UNCERTAINTY QUANTIFICATION AND RELIABILITY ANALYSIS METHODS APPLIED TO AIRCRAFT STRUCTURES

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Abstract

While deterministic methods have been and still are used in the design of aircraft components, it has been recognized that the inherent uncertainty involved in those structures requires the consideration of alternative procedures to deal more accurately with structural reliability. The necessity for a competitive industry to design efficient and reliable products, implies taking into account these methodologies applied to realistic computer models of aircraft components. In this work, a revision of some existing procedures of reliability analysis and their capabilities is carried out, along with a practical example showing the performance of the methods used in the study. The reliability analysis methods selected are the FOSM, the FORM and TANA3, based on the approximation of the limit state function and the Monte Carlo, Latin Hypercube and importance sampling. Results show that some differences exist in the performance of each method. Limit state approximation methods behave much better in those cases with a reduced number of random variables or limit state functions. However, in those cases having a high number of random variables and/or limit state functions, sampling methods become an alternative, with computational costs quite similar to the limit state approximation methods.

1 Introduction

Uncertainty quantification of structural response is an essential task in the design of aircraft components. Conventional methods, involving deterministic design, define safety factors to deal with the inherent uncertainties found in every system. However, in reliability analysis the goal is the obtainment of the probability of failure when design criteria or limit states are not satisfied. The main advantage of these methods over deterministic design is that structural safety can be defined more accurately and, at the same time, the performace of conventional procedures is enhanced, because in each situation its specific uncertainties are taken into account.

The word uncertainty is used to identify the fact that it is impossible to know exactly the value of a quantity that is susceptible to be measured, but it is possible to know its most probable value. This probabilistic information, related to the variable, can be estimated and when the magnitude is considered from this point of view, then is qualified as a random variable.

Oberkampf et al. [2002] oppose the concept of uncertainty, as a potential difference in any phase of the design process, due to the lack of information, with the concept of error, which is a recognizable failure not attributable to the lack of information neither to a random component.

Nowadays, it is accepted a classification that divides the uncertainty in epistemological and random. According to Oberkampf et al. [2002, 2004], epistemological uncertainty refers to the lack of knowledge about a phenomenon and it can be decreased with further investigation. It comes from some degree of ignorance or incomplete information about the system or its environment.

Random uncertainty is a measure of the heterogeneity or variability of the physical system considered. It cannot be decreased with additional investigation.

Thoft-Christensen and Baker [1982] divide the uncertainties related to structural analysis in three types, depending on the modeling process. The first type is the physical uncertainty, due to the natural variability of magnitudes involved in the structure, as loads, material properties and dimensions. This uncertainty can be decreased in some cases, specially in those quantities subjected to quality control, as materials and geometrical dimensions, but it cannot be removed because it is a random uncertainty.

The next type of uncertainty comes from the probabilistic characterization of the physical uncertainty. This type is known as statistical or parametric uncertainty and is caused by the lack of data to accurately estimate the statistical behaviour of the physical magnitudes related to the structure. It is mainly a random uncertainty.

The third type is the model uncertainty, also known as formal uncertainty, which is mainly epistemological and is caused by the simplifying hypothesis required to build the mathematical model of the structure.

A special kind of epistemological uncertainty, called phenomenological uncertainty, which is of special interest in structural engineering, is described by Melchers [1999]. It occurs when the technology used in any stage of design, analysis or contruction causes uncertainty about any aspect of structural performance. It is of special importance when novel techniques are applied or when they are not fully tested. As it happens with the other uncertainties of this type, its effects can only be estimated subjectively.

On the other hand, according to Schueller [2007], it is advisable to take into account all the posible uncertainties, instead of selecting the most important ones from an aprioristic criterion. Thus, errors coming from not considering the factors having great sensitivity on the response are avoided. The drawback of this procedure is the impact on computational cost when a high number of random variables are considered.

Reliability is related with the probability of verifying a certain condition, and so, it cannot be established with total certainty that a design will fulfill a limit state condition. Instead, there is some probability p_f that the limit state will not be verified. This is known as the probability of failure.

