A STUDY ON THE AXIAL STRESSES OF P-FGM, S-FGM AND E-FGM PLATES UNDER PRESSURE LOADING USING THE ENERGY CONCEPT

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Abstract

In this study the energy concept along with the classical plate theory (CPT), first and third order shear deformation theories (FSDT and TSDT) are used to predict the large deflection and through the thickness stresses of a FGM plate. For defining the volume fraction of the FGM constituent materials three different functions are considered; simple power-law (P-FGM), exponential (E-FGM) and sigmoid (S_FGM) functions. Power-law and exponential functions are commonly used to control the variations of properties of FGMs. However, with both functions, a stress concentration appears due to abrupt change of the volume fraction of the constituents. Therefore, a sigmoid FGM is used to define a new distribution for the volume fraction. The aim of this paper is to investigate and discuss the differences between these distribution functions for the constituents' volume fraction. For any of these functions, the results for different "n" and different aspect ratios are obtained and the relationships between them and the produced stress curves will be discussed. So, in this way it will be possible to predict the appropriate distribution functions considering the range of "n" and geometry (aspect ratio) of the plate.

1 Introduction

The conventional laminate composites which are usually comprised of two different materials have been widely used to satisfy the increasing high performance industrial demands. However, stress singularities in such composites may occur at the interface between two different materials, due to the mismatch properties of the constituent materials. Particularly, in a high-temperature environment, such as engine combustion chamber the relatively high mismatch in thermal expansion coefficients will induce high residual stresses. Consequently, these large inter-laminar stresses will lead to delamination. Furthermore, large plastic deformations at the interface may trigger the initiation and propagation of cracks. One way to overcome these problems is to use "functionally graded materials" [1, 2].

Functionally graded materials (FGMs) are a kind of composite material formed by two or more constituent phases with a continuously variable composition. FGMs possess a number of advantages that make them attractive in potential applications, including reduction of in-plane and transverse or through-the-thickness stresses, an improved residual stress distribution, enhanced thermal properties, higher fracture toughness, and reduced stress intensity factors. A wide range of results on linear behavior of functionally graded plates with different material function models are available in the literature [3-5]. However, nonlinear investigations of FGM plates under mechanical loading are limited in number.

In this study, a simply supported elastic rectangular FGM plate subjected to pressure loadings is considered. The material properties of the FGM plates are assumed to change continuously throughout the thickness of the plate, according to the volume fraction of the constituent materials based on the power-law, exponential, or sigmoid functions. The material properties of the FGM plate, except for the Poisson's ratio which is constant, are assumed to vary continuously throughout the thickness of the plate. However, considering the assumed loading the variations in Young's modulus (E) is only important .The constitutive equations for rectangular plates of FGM are obtained using the Von-Karman theory for large deflections and the solution was obtained by minimization of the total potential energy.

2 Theoretical formulation

A FGM can be defined by the variation in the volume fraction of its constituents. Most researchers use the power-law function or exponential function but here a sigmoid function is also considered to describe the volume fractions for the FGM. The configuration of elastic rectangular plates is considered as shown in Fig. 1. The material properties i.e. Young's modulus (E), are normally considered to be varied from upper to the lower surface of the plate such that the top surface (i.e. z=h/2) is ceramic-rich, whereas the bottom surface (i.e. z=-h/2) is metal-rich. Since the effect of Poisson's ratio on the deformation is much less than that of Young's modulus [6] its value is assumed to be constant.

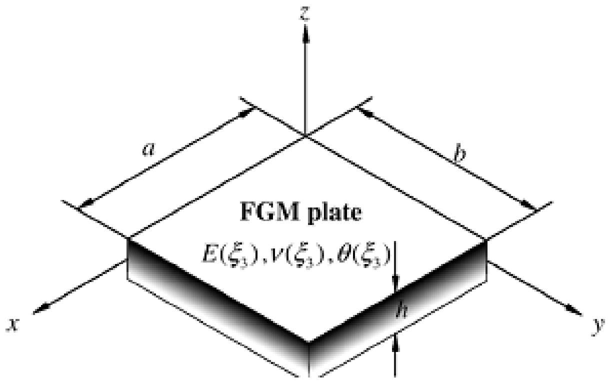

Fig. 1. Geometry of FGM plate

In the following sections responses of the FGM plate under pressure loading based on simple power-law, sigmoid and exponential functions will be investigated. At first, the general characteristics of the FGM plates based on these functions are described.

