frac{\partial}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\Delta t\end{aligned} \tag{128}$$

The different parts of $[\frac{\partial}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \boldsymbol{\Omega})$, which is the second combination of terms in Equation 128, will now be considered.

The transient term in the equation above can be expanded on with the use of the product rule for partial differential equations:

$$\frac{\partial}{\partial t}(\mathbf{x} \wedge \boldsymbol{\Omega}) = \frac{\partial \mathbf{x}}{\partial t} \wedge \boldsymbol{\Omega} + (\mathbf{x} \wedge \frac{\partial \boldsymbol{\Omega}}{\partial t}) \tag{129}$$

In the equation above the first term on the right hand side is zero because the magnitude of $\mathbf{x}$ is constant, it does not change its magnitude with respect to the origin. The second term is not equal to zero in this case and has to be taken into account since it represents the unsteady rotation, this is called the Euler fictitious force.

The terms $[(\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \boldsymbol{\Omega})$ has already been shown in the previous section to be equal to zero:

$$[(\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \boldsymbol{\Omega}) = 0 \tag{130}$$

The relation $[\frac{\partial}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \boldsymbol{\Omega})$ will, for this case, simplify to:

$$[\frac{\partial}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \boldsymbol{\Omega}) = \mathbf{x} \wedge \dot{\boldsymbol{\Omega}} \tag{131}$$

The different parts of $[\frac{\partial}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\Delta t$, which is the third combination of terms in Equation 128, will now be considered.

The transient component of the terms can be expanded again using the product rule. In this case the terms are all equal to zero:

$$\frac{\partial}{\partial t}(\mathbf{x} \wedge \dot{\boldsymbol{\Omega}}) = \frac{\partial \mathbf{x}}{\partial t} \wedge \dot{\boldsymbol{\Omega}} + (\mathbf{x} \wedge \frac{\partial \dot{\boldsymbol{\Omega}}}{\partial t}) = 0 \tag{132}$$

The first term in the equation above is equal to zero because the magnitude of $\mathbf{x}$ is constant. The second term in the equation above is equal to zero because constant acceleration is considered in this case. In the case where the acceleration is not constant, the second term will not be zero. However, this entire term will fall away in the limit as Δt, in the main equation, tends to zero.

Now consider the term $[(\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\Delta t$. If the identity:

$$(\mathbf{a} \cdot \nabla)(\mathbf{x} \wedge \dot{\boldsymbol{\Omega}}) = \mathbf{a} \wedge \dot{\boldsymbol{\Omega}} \tag{133}$$

is considered, it can be shown that the entire term is equal to zero:

$$\begin{aligned}&[(\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla - \boldsymbol{\Omega}\wedge](\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\\&\quad = (\boldsymbol{\Omega} \wedge \mathbf{x}) \wedge \dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega} \wedge (\mathbf{x} \wedge \dot{\boldsymbol{\Omega}}) = 0\end{aligned} \tag{134}$$

The entire third combination of terms in Equation 128 falls away, if not due to the nature of the rotational motion, it will fall away in the limit due to the Δt term.

The above leads to the final description of the unsteady terms in the momentum equation for constant rotation:

$$\begin{aligned}\frac{\partial \hat{\mathbf{u}}}{\partial t}(\hat{\mathbf{x}}_t, t) &= R^{M^t}[\frac{\partial}{\partial t} + (\boldsymbol{\Omega} \wedge \mathbf{x}) \cdot \nabla\\&- \boldsymbol{\Omega}\wedge](\mathbf{u}(\mathbf{x}_t, t)) + R^{M^t}(\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\end{aligned} \tag{135}$$

At this stage it must noted that in any pure rotation, this is the form the unsteady component of the equation will always take. Additional terms that appear in rotation will become negligible in the limit due to the Δt.

Take note in this equation of the appearance of a part of the Coriolis effect and the Euler effect.