In a probabilistic analysis, the uncertainties in the basic magnitudes of the structure are considered directly in the analysis, changing from fixed quantities to random variables. In the case of an aircraft component with resistance R supporting some external loads which provoke a structural response S, the probability of failure is:

$$\begin{aligned} p_f =& P(R \leqslant S) = P(R - S \leqslant 0) = \\ =& P[g(r,s) \leqslant 0] \end{aligned} \tag{1}$$

Where $g(r,s)$ is the limit state function. If f_R and f_S are the probability density functions of R y S, respectively, and f_{RS} is its joint probability density function (fig. 1), then:

$$p_f = P[g(r,s) \leqslant 0] = \int_{-\infty}^{\infty}\int_{-\infty}^{s} f_{RS}(r,s)drds \tag{2}$$

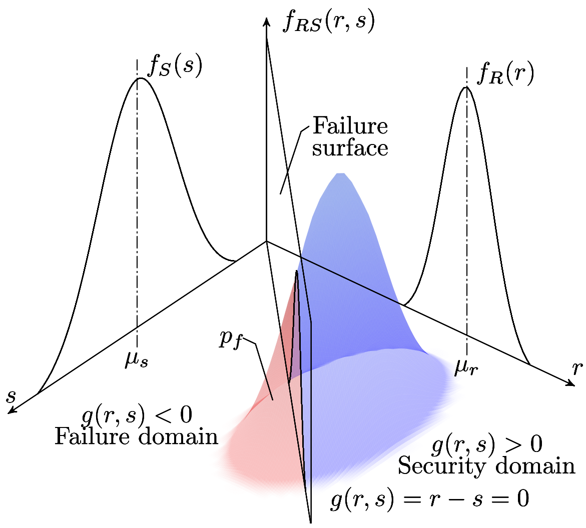

Fig. 1 Joint probability density function of R and S

And when R and S are statistically independent:

$$p_f = \int_{-\infty}^{\infty}\int_{-\infty}^{s} f_R(r) f_S(s)drds \tag{3}$$

The limit state function defines if a design belongs to the failure domain, where the limit state is not verified, or to the safety domain, where it is. If **a** is the vector of basic variables which contains the n random variables of the structure, then the domains are defined as follows:

$$\text{Failure domain: } F = \{\mathbf{a} \mid g(\mathbf{a}) < 0\} \tag{4}$$

$$\text{Security domain: } S = \{\mathbf{a} \mid g(\mathbf{a}) \geqslant 0\} \tag{5}$$

The boundary between both domains is known as the failure surface or limit state surface, which generally is an hypersurface of $n-1$ dimensions in the n-dimensional space of basic variables. The safety margin is now defined as a random variable which can be identified with the value of the limit state function:

$$M = g(\mathbf{a}) \tag{6}$$

From the previous considerations, it can be generalized the expression (2) corresponding to the probability of failure, which now can be formulated as:

$$p_f = P[g(\mathbf{a}) \leqslant 0] = \int \cdots \int_{g(\mathbf{a}) \leqslant 0} f_{\mathbf{A}}(\mathbf{a})\, d\mathbf{a} \tag{7}$$

The equation (7) is known as the fundamental equation of reliability, where $f_{\mathbf{A}}(\mathbf{a})$ is the joint probability density function of all the basic variables involved in the response of the system.

Except in some particular cases, the integral (7) cannot be resolved analytically, because of the nonlinearity of $f_{\mathbf{A}}(\mathbf{a})$, and also due to the fact that the number of random variables usually employed is high, and therefore the dimension of the problem. Several methods have been proposed to solve this problem. In this paper, the most appropiate methods for the design of aircraft components are selected and explained. Also, a practical example is considered to demonstrate the application of the methods and their performance.

2 Methods

The uncertainty quantification methods considered in this paper can be divided in two types. The first one covers the methods involving approximations of limit state surface. The considered methods, based on the Taylor series expansion of the limit state function and other approximations, are the FOSM or first order second-moment method, the FORM or first order reliability method and the TANA3 or two point adaptive nonlinear approximation. Those methods require information about the value of the limit state function and its derivatives in the vicinity of the design point.