2.1 Characteristics of P-FGM plates

The volume fraction of the P-FGM is assumed to obey a power-law function:

$$\vartheta(z) = \left(\frac{z+\frac{h}{2}}{\frac{h}{2}}\right)^n \tag{1}$$

where "n" is the material parameter and "h" is the thickness of the plate. The material properties of a P-FGM can be determined by the rule of mixture using $\vartheta(z)$ as shown below:

$$E(z) = \vartheta(z)E_1 + \big(1 - \vartheta(z)\big)E_2 \tag{2}$$

Where E_1 and E_2 are the Young's moduli of the bottom and top surfaces of the FGM plate, respectively. The variation of Young's modulus in the thickness direction of the P-FGM plate is depicted in Fig. 2, which shows that the Young's modulus changes rapidly near the bottom surface for n>1,and increases quickly near the top surface for n <1.

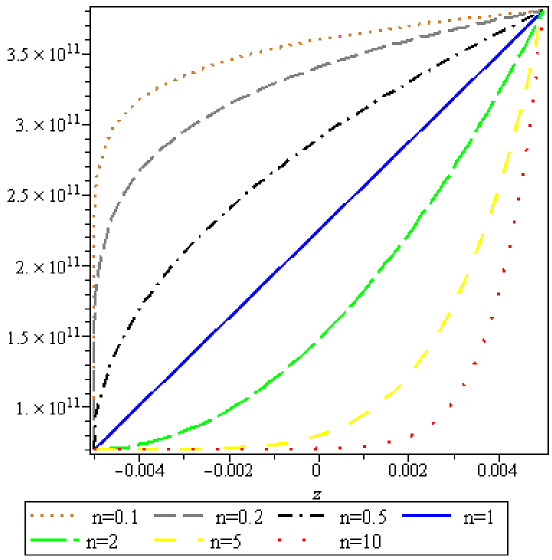

Fig. 2. Variation of Young's modulus in P-FGM plate.

2.2 Characteristics of S-FGFM plates

For a FGM of a single power-law function with an extreme high or extreme low "n" value stress concentration like laminated composite materials occurs on the interface of the ceramic/metal constituents. Note that such a material looks like a metal base with a thin layer of ceramic as coating or vice versa. To eliminate or reduce the stress concentration and ensure the smooth distribution of stresses among all the interfaces, volume fraction of the constituents is defined by two power-law functions which simulate the behavior of a sigmoid function. The two power-law functions are defined by:

$$\vartheta_1(z) = \frac{1}{2}\left(\frac{\frac{h}{2}+z}{\frac{h}{2}}\right) \tag{3}$$

$$for\ -\left(\frac{h}{2}\right) \leq z \leq 0$$

$$\vartheta_2(z) = 1 - \frac{1}{2}\left(\frac{\frac{h}{2}-z}{\frac{h}{2}}\right) \tag{4}$$

$$for\ 0 \leq z \leq -h/2$$

By using the rule of mixture, the Young's modulus of the S-FGM can be evaluated as follows:

$$E(z) = \vartheta_1(z)E_1 + \left(1-\vartheta_1(z)\right)E_2 \tag{5}$$

$$for\ -\left(\frac{h}{2}\right) \leq z \leq 0$$

$$E(z) = \vartheta_2(z)E_1 + \left(1-\vartheta_2(z)\right)E_2 \tag{6}$$

$$for\ 0 \leq z \leq \left(\frac{h}{2}\right)$$

The variation of Young's modulus in such a S-FGM plate is depicted in Fig .3.