The advection term in the non-inertial Navier-Stokes equation for constant rotation will be transformed in the following paragraphs. The relation between the inertial and rotational frames can be described by the equation below:

$$\begin{aligned}(\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} &= R^{M^t} G^{M^t} (\mathbf{u} \cdot \nabla)\mathbf{u} \\ &= R^{M^t} (G^{M^t}\mathbf{u} \cdot \nabla) G^{M^t}\mathbf{u}\end{aligned} \tag{136}$$

With the use of Equation 102, the equation above is expanded into:

$$\begin{aligned}&(\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} \\ &= R^{M^t}[(\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\Omega} + ((\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\Delta t) \cdot \nabla]\mathbf{u} \\ &+ R^{M^t}[(\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\Omega} + ((\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\Delta t) \cdot \nabla](\mathbf{x} \wedge \boldsymbol{\Omega}) \\ &+ R^{M^t}[(\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\Omega} + ((\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\Delta t) \cdot \nabla](\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\Delta t\end{aligned} \tag{137}$$

The equation above can be simplified by considering it in the limit of Δt and using the identity in Equation 37. this will lead to the equation:

$$\begin{aligned}&(\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} \\ &= R^{M^t}[(\mathbf{u} \cdot \nabla)\mathbf{u} + ((\mathbf{x} \wedge \boldsymbol{\Omega}) \cdot \nabla)\mathbf{u} \\ &+ (\mathbf{u} \wedge \boldsymbol{\Omega}) + (\mathbf{x} \wedge \boldsymbol{\Omega}) \wedge \boldsymbol{\Omega}]\end{aligned} \tag{138}$$

In this case a note can again be made that this is the form that the advection terms of the momentum equation will always taken in rotation, irrespective if the rotation is constant or variable.

Note in this equation the appearance of the other part of the Coriolis effect and the centrifugal effect.

The pressure gradient term will be considered next. The relation between the inertial and rotational frame for the pressure gradient can be expressed in the manner below:

$$\hat{\nabla}\hat{p} = R^{M^t} G^{M^t} \nabla p \tag{139}$$

Since scalars are invariant under transformation the equation can be simplified to:

$$\hat{\nabla}\hat{p} = R^{M^t} \nabla p \tag{140}$$

The last term in the momentum equation that must be transformed is the diffusion term. The relation between the inertial and non-inertial frames is described below and Equation 102 is used to expand on the relation:

$$\begin{aligned}&\nu\hat{\nabla}^2\hat{\mathbf{u}} \\ &= R^{M^t} G^{M^t} \nu\nabla^2\mathbf{u} \\ &= R^{M^t} \nu\nabla^2 G^{M^t}\mathbf{u} \\ &= R^{M^t} \nu\nabla^2[\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\Omega} + (\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\boldsymbol{\Delta}\mathbf{t}]\end{aligned} \tag{141}$$

If it is consider that:

$$\begin{aligned}&\nabla^2(\mathbf{x} \wedge \boldsymbol{\Omega}) = 0 \\ &\nabla^2((\mathbf{x} \wedge \dot{\boldsymbol{\Omega}})\Delta t) = 0\end{aligned} \tag{142}$$

The advection terms of the equation becomes:

$$\nu\hat{\nabla}^2\hat{\mathbf{u}} = R^{M^t}\nu\nabla^2\mathbf{u} \tag{143}$$

Note that the pressure and viscous term is Galilean invariant in this instance and the two components can be combined into a single variable $\mathbf{f}$:

$$\mathbf{f}(\mathbf{x}, t) = -\nabla p + \nu\nabla^2\mathbf{u} \tag{144}$$

The relation for f between the inertial and rotational frames can therefore be described by:

$$\hat{\mathbf{f}}(\hat{\mathbf{x}}, t) = R^{M^t}\mathbf{f}(\mathbf{x}, t) \tag{145}$$

Expressions have been obtained for all the parts of the momentum equation that relates the inertial frame to the rotational frame. The transformed equation is obtained by the summation of the transient and advection components as derived in Equation 135 and Equation 138:

$$\begin{aligned}&\frac{\partial\hat{\mathbf{u}}}{\partial t} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} \\ &= R^{M^t}[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{u} \wedge \boldsymbol{\Omega} \\ &+ \mathbf{x} \wedge \boldsymbol{\Omega} \wedge \boldsymbol{\Omega} + \mathbf{x} \wedge \dot{\boldsymbol{\Omega}}] \\ &= R^{M^t}[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}] \\ &+ R^{M^t}[2\mathbf{u} \wedge \boldsymbol{\Omega} + \mathbf{x} \wedge \boldsymbol{\Omega} \wedge \boldsymbol{\Omega} + \mathbf{x} \wedge \dot{\boldsymbol{\Omega}}]\end{aligned} \tag{146}$$