The second type is made up of the simulation methods. This category includes Monte Carlo simulation and its modifications, aimed to reduce the elevated computational requirements associated with them. Those methods are the latin hypercube sampling and importance sampling. In simulation methods, samplings of the random properties are generated and feeded as an input to the system, obtaining a response population where statistical data is measured. A brief description of all the aforementioned methods is presented next.

2.1 FOSM method

The FOSM (First Order Second-Moment Method), proposed by Cornell [1969], assume the approximation of limit state surface by the tangent hyperplane at the point $\boldsymbol{\mu}_{\mathbf{A}}$, defined by the mean value of the random variables:

$$M = g(\mathbf{a}) \simeq g(\boldsymbol{\mu}_{\mathbf{A}}) + \nabla g^T(\boldsymbol{\mu}_{\mathbf{A}})(\mathbf{a} - \boldsymbol{\mu}_{\mathbf{A}}) \tag{8}$$

The mean value of the previous expression is:

$$\mu_M = E[g(\mathbf{a})] \simeq g(\boldsymbol{\mu}_{\mathbf{A}}) \tag{9}$$

And the variance has the value:

$$\sigma_M^2 = Var[g(\mathbf{a})] \simeq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial g(\boldsymbol{\mu}_{\mathbf{A}})}{\partial a_i} \frac{\partial g(\boldsymbol{\mu}_{\mathbf{A}})}{\partial a_j} \sigma_{A_i A_j} \tag{10}$$

From the equations (9) and (10), it is possible the determination of the reliability index as:

$$\beta = \frac{\mu_M}{\sigma_M} \tag{11}$$

The probability of failure is then obtained as:

$$p_f = 1 - \Phi(\beta) \tag{12}$$

Where Φ is the cumulative distribution function of the standard normal variable. The previous procedure has several drawbacks. The most important is that the results is not invariant with respect to the formulation of the limit state function. Besides, it only gives accurate results in certain situations as, for instance, when all the variables $\mathbf{a}$ are statistically independent, follow a Gauss distribution and $g(\mathbf{a})$ is a linear function. In this case, M follows a Gauss distribution too and the reliability index (11) has an exact value.

Unfortunately, in most of the situatons, these hypothesis cannot be verified and so the equation eq. (11) only gives an approximation to the reliability of the design.

2.2 FORM method

The FORM method (First Order Reliability Method), proposed by Hasofer and Lind [1974], uses the information of the first two statistical moments of the random variables. It assumes that the random variables are statistically independent and follow a normal distribution. This does not reduce the generality of the approach, because by means of some transformations it is possible the approximation of any type of distribution in this way. Those tranformations can be linear, like the ones by Rackwitz and Fiessler [1978], Chen and Lind [1983] and Wu and Wirsching [1987] or nonlinear, as in the case of the developed by Rosenblatt [1952], Nataf [1962] and Box and Cox [1964].

Taking into account those considerations, the standard normal properties $\mathbf{a}'$ can be defined from the transformation of the original random variables to a standard Gauss distribution as:

$$a'_i = \frac{a_i - \mu_{A_i}}{\sigma_{A_i}} \tag{13}$$

The FORM consists in the search of the most probable point of failure (MPP) in the standardised domain, in order to allow the substitution of the limit state function by its Taylor series expansion of first order at that point:

$$g(\mathbf{a}') \simeq g(\mathbf{a}'_\mathbf{f}) + \nabla g(\mathbf{a}'_f)^T(\mathbf{a}' - \mathbf{a}'_f) \tag{14}$$

Where $\mathbf{a}'_f$, the most probable point of failure, is the point of minimum distance to the origin from the limit state surface. Geometrically, the method supposes the approximation of the limit state surface by the tangent hyperplane at the most probable point of failure (fig. 2). The reliability index, which it is also known as Hasofer and Lind index, is related now to the failure surface, but it is invariant with respect to the formulation of the limit state function.