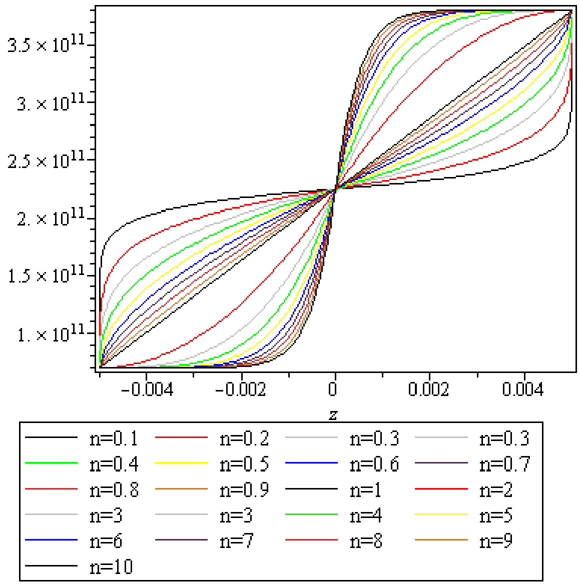

Fig. 3. Variation of the Young's modulus in S-FGM plate

2.3 Characteristics of the E-FGM plate

In some references another function rather than the simple power-law and sigmoid is proposed to describe the material properties of FGMs called exponential function shown below:

$$E(z) = Ae^{B\left(z+\frac{h}{2}\right)} \tag{7}$$

where $A = E_2$ and $B = \frac{1}{h}\ln\left(\frac{E_1}{E_2}\right)$.

The material distribution in the thickness direction of the E-FGM plate is plotted in Fig .4.

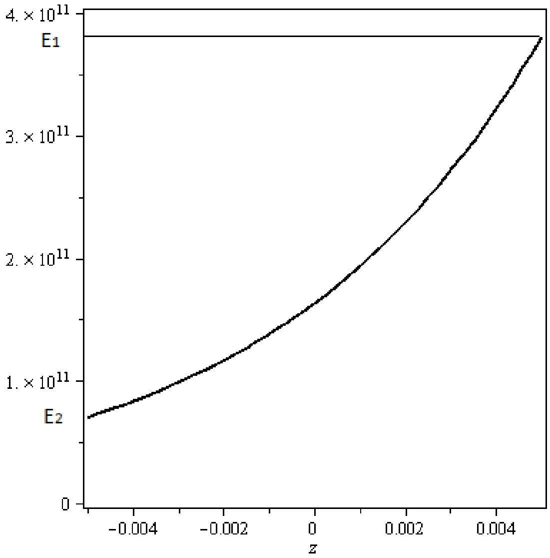

Fig. 4.Variation of Young's modulus in E-FGM plate

3 The solution procedure

A linear elastic simply supported FGM plate under pressure loading is considered. The displacement field associated with classical plate theory (CPT), first and third order shear deformation theories (FSDT and TSDT) were employed and the related constitutive equations were obtained using the Von-Karman theory for large deflections. Consequently, the total potential energy is calculated and then minimized in order to obtain the solution. The general displacement field for CPT, FSDT, and TSDT can be written as [7]

$$u(x,y,z) = u_0(x,y) + f(z)\frac{\partial w(x,y)}{\partial x} + g(z)\phi_x(x,y) \tag{8}$$

$$v(x,y,z) = v(x,y) + f(z)\frac{\partial w(x,y)}{\partial y} + g(z)\phi_y(x,y) \tag{9}$$

$$w(x,y) = w_0(x,y) \tag{10}$$

where $(u\,,v,w)$ are the displacements corresponding to the coordinate system and are functions of the spatial coordinates; $(u_0\,,v_0,w_0\,)$ are the displacements along the respective axes of x, y, and z, and ϕ_x and ϕ_y are the rotations about y and x-axes, respectively. Note that for each one of the theories considered the functions f(z) and g(z) are defined as below:

I. For the CPT: f(z)=z and g(z)=0
II. For the FSDT: f(z)=0 and g(z)=z
III. For the TSDT: f(z)=$4z^3/3h^2$ and $g(z) = z - f(z)$

According to the non-linear strain-displacement relationships [8] and the stress-strain relationships the stresses in the plate using the energy concept can be evaluated. By definition, the total potential energy is the summation of strain energy and the change in potential energy of the applied uniform pressure which can be written as below:

$$\Pi = U + V \tag{11}$$

where U and V are defined as below:

$$U = \frac{1}{2}\int_0^a\int_0^a\int_{-h/2}^{h/2}\sigma^T\epsilon\; dxdydz \tag{12}$$

$$V = \int_0^a\int_0^a q\; w(x,y)dxdy \tag{13}$$

where q is the uniformly distributed load.

By applying suitable boundary conditions and guessing the appropriate displacement and rotation fields, it is possible to evaluate the constants of the functions by minimizing the total potential energy. It is assume that the constants of displacement and rotation fields are u_0, w_0 and ϕ_0 which are respectively for displacements along x and y-axis, displacement along z-axis and the rotation about x and y-axis.

$$\frac{\partial \Pi}{\partial (u_0, w_0, \phi_0)} = 0 \qquad (14)$$

Eq. (13) provides a set of three non-linear equilibrium equations in terms of some constants which should be found.
After calculation of these constants and finding the displacement and rotation fields the through the thickness stresses are evaluated.

4 Results and discussion

In this section the responses of P-FGM and S-FGM plates for different value of "n" and different aspect ratios are compared but the E-FGM is compared by the other two functions for n=1 because of its characteristic of being independent of "n". Here, a ceramic-metal FGM plate is considered. Young's moduli for ceramic and Aluminum are 380GPa and 70GPa, respectively. As stated before, the Poisson's ratio is assumed to be constant and equal to 0.3. The analytical results are presented in terms of dimensionless deflection and stress. The dimensionless parameters used here are as follows [9]:

Aspect ratio AR=a/h;
Dimensionless axial stress $\sigma = \sigma_x a^2/(E_1 h^2)$;
Load parameter $Q = q\, a^4/(E_1 h^4)$;
Dimensionless thickness coordinate Z=z/a

4.1 Comparison the responses of P-FGM and S-FGM plates for different "n" and aspect ratios

First of all, it should be noted that in P-FGMs for small "n", the plate will be rich in ceramic (alumina), which has a large Young's modulus, and as a result its deflection will be small. However, for large "n", the plate will be rich in metal and the deflections will be larger.

For the stress field, in both small and large "n" cases one side of the plate will experience stress singularity because of existing a thin layer of metal or ceramic, respectively. However, this phenomenon will occur in S-FGM in both sides of the plate for each value of "n". Note that if "n" is very small for this material, we will have two thin layers of ceramic and metal in top and bottom surfaces of the plate and a large area composed of ceramic/metal mixture placed between the two layers. On the other hand, if "n" is too large, two thick layers of ceramic and metal in the top and bottom surfaces of the plate will form and only a very thin layer of mixture remains in the mid-region of the plate which results in stress jumps when moving across the interfaces (for more details see Fig. 5 to 10).