First grouping of terms on the right hand side of the equation above can be simplified using Equations 112, 144 and 145:

$$\begin{aligned}&R^{M^t}[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}] \\ &= R^{M^t}[-\nabla p + \nu\nabla^2\mathbf{u}] \\ &= R^{M^t}\mathbf{f}(\mathbf{x}, t) \\ &= \hat{\mathbf{f}}(\hat{\mathbf{x}}, t)\end{aligned} \tag{147}$$

Second grouping of terms in the transformed equation can be simplified using Equation 102 and with some manipulation the equation will become:

$$\begin{aligned} &R^{M^t}[2\mathbf{u}\wedge\mathbf{\Omega}+\mathbf{x}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}+\mathbf{x}\wedge\dot{\mathbf{\Omega}}] \\ &= 2\hat{\mathbf{u}}\wedge\mathbf{\Omega}-\hat{\mathbf{x}}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}+\hat{\mathbf{x}}\wedge\dot{\mathbf{\Omega}} \end{aligned} \tag{148}$$

The above simplifications can be substituted into Equation 146 and will result in the non-inertial momentum equation in a rotational frame:

$$\begin{aligned} &\frac{\partial\hat{\mathbf{u}}}{\partial t}+(\hat{\mathbf{u}}\cdot\hat{\nabla})\hat{\mathbf{u}} \\ &= -\hat{\nabla}\hat{p}+\nu\hat{\nabla}^2\hat{\mathbf{u}} \\ &+ 2\hat{\mathbf{u}}\wedge\mathbf{\Omega}-\hat{\mathbf{x}}\wedge\mathbf{\Omega}\wedge\mathbf{\Omega}+\hat{\mathbf{x}}\wedge\dot{\mathbf{\Omega}} \end{aligned} \tag{149}$$

It can be noted that the only difference between the momentum equation in constant and variable rotation is the appearance of the Euler term.

The Euler term $\hat{\mathbf{x}}\wedge\dot{\mathbf{\Omega}}$ represents the unsteady rotational acceleration of the point P around the axis. It has also been shown that the equation above will always take this form whether the acceleration in rotation is constant or variable. In the case where there is no acceleration, the Euler term will fall away.

3.2.3 Energy Equation

Consider the energy equation in the inertial frame:

$$\frac{\partial\rho\varepsilon}{\partial t}+(\nabla\cdot\rho\varepsilon\mathbf{u}) = -p(\nabla\cdot\mathbf{u})+\nabla\cdot(k\nabla T) \tag{150}$$

The various terms can be transformed to the rotational frame separately and then combined to obtained the energy equation in the rotational frame.

The time dependant term can be related between the frames as shown below since scalars are invariant under transformation:

$$\frac{\partial\hat{\rho}\hat{\varepsilon}}{\partial t} = R^{M^t}\frac{\partial\rho\varepsilon}{\partial t} \tag{151}$$

The relation for the convective term between the inertial and rotational frame is shown below. This equation can be expanded upon with the used of Equation 102:

$$\begin{aligned} &(\hat{\nabla}\cdot\hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) \\ &= R^{M^t}G^{M^t}(\nabla\cdot\rho\varepsilon\mathbf{u}) \\ &= R^{M^t}[\nabla\cdot\rho\varepsilon(\mathbf{u}+\mathbf{x}\wedge\mathbf{\Omega}+((\mathbf{x}\wedge\dot{\mathbf{\Omega}})\Delta t))] \\ &= R^{M^t}[\nabla\cdot\rho\varepsilon\mathbf{u}+\nabla\cdot\rho\varepsilon(\mathbf{x}\wedge\mathbf{\Omega}) \\ &+ \nabla\cdot\rho\varepsilon(\mathbf{x}\wedge\dot{\mathbf{\Omega}})\Delta t] \end{aligned} \tag{152}$$

The second and third terms on the right hand side of the equation above was shown in Equation 107 to be equal to zero. The convective term therefore becomes Galilean invariant under transformation:

$$(\hat{\nabla}\cdot\hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = R^{M^t}(\nabla\cdot\rho\varepsilon\mathbf{u}) \tag{153}$$

The term that represents the rate of work done by the normal force can be related in the inertial and rotational frames as shown below. This term can be expanded upon using Equation 102.