$$\beta = -\frac{\mathbf{a}'^T_f \nabla g(\mathbf{a}'_f)}{\sqrt{\nabla g(\mathbf{a}'_f)^T \nabla g(\mathbf{a}'_f)}} \tag{15}$$

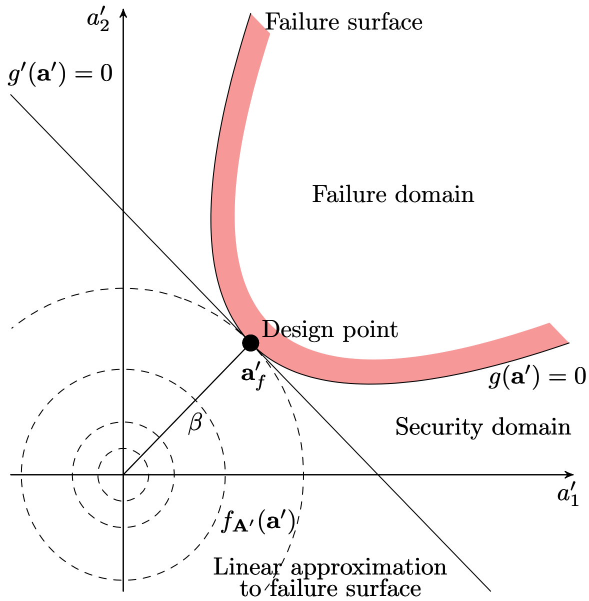

Fig. 2 First Order Reliability Method

Several methods exist which calculate the MPP position. In that regard, the alternatives suggested by Rackwitz [1976], Rackwitz and Fiessler [1978] and Ayyub and Haldar [1984] can be recommended.

2.3 TANA3 method

Instead of using information about the limit state function and its derivatives till a certain order, like in the rest of the limit state approximation methods, adaptive approximations use information generated in several points. Thus, a more precise estimate of nonlinear limit state functions can be calculated without using derivatives of higher order, which reduces the computational cost.

The TANA3 (Two-point Adaptive Nonlinear Approximations) method [Xu and Grandhi, 1998] is based on an exponential approximation which uses information of the current iteration k and also of the previous one $k-1$. So, the limit state surface can be approximated as:

$$g(\mathbf{a}) \simeq g(\mathbf{a}_k) + \sum_{i=1}^{n} \frac{\partial g(\mathbf{a}_k)}{\partial a_i} \frac{(a_{i,k})^{(1-r_i)}}{r_i} \left(a_i^{r_i} - a_{i,k}^{r_i}\right) + \\ + \frac{\varepsilon_2}{2} \sum_{i=1}^{n} \left(a_i^{r_i} - a_{i,k}^{r_i}\right)^2 \tag{16}$$

Where the nonlinear index r_i and the parameter ε_2 can be defined as:

$$r_i = 1 + \frac{\ln\left(\dfrac{\partial g(\mathbf{a}_{k-1})}{\partial a_i}\right) - \ln\left(\dfrac{\partial g(\mathbf{a}_k)}{\partial a_i}\right)}{\ln(a_{i,k-1}) - \ln(a_{i,k})} \tag{17}$$

$$\varepsilon_2 = \frac{2\left[g(\mathbf{a}_{k-1}) - g(\mathbf{a}_k)\right]}{\sum_{i=1}^{n}\left(a_i^{r_i} - a_{i,k-1}^{r_i}\right)^2 + \sum_{i=1}^{n}\left(a_i^{r_i} - a_{i,k}^{r_i}\right)^2} - \\ - \frac{2\left[\sum_{i=1}^{n} \dfrac{a_{i,k}^{1-r_i}}{r_i} \dfrac{\partial g(\mathbf{a}_k)}{\partial a_i}\left(a_{i,k-1}^{r_i} - a_{i.k}^{r_i}\right)\right]}{\sum_{i=1}^{n}\left(a_i^{r_i} - a_{i,k-1}^{r_i}\right)^2 + \sum_{i=1}^{n}\left(a_i^{r_i} - a_{i,k}^{r_i}\right)^2} \tag{18}$$

2.4 Monte Carlo method

The Monte Carlo Sampling method (MCS)[Sobol, 1994] provides an estimation of the reliability by means of statistical simulations of the random variables. The procedure consists in the selection of a high number of samples m of the random properties, according to their probability distribution, and perform deterministic analysis with those values in order to obtain the structural response for each one of the samples. By means of the processing of those results, the statistical moments of the structural response can be obtained.