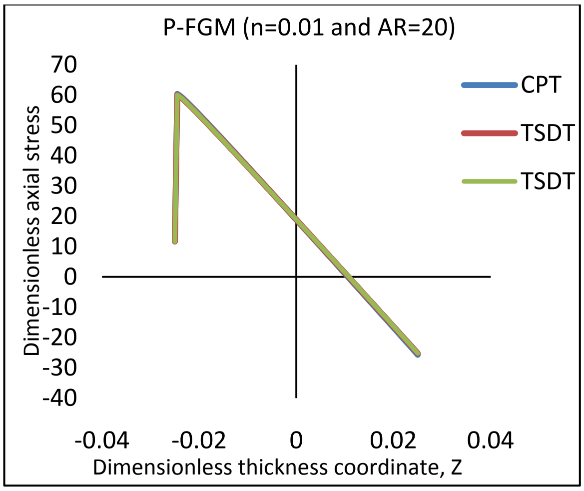

Fig. 5. σ at the center of the plate under load Q=-400 for AR=20 in P-FGM plate

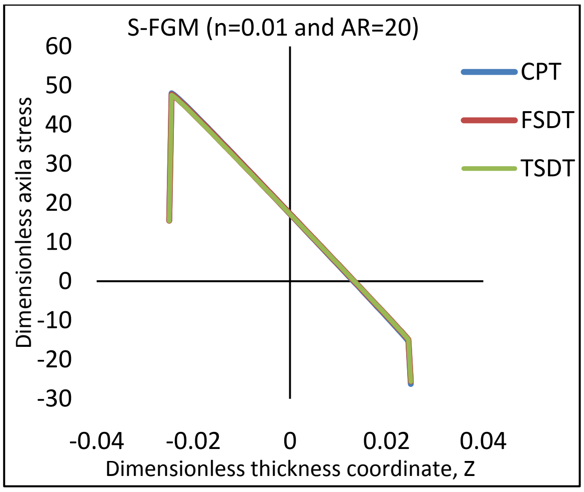

Fig. 6. σ at the center of the plate under load Q=-400 for AR=20 in S-FGM plate

As shown in Figs. 5 and 6, the sigmoid function leads to smaller axial stress values with respect to simple power-law in both tension and compression parts of the stress field. This phenomenon will occur for all values of “n” and all aspect ratios. Another important finding is that for large aspect ratios, which will occur for thin plates, all the theories (CPT, FSDT and TSDT) predict the same results. It means that using the CPT is adequate for stress analysis of thin plates and there is no need to use higher order shear deformation theories. Nevertheless, decreasing the aspect ratio will result in the deviation of predictions by CPT from other theories’ (see Figs. 7 and 8). As mentioned in [9] the compressive nature of stress at the top surface predicted by the CPT for thick plates appears to be tensile if one applies higher order theories to the problem. This is clearly shown in Figs. 7 and 8. It can be observed from these two figures that in S-FGM materials this behavior appears more quickly and obviously in small values of “n”. If the aspect ratio decreases more, i.e. AR=1.5, the FSDT and TSDT predict a pure tensile stress instead of compressive stress in thinner plates. This phenomena is more apparent for large values of “n” (see Figs. 9-12)