$$\begin{aligned} &-\hat{p}(\hat{\nabla}\cdot\hat{\mathbf{u}}) \\ &= R^{M^t}G^{M^t}[-p(\nabla\cdot\mathbf{u})] \\ &= R^{M^t}[-p\nabla\cdot(\mathbf{u}+\mathbf{x}\wedge\mathbf{\Omega}+(\mathbf{x}\wedge\dot{\mathbf{\Omega}})\Delta t] \\ &= R^{M^t}[-p\nabla\cdot\mathbf{u}-p\nabla\cdot(\mathbf{x}\wedge\mathbf{\Omega}) \\ &-p\nabla\cdot(\mathbf{x}\wedge\dot{\mathbf{\Omega}})\Delta t] \end{aligned} \tag{154}$$

Showing that the second and third terms is again equal to zero, the same as above and indicated in Equation 107, this transformation is also invariant.

$$-\hat{p}(\hat{\nabla}\cdot\hat{\mathbf{u}}) = R^{M^t}[-p\nabla\cdot\mathbf{u}] \tag{155}$$

The diffusive term in the rotational frame can be expressed in the inertial frame with the following relation:

$$\hat{\nabla}\cdot(\hat{k}\hat{\nabla}\hat{T}) = R^{M^t}G^{M^t}[\nabla\cdot(k\nabla T)] \tag{156}$$

Since k and T are scalars the relation is invariant under transformation:

$$\hat{\nabla}\cdot(\hat{k}\hat{\nabla}\hat{T}) = R^{M^t}[\nabla\cdot(k\nabla T)] \tag{157}$$

The full relation between the rotational and inertial frames for the energy equation can be obtained by summation of the components obtained above. This leads to the equation:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) + \hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) - \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) = R^{M^t}[\frac{\partial \rho\varepsilon}{\partial t} + (\nabla \cdot \rho\varepsilon\mathbf{u}) + p(\nabla \cdot \mathbf{u}) - \nabla \cdot (k\nabla T)] \quad (158)$$

The right hand side of the equation is equal to zero, this can be seen from re-arrangement of the terms in Equation 150. The non-inertial energy equation is invariant under transformation in this specific case for constant acceleration in rotation, but it can be seen that this equation will remain in this form even if the acceleration is not constant. Equation 110 is further used to arrive at:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) \quad (159)$$

3.3 Compressible Flow Conditions

In this section the non-inertial Navier-Stokes equations for conservation of mass, momentum and energy for variable rotation in compressible flow will be derived using an Eulerian approach.

3.3.1 Continuity Equation

The continuity equation was shown to be invariant under the modified Galilean transformation for constant rotational acceleration in Equation 109. Since the compressible case is considered here, the transformed equation becomes:

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = 0 \quad (160)$$

The continuity equation has thus proven to be invariant in all instances of rotation.

3.3.2 Momentum Equation

The difference between the compressible and incompressible momentum equation was discussed in the previous section. It was shown in Section 2.3.2 that the difference between the incompressible and compressible case only manifests in the diffusion term. In a similar manner the equation for variable rotation will take a form similar to Equation 80 but with the inclusion in this case of the Euler term as seen in Equation 149:

$$\frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t} + \hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) = -\hat{\nabla}\hat{P} + \hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla} \cdot \hat{\mathbf{u}})\hat{\mathbf{I}}] + 2\rho\hat{\mathbf{u}} \wedge \mathbf{\Omega} - \rho\hat{\mathbf{x}} \wedge \mathbf{\Omega} \wedge \mathbf{\Omega} + \rho\hat{\mathbf{x}} \wedge \dot{\mathbf{\Omega}} \quad (161)$$

In the case only pure rotations was considered, if the rotation was not pure, an additional fictitious term would have been present in the momentum equation. This term has its origin from the transient term as shown in Equation 129. The resulting momentum equation would be:

$$\frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t} + \hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) = -\hat{\nabla}\hat{P} + \hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla} \cdot \hat{\mathbf{u}})\hat{\mathbf{I}}] + 2\rho\hat{\mathbf{u}} \wedge \mathbf{\Omega} - \rho\hat{\mathbf{x}} \wedge \mathbf{\Omega} \wedge \mathbf{\Omega} + \rho\hat{\mathbf{x}} \wedge \dot{\mathbf{\Omega}} + \rho\dot{\mathbf{x}} \wedge \mathbf{\Omega} \quad (162)$$

This equation applies to all bodies in rotational acceleration. It has been shown that no further terms will be added to the non-inertial formulation even if the acceleration is unsteady.

3.3.3 Energy Equation

The energy equation remains invariant in the non-inertial frame as shown in Equation 159. The non-inertial energy equation in compressible flow therefore becomes:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = -\hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) + \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) \quad (163)$$

It was shown in this paper that the energy equation is invariant under transformation in all cases of rotation.