The application of Monte Carlo method to obtain the solution of equation (7) requires the introduction in the integrand of the function $\upsilon(\mathbf{a})$:

$$\upsilon(\mathbf{a}) = \begin{cases} 1 & , \text{si } g(\mathbf{a}) \leqslant 0 \\ 0 & , \text{si } g(\mathbf{a}) > 0 \end{cases} \tag{19}$$

Now, the domain of integration in the expression (7) includes the whole real domain and the probability of failure can be formulated as:

$$p_f = P\left[g(\mathbf{a}) \leqslant 0\right] = \int \cdots \int \upsilon(\mathbf{a}) f_{\mathbf{A}}(\mathbf{a})\, d\mathbf{a} \tag{20}$$

If a Monte Carlo sampling is applied to the previous equation, so that m samples are generated, of which m_f provoke that $g(\mathbf{a}) \leqslant 0$, then the probability of failure $p_{f,e}$ can be estimated as:

$$p_{f,e} = \frac{1}{m} \sum_{j=1}^{m} \upsilon(\mathbf{a}_j) = \frac{m_f}{m} \tag{21}$$

The accuracy of the previous estimation increases with the number of samples m, although it is related with the probability of failure too, since a very low value of p_f requires a higher number of samples to achieve some results in the failure domain.

Shooman [1968] suggest the following expression to estimate, with a confidence of 95 %, the probability of failure p_f with an error $\varepsilon_{p_{f,e}}$:

$$m = \frac{4(1-p_f)}{\varepsilon_{p_{f,e}}^2 p_f} \tag{22}$$

As can be seen, the obtention of precise results with low values of probability of failure entails a large number of simulations. Thus the application of direct Monte Carlo simulation is restricted to simple problems, not requiring excessive computational resources, or to structures where either the failure criteria can be relaxed or the accuracy of the results or both.

On the other hand, the equation (22) clearly shows that the error not depends on the dimension n of the problem, and so the method is suitable for those cases with a high number of random variables, where the computational cost compares with the required by the limit state approximation methods. Moreover, as each one of the deterministic simulations in Monte Carlo sampling is independent from the others, the algorithm can be easily parallelized, reducing the analysis time by a factor related with the number of simultaneous jobs than can be executed.

2.5 Latin hypercube sampling

The random sampling in Monte Carlo method can be improved with some techniques of sampling selection. One of these methods is the Latin Hypercube Sampling (LHS), proposed by McKay et al. [1979]. In LHS, the domain where random variables are defined, is divided in subdomains of equal probability and the samples are selected so the design region is evenly covered.

If m is the number of samples and n is the number of random variables, then the domain of each variable is divided in m subdomains of equal probability. The samples are selected randomly in each subdomain and only a sample is taken per subdomain, and so each row and column in the hypercube of partitions has only a sample. Figure 3 shows an example with $m = 10$ samples and $n = 2$ random variables.

An advantage of LHS sampling over Monte Carlo sampling is that, if the structural response is dominated by only one parameter, then all the levels of the response are evaluated. This is not guaranteed by direct sampling. However, if the response is controled by multiple parameters, then LHS sampling does not provide a significant advantage over MCS.