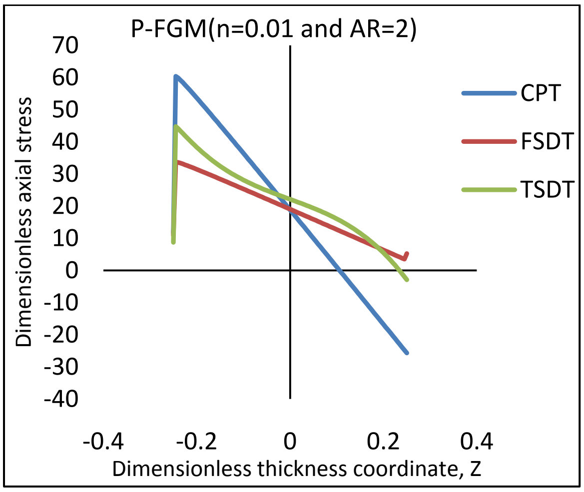

Fig. 7. σ at the center of the plate under load Q=-400 for AR=2 in P-FGM plate

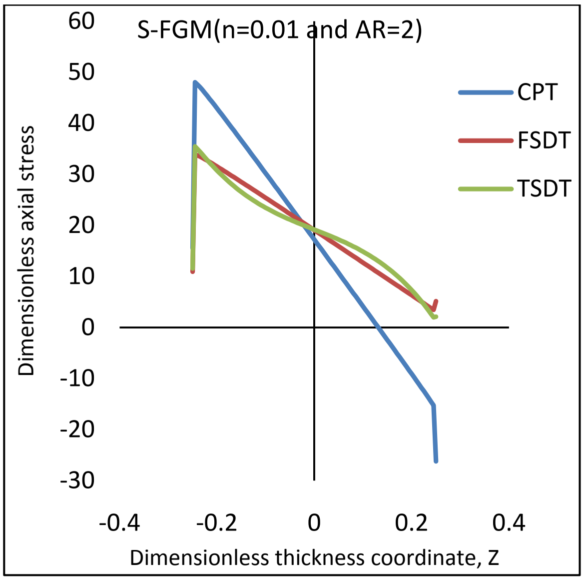

Fig. 8. σ at the center of the plate under load Q=-400 for AR=2 in S-FGM plate

For large values of “n” the P-FGM plate will be rich in metal and stress singularity will take place in top surface because of a thin layer of remaining ceramic. The results for this material are similar to the results of Fig. 9. For the same situation for S-FGM, stress singularities occur at the interfaces of the mid-layer (composed of a mixture of ceramic and metal) with the pure metal and ceramic layers (see Fig. 10). If the value of “n” increases further, the two pure metal and ceramic layers look like two breaks in thickness and get closer to each other which means that the thickness of mid-layer tends to decrease.