3.4 Constant Rotation Equations as a Special Case of the Variable Rotation Equations

In the previous sections it was shown that the continuity and conservation of energy equations are invariant under transformation for all cases of rotation. This is not unexpected since mass and energy are scalar values that is invariant, however this is an important result to show since it is not obvious when deriving the equations using the fluid parcel approach. The derivation of the momentum equation for the various cases not only provided the appropriate fictitious forces for each case, but it also showed from which transformations the forces originated.

As was shown, the derivations are directly reliant on the Tailor series expansion of the motion of the observed point P. The constant rotation case is therefore a special case of the variable rotation case since the order at which the Taylor series was truncated affects the fictitious force involved. If the derivations was done correctly, the variable rotation case will lead directly to the constant rotation case in constant rotation conditions is applied to it. Since the conservation of mass and energy equations is invariant, this will be shown using the momentum equation.

Consider the momentum equation for variable, pure rotation in the rotational frame:

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} = -\hat{\nabla}\hat{p} + \nu\hat{\nabla}^2\hat{\mathbf{u}} + 2\hat{\mathbf{u}} \wedge \mathbf{\Omega} - \hat{\mathbf{x}} \wedge \mathbf{\Omega} \wedge \mathbf{\Omega} + \hat{\mathbf{x}} \wedge \dot{\mathbf{\Omega}} \tag{164}$$

The fictitious forces involved are the Coriolis force, centrifugal force and Euler force. If it is considered that the rotational is constant, the Euler force should fall away. The equation then becomes:

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} = -\hat{\nabla}\hat{p} + \nu\hat{\nabla}^2\hat{\mathbf{u}} + 2\hat{\mathbf{u}} \wedge \mathbf{\Omega} - \hat{\mathbf{x}} \wedge \mathbf{\Omega} \wedge \mathbf{\Omega} \tag{165}$$

This is the same as Equation 53 which describes the momentum equation for constant rotation as seen from the rotational frame. This indicates consistency in the derivations.

4 Conclusion

This paper presented an Eulerian derivation of the non-inerital Navier-Stokes equations for compressible flow in arbitrary rotational conditions.

It was shown that the continuity equation and the energy equation is invariant under transformation in all cases. Some instances have been observed in the literature where the fictitious effects were added to the energy equation due to misconception that arise when using the fluid parcel (Lagrangian) approach. This work indicates that no fictitious effects are present in the energy equation.

In the derivation of the non-inertial momentum equation the origin of the fictitious forces could be observed. The Coriolis force originates from the transformation of both the transient and the adjective terms. This term is present in all cases of rotation. The centrifugal force originates from the transformation of the advection term and is also present in all rotation cases. The Euler force originates from the transformation of the transient term and is not present when the rotational speed is constant. There is a fourth term that is closely linked to the Euler force, $\dot{\mathbf{x}} \wedge \mathbf{\Omega}$, which is not present in cases of pure rotation, but in full arbitrary, non-constant rotations.

It has further been shown in this paper that changes in rotational acceleration does not introduce further fictitious forces in the momentum equation. The four effects identified includes the full range of terms that can be added in the rotational frame.

The Eulerian approach does not allow for the misconceptions that can arise when using the Lagrangian approach. The method is mathematically rigorous, but more so the meaning of the terms is clear and leads to a improved understanding of the origin of the fictitious effects in the rotational frame.

5 References

[1] Kageyama A and Hyodo M. Eulerian derivation of the Coriolis force. *Geochemistry, Geophysics and Geosystems*, Vol. 7, No. 2, pp 1-5, 2006.

[2] White FM. *Viscous Fluid Flow*. Third Edition, McGraw-Hill, 2006.