2.6 Importance sampling

Importance sampling method was proposed by Kahn and Marshall [1953] and was applied later to reliability structural analysis [Augusti et al., 1984]. The objective in this method is to concentrate the distribution of samples in the region which has more contribution to the probability of failure, instead of spreading them over the whole domain (fig. 4). In order to do that, an auxiliar probability density function $h(\mathbf{a})$ is defined, with the purpose of generate samples in the region which most contributes to the value of integral (20):

$$p_f = P\left[g\left(\mathbf{a}\right) \leqslant 0\right] = \int \cdots \int \upsilon(\mathbf{a}) \frac{f_{\mathbf{A}}\left(\mathbf{a}\right)}{h_{\mathbf{A}}\left(\mathbf{a}\right)} h_{\mathbf{A}}(\mathbf{a}) d\mathbf{a} \tag{23}$$

According to that, the estimation of probability of failure $p_{f,e}$ is now as follows:

$$p_{f,e} = \frac{1}{m} \sum_{j=1}^{m} \upsilon(\mathbf{a}_j) \frac{f_{\mathbf{A}}\left(\mathbf{a}_j\right)}{h_{\mathbf{A}}\left(\mathbf{a}_j\right)} \tag{24}$$

The main disadvantage of this method is that the selection of $h_{\mathbf{A}}(\mathbf{a})$ is conditioned by the shape of failure domain.

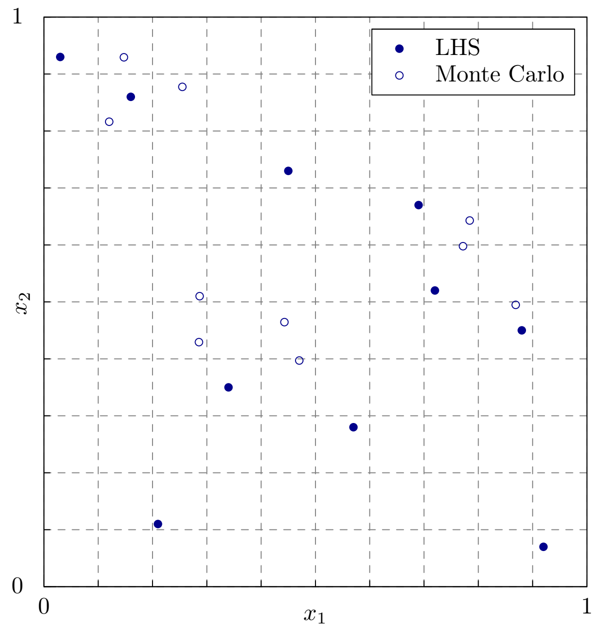

Fig. 3 Comparison of MCS and LHS sampling

3 Application example

This section describes the application of the uncertainty quantification procedures previously mentioned to a finite element model of an aicraft panel (fig. 5). The panel is made up of an aluminum skin reinforced with four frames and four stiffeners. The nominal values of the panel properties are shown in table 1.

Variable	μ	σ	δ (%)	Distribution
Skin thickness (mm)	1.5	0.225	15	Lognormal
Frame thickness (mm)	2.5	0.375	15	Lognormal
Stiffener thickness (mm)	2.5	0.375	15	Lognormal
Frame pitch (mm)	200	37.5	15	Lognormal
Stiffener pitch (mm)	250	30	15	Lognormal
Frame height (mm)	50	7.5	15	Lognormal
Stiffener height (mm)	30	4.5	15	Lognormal
Skin modulus (GPa)	72	3.6	5	Normal
Frame modulus (GPa)	71	3.55	5	Normal
Stiffener modulus (GPa)	71	3.55	5	Normal
Shear load (kN/m)	5	1.5	30	Gumbel
Axial load (kN/m)	10	3	30	Gumbel