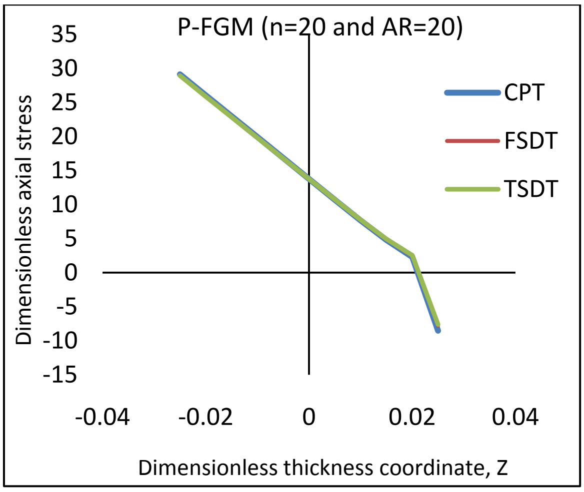

Fig. 9. σ at the center of the plate under load Q=-400 for AR=20 in P-FGM plate

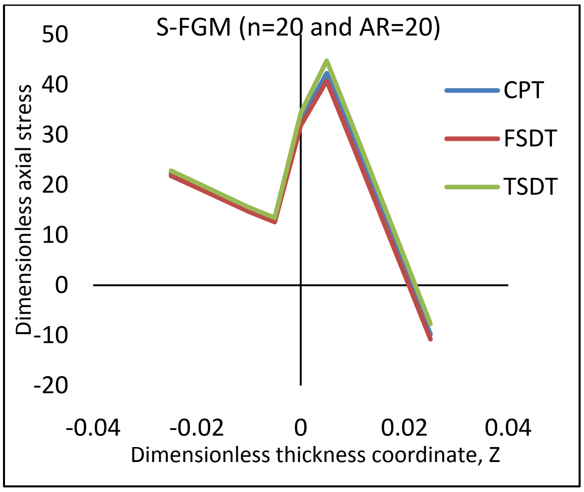

Fig. 10. σ of the plate under load Q=-400 for AR=20 in S-FGM plate

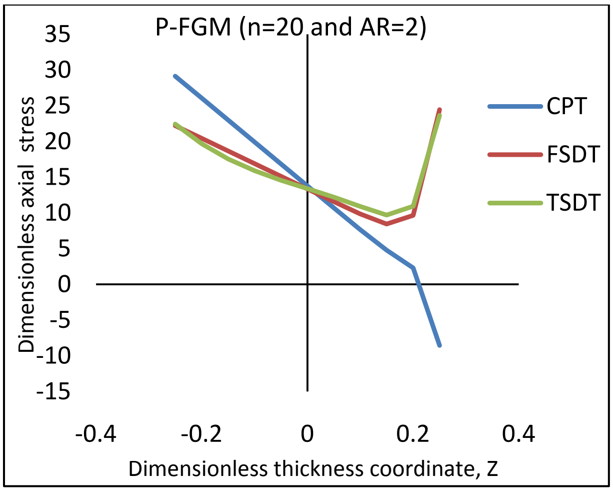

Fig. 11. σ at the center of the plate under load Q=-400 for AR=2 in P-FGM plate

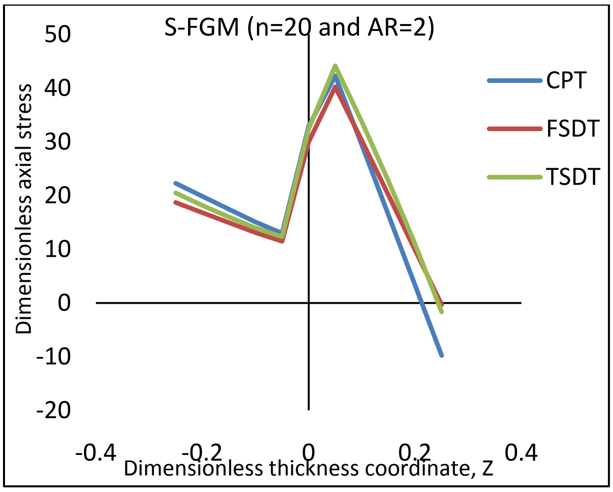

Fig. 12. σ at the center of the plate under load Q=-400 for AR=2 in S-FGM plate

Comparisons of Figs. 9 and 11 and Figs. 10 and 12 show that for thicker plates or small aspect ratios, no compressive stress is observed in top layers (the thing which is seen in thin layers) by higher order shear deformation theories. This is what would expect from deformation of a thick plate. Therefore, it would be concluded that besides the existence of shear effect there is one more reason for CPT not to be valid for thick plates.

4.2 Comparison of P-FGM, S-FGM and E-FGM responses in n=1

As shown in Figs.13 through 16, the exponential function predicts less stresses that the other two functions. For all aspect ratios and different deflection theories the E-FGM plate experiences lower stress values that P-FGM and S-FGM plates and the magnitude of stress in both P-FGM and S-FGM are the same.