6 Nomenclature

6.1 Super Scripts and Sub Scripts

$'$	Orientation preserving frame
$\hat{}$	Rotational frame
rel	Relative conditions
t	Time
Δt	Change in time

6.2 Alphabet

$\mathbf{a}$	Acceleration vector
$\mathbf{b}$	Vector
k	Heat transfer coefficient
p	Pressure per unit mass
t	Time
$\mathbf{u}$	Veloctiy vector
x	Distance in x-direction
$\mathbf{x}$	Position vector

y	Distance in y-direction
z	Distance in z-direction
G	Galilean operator
$\mathbf{I}$	Identity matrix
O	Frame designations
P	Pressure
R	Rotational transform operator
T	Temperature
V	Velocity in x-direction
$\mathbf{V}$	Velocity vector
$\mathbf{X}$	Position vector

6.3 Greek Letters

ε	Internal energy
λ	Second viscosity
μ	Dynamic viscosity
ν	Kinematic viscosity
ρ	Density
Ω	Rotational speed around the z-axis
$\mathbf{\Omega}$	Rotational speed vector

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Appendix A - Proof of Identities

Identity 1

$$\begin{aligned}
&\nabla\cdot(\mathbf{x}\wedge\mathbf{\Omega})\\
&=\frac{\partial}{\partial x_i}[x_j\Omega_k - x_k\Omega_j]\mathbf{i}\\
&-\frac{\partial}{\partial x_j}[x_i\Omega_k - x_k\Omega_i]\mathbf{j}\\
&+\frac{\partial}{\partial x_k}[x_i\Omega_j - x_j\Omega_i]\mathbf{k}\\
&=0
\end{aligned}$$

Identity 2

$$\begin{aligned}
&\nabla\cdot(\mathbf{x}\wedge\dot{\mathbf{\Omega}})\\
&=\frac{\partial}{\partial x_i}[x_j\dot\Omega_k - x_k\dot\Omega_j]\mathbf{i}\\
&-\frac{\partial}{\partial x_j}[x_i\dot\Omega_k - x_k\dot\Omega_i]\mathbf{j}\\
&+\frac{\partial}{\partial x_k}[x_i\dot\Omega_j - x_j\dot\Omega_i]\mathbf{k}\\
&=0
\end{aligned}$$

Identity 3

$$\begin{aligned}
&\nabla^2(\mathbf{x}\wedge\mathbf{\Omega})\\
&=(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial x_k^2})[x_j\Omega_k - x_k\Omega_j]\mathbf{i}\\
&-(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial x_k^2})[x_i\Omega_k - x_k\Omega_i]\mathbf{j}\\
&+(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial x_k^2})[x_i\Omega_j - x_j\Omega_i]\mathbf{k}\\
&=0
\end{aligned}$$

Identity 4

$$\begin{aligned}
&\nabla^2(\mathbf{x}\wedge\dot{\mathbf{\Omega}})\\
&=(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial x_k^2})[x_j\dot\Omega_k - x_k\dot\Omega_j]\mathbf{i}\\
&-(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial x_k^2})[x_i\dot\Omega_k - x_k\dot\Omega_i]\mathbf{j}\\
&+(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial x_k^2})[x_i\dot\Omega_j - x_j\dot\Omega_i]\mathbf{k}\\
&=0
\end{aligned}$$

Identity 5

$$(\mathbf{c}\cdot\nabla)(\mathbf{x}\wedge\mathbf{\Omega}) = \mathbf{c}\wedge\mathbf{\Omega}$$

$$\begin{aligned}
LH &= (\mathbf{c}\cdot\nabla)(\mathbf{x}\wedge\mathbf{\Omega})\\
&=(c_i\frac{\partial}{\partial x_i}+c_j\frac{\partial}{\partial x_j}+c_k\frac{\partial}{\partial x_k})[(x_j\Omega_k - x_k\Omega_j)\mathbf{i}\\
&-(x_i\Omega_k - x_k\Omega_i)\mathbf{j}+(x_i\Omega_j - x_j\Omega_i)\mathbf{k}]\\
&=(c_j\Omega_k - c_k\Omega_j)\mathbf{i}-(c_i\Omega_k - c_k\Omega_i)\mathbf{j}\\
&+(c_i\Omega_j - c_j\Omega_i)\mathbf{k}]
\end{aligned}$$

$$(\mathbf{c}\cdot\nabla)(\mathbf{x}\wedge\mathbf{\Omega}) = \mathbf{c}\wedge\mathbf{\Omega}$$

$$\begin{aligned} RH &= \mathbf{c}\wedge\mathbf{\Omega} \\ &= (c_j\Omega_k - c_k\Omega_j)\mathbf{i} - (c_i\Omega_k - c_k\Omega_i)\mathbf{j} \\ &+ (c_i\Omega_j - c_j\Omega_i)\mathbf{k}] \end{aligned}$$

$$LH = RH$$

Identity 6

$$(\mathbf{c}\cdot\nabla)(\mathbf{x}\wedge\dot{\mathbf{\Omega}}) = \mathbf{c}\wedge\dot{\mathbf{\Omega}}$$