Table 1 Panel properties

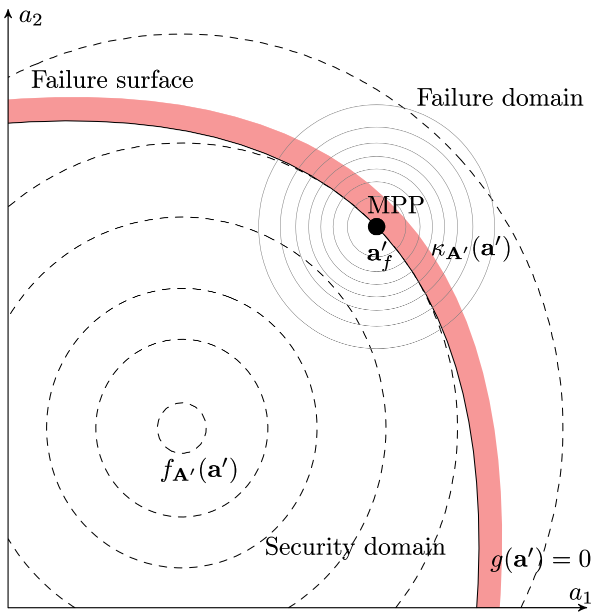

Fig. 4 Importance sampling method

The mesh consists of four node shell elements to modelate the skin, and beam elements for the frames and the stiffeners (fig. 6), resulting in a total of 2329 elements, 2432 nodes and 14,592 degrees of freedom (DOF). The model has its rotations restrained at the four edges and also its displacements are restrained in one of the borders. The loads applied consist of a shear load in the borders with a value of 0.5 kN/m and an axial load of 1.0 kN/m on the edge opposite to the restrained border.

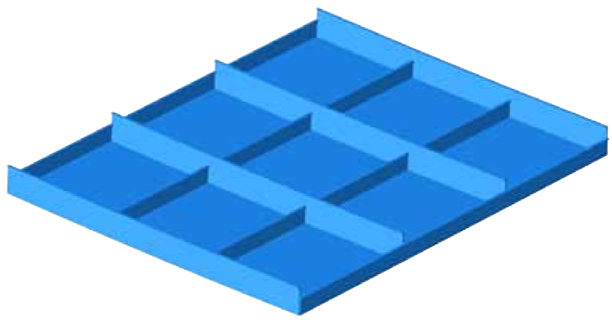

Fig. 5 Geometry of the panel

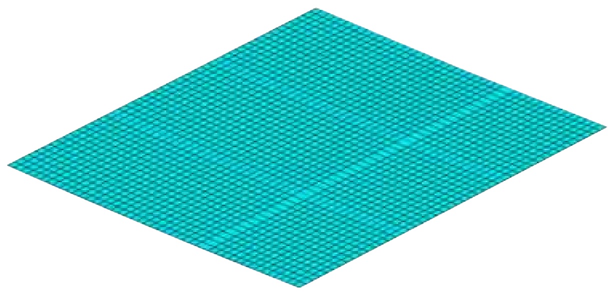

Fig. 6 Finite element mesh of the panel

For carrying out the calculations, a HPC cluster of 8 nodes with 48 processors of 64 bits and 192 GB of physical memory has been used. The peak performance provided by this machine is 237 GFlops.

In order to evaluate the structural reliability of the panel, several uncertainties have been considered. Those uncertainties have been classified according to their type into one of the categories of material, geometry or loads. The type of probability distribution for each one of the 12 random variables has been chosen taken into account the physical phenomena involved with them (table 1). On the other hand, all the variables are statistically independent.

The limit state selected to verify the structural reliability evaluates the buckling factor λ, which must be above the minimum value $\lambda_{min} = 100$. This condition can be expressed as:

$$g(\mathbf{a}) = \frac{\lambda}{\lambda_{min}} - 1 \geqslant 0 \tag{25}$$

Prior to the evaluation of the structural reliability, a deterministic analysis is conducted with the mean values of the random properties. Figure 7 shows the first buckling mode obtained with those values.

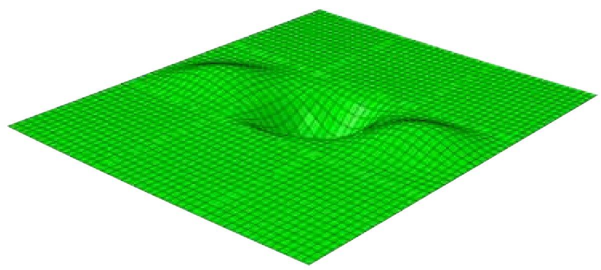

Fig. 7 First buckling mode

The table 2 shows the results of probability of failure and reliability index for each one of the methods considered. Besides, to establish some conclusions regarding to computational cost of each method, the number of iterations is also presented, including the required ones to calculate the gradients of limit state function with finite differences in limit state approximation methods. Finally, the fig. 8 compares the computational times of each method.