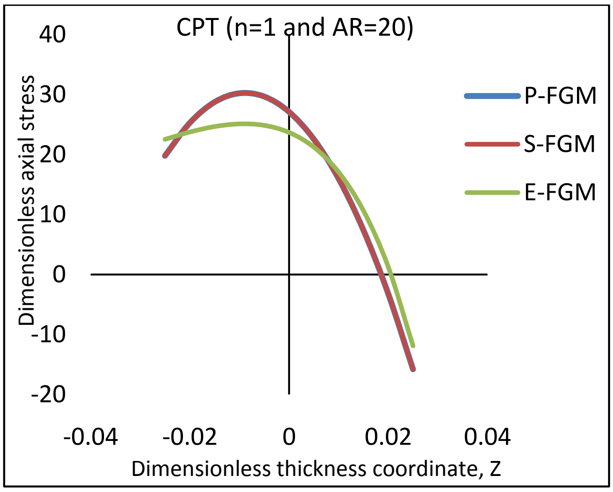

Fig. 13. Comparison of axial stresses for P-FGM, S-FGM and E-FGM based on CPT

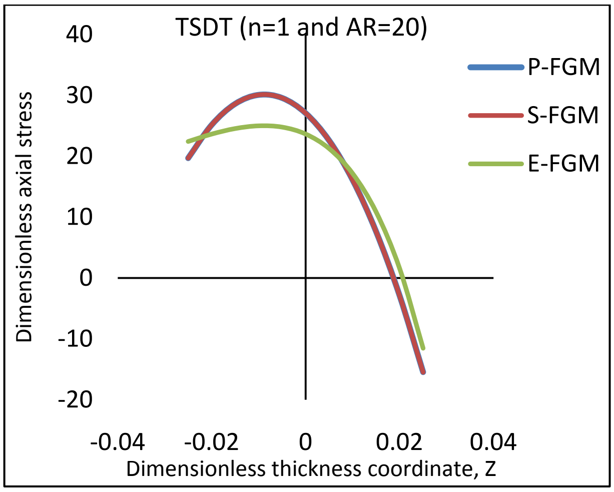

Fig. 14. Comparison of axial stresses for P-FGM, S-FGM and E-FGM based on TSDT

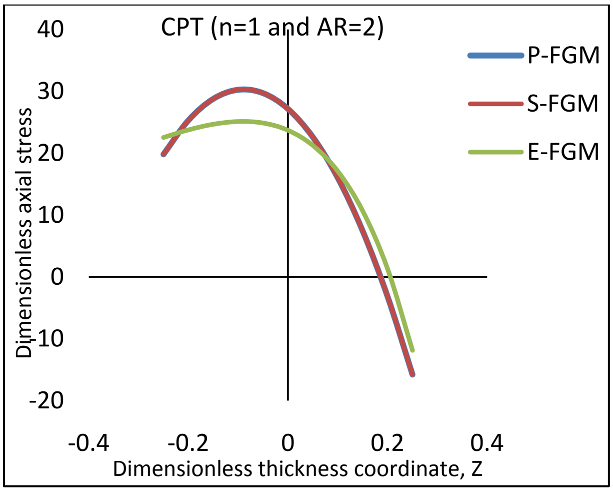

Fig. 15. Comparison of axial stresses for P-FGM, S-FGM and E-FGM based on CPT

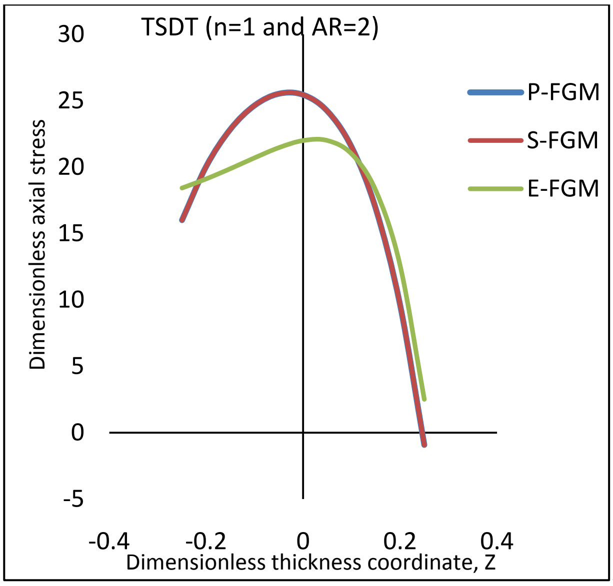

Fig. 16. Comparison of axial stresses for P-FGM, S-FGM and E-FGM based on TSDT

5 Conclusions

As discussed in the previous sections, for all values of “n” the CPT becomes invalid with increasing the plate thickness or decreasing the aspect ratio. Also, the predicted stresses in S-FGM are lower than those of P-FGM plates. While, for a fixed aspect ratio, by moving from small values of “n” towards its larger values the total behavior of the plate under pressure based on different distribution functions does not change significantly, rather than the shape of the curves and the points of singularities. For small “n”, depending on distribution function there are some breaks in one or both sides of the curves. For large “n”, the breaks in P-FGM plate move to another side and in S-FGM plate they will move closer to the mid-region of the plate. The behavior of P-FGM and S-FGM plates in converting the compression stress into tension for thick plates are similar with only slight differences. For n=1 it was shown that the stresses predicted by exponential distribution function is lower than the two other functions.

6 References

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