$$\begin{aligned} LH &= (\mathbf{c}\cdot\nabla)(\mathbf{x}\wedge\dot{\mathbf{\Omega}}) \\ &= (c_i\frac{\partial}{\partial x_i} + c_j\frac{\partial}{\partial x_j} + c_k\frac{\partial}{\partial x_k})[(x_j\dot{\Omega}_k - x_k\dot{\Omega}_j)\mathbf{i} \\ &- (x_i\dot{\Omega}_k - x_k\dot{\Omega}_i)\mathbf{j} + (x_i\dot{\Omega}_j - x_j\dot{\Omega}_i)\mathbf{k}] \\ &= (c_j\dot{\Omega}_k - c_k\dot{\Omega}_j)\mathbf{i} - (c_i\dot{\Omega}_k - c_k\dot{\Omega}_i)\mathbf{j} \\ &+ (c_i\dot{\Omega}_j - c_j\dot{\Omega}_i)\mathbf{k}] \end{aligned}$$

$$\begin{aligned} RH &= \mathbf{c}\wedge\dot{\mathbf{\Omega}} \\ &= (c_j\dot{\Omega}_k - c_k\dot{\Omega}_j)\mathbf{i} - (c_i\dot{\Omega}_k - c_k\dot{\Omega}_i)\mathbf{j} \\ &+ (c_i\dot{\Omega}_j - c_j\dot{\Omega}_i)\mathbf{k}] \end{aligned}$$

$$LH = RH$$

Identity 7

$$\nabla(\mathbf{x}\wedge\mathbf{\Omega}) + \nabla(\mathbf{x}\wedge\mathbf{\Omega})^T = 0$$

$$\nabla(\mathbf{x}\wedge\mathbf{\Omega}) =$$

$$\begin{Bmatrix} \frac{\partial}{\partial x_i}(x_j\Omega_k - x_k\Omega_j) & \dots & \frac{\partial}{\partial x_k}(x_j\Omega_k - x_k\Omega_j) \\ \frac{\partial}{\partial x_i}(-x_i\Omega_k + x_k\Omega_i) & \dots & \frac{\partial}{\partial x_k}(-x_i\Omega_k + x_k\Omega_i) \\ \frac{\partial}{\partial x_i}(x_i\Omega_j - x_j\Omega_i) & \dots & \frac{\partial}{\partial x_k}(x_i\Omega_j - x_j\Omega_i) \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & \Omega_k & -\Omega_j \\ -\Omega_k & 0 & \Omega_i \\ \Omega_j & -\Omega_i & 0 \end{Bmatrix}$$

$$\nabla(\mathbf{x}\wedge\mathbf{\Omega}) + \nabla(\mathbf{x}\wedge\mathbf{\Omega})^T = 0$$

$$\begin{Bmatrix} 0 & \Omega_k & -\Omega_j \\ -\Omega_k & 0 & \Omega_i \\ \Omega_j & -\Omega_i & 0 \end{Bmatrix} + \begin{Bmatrix} 0 & -\Omega_k & \Omega_j \\ \Omega_k & 0 & -\Omega_i \\ -\Omega_j & \Omega_i & 0 \end{Bmatrix} = 0$$

Identity 8

$$\mathbf{x}\wedge\mathbf{\Omega} = -\mathbf{\Omega}\wedge\mathbf{x}$$

$$\begin{aligned} LH &= \mathbf{x}\wedge\mathbf{\Omega} \\ &= (x_j\Omega_k - x_k\Omega_j)\mathbf{i} - (x_i\Omega_k - x_k\Omega_i)\mathbf{j} \\ &+ (x_i\Omega_j - x_j\Omega_i)\mathbf{k} \end{aligned}$$

$$\begin{aligned} RH &= -\mathbf{\Omega}\wedge\mathbf{x} \\ &= -[(x_k\Omega_j - x_j\Omega_k)\mathbf{i} - (x_k\Omega_i - x_i\Omega_k)\mathbf{j} \\ &+ (x_j\Omega_i - x_i\Omega_j)\mathbf{k}] \\ &= (x_j\Omega_k - x_k\Omega_j)\mathbf{i} - (x_i\Omega_k - x_k\Omega_i)\mathbf{j} \\ &+ (x_i\Omega_j - x_j\Omega_i)\mathbf{k} \end{aligned}$$

$$LH = RH$$

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