In view of the previous results, several considerations can be done. The first one is that all the results are quite similar, except for the FOSM method, which show some diferences. Those results were expected, as this method only gives an

Method	β	p_{fn}	Iterations
FOSM	1.177	1.195×10^{-1}	23
FORM	2.451	7.123×10^{-3}	644
TANA3	2.133	1.645×10^{-2}	185
MCS	2.878	2.000×10^{-3}	10,000
LHS	2.652	4.000×10^{-3}	1000
VRT-I	2.927	1.711×10^{-3}	745

Table 2 Structural reliability results for each method

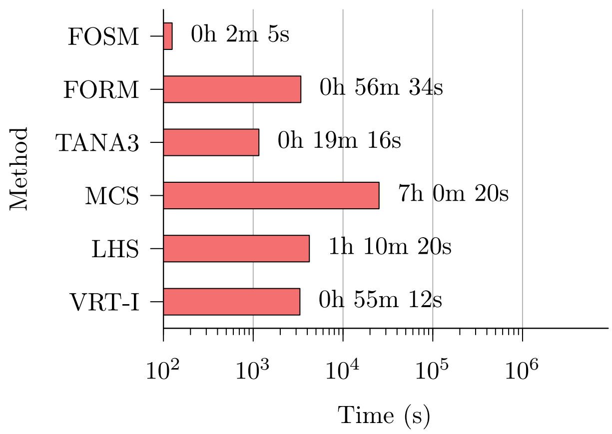

Fig. 8 Computational times for each method

approximation to the structural reliability. On the other hand, the number of iterations is very low, so this method can be considered as an alternative to evaluate the probability of failure at early stages of the design, when precise values are not required.

The results from the rest of the methods will be compared against the Monte Carlo method, as the reliability result in this case has been obtained from a population with 10,000 samples. This sampling size has been selected based on previous calculations which concluded that this size gives acceptable results in most of the analysis cases, with a processing time not too long.

The FORM method gives an accurate approximation to the probability of failure at a reasonable cost. This method is much more precise than the FOSM and also has the advantage of being invariant with respect to the formulation of the limit state function. However, although this is not the case, the computational time increases very quickly with the number of random variables and limit state functions. This is because the values of derivatives are usually calculated with procedures which increase the computational cost, like finite differences. In those cases, sampling methods become an alternative, because, except for the importance sampling, they are independent of the size of the problem.

The adaptive approximation TANA3 gives a more conservative value of the probability of failure, which can be considered as a better approximation than the one provided by the FOSM method, but not as good as the one by the FORM. In that regard, TANA3 can be view as a compromise between those methods.

The latin hypercube sampling has performed properly. This method gives a result quite similar to the Monte Carlo sampling, although requiring far less iterations and computational time. The imporance sampling is also a very good alternative when considering sampling methods, as their results are quite accurate with a cost significantly less than direct sampling methods MCS and LHS.

4 Conclusions

In this work, a revision of some existing procedures of reliability analysis and their capabilities has been carried out. A practical example has been used to show the performance of the methods used in the study. Finally, some conclusions can be drawn.

The FOSM method gives an approximation to calculate the probability of failure at early stages of the design, when precise values are not required, but it is not an alternative to evaluate the security of the structure.

Considering the performance of each method, it is reasonable the recommendation of using limit state approximation methods in those cases with a reduced number of random variables or limit state functions, when the values of derivatives must be calculated with procedures which increase the computational cost. In this situation, sampling methods can not beat the performance of limit state approximations.

However, in those cases having a high number of random variables and/or limit state functions, the number of iterations required to evaluate the first derivatives, causes that sampling methods become an alternative, with computational costs quite similar to the limit state approximation methods. Moreover, those methods are not penalized by the problem size and can be easily parallelized, increasing their performance.

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