PROBABILISTIC DESIGN OF ADVANCED COMPOSITE MATERIALS FOR AEROSPACE STRUCTURES

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Keywords: composite variability, FORM, probabilistic design, reliability, SORM

Abstract

In the present work the authors outline a procedure to evaluate uncertainty propagation in composite mechanical characteristics and to compute probability of failure of a composite laminate under generic static loads in accordance to the available literature. A review of probabilistic analysis basic concepts and methods is firstly introduced. A software is developed by authors, based on FORM and SORM, in order to perform analysis on a test case. Preliminary results obtained for a test case, are presented in the paper and compared with Monte Carlo simulation.

Nomenclature

\( f_\xi(\xi) \): probability density function (pdf) of \( \xi \)
\( F_\xi(\xi) \): cumulative distribution function (CDF) of \( \xi \)
\( \mu_\xi \): mean value of \( \xi \)
\( \sigma_\xi \): standard deviation of \( \xi \)
\( \phi \): standard normal probability density function
\( \Phi \): standard normal probability density function
\( \bar{X} \): set of independent normal random variables
\( \bar{U} \): independent standard normal random var.
\( \bar{Y} \): independent random variables
\( G \): limit state function
\( \beta \): reliability index
\( E_f \): fibre longitudinal elastic modulus
\( E_m \): matrix longitudinal elastic modulus
\( G_f \): fibre shear elastic modulus
\( G_m \): matrix shear elastic modulus
\( \%V_f \): fibre volume percentage
\( v_f \): fibre Poisson ratio
\( v_m \): matrix Poisson ratio

1: ply-angle
\( s_{tot} \): laminate thickness
\( Q_{11}, Q_{12}, Q_{22}, Q_{66} \): ply reduced stiffnesses
A,B,D: laminate extensional, coupling and bending stiffness matrices
\( N_x, N_y, N_{xy} \): loads per unit length
\( S_{i,c/t} \): compression/tensile strength of lamina in i direction
\( \nabla (\bullet) \): gradient of the function (\( \bullet \))
\( \nabla^2 (\bullet) \): Hessian of the function (\( \bullet \))
\( TR(M) \): trace of the matrix M

1 Introduction

The use of safety factor in structural design leads to an increase of weight with a not-quantified increase of structural reliability. Furthermore the amount of scatter observed in composite material testing tends to be high relative to metals; variability in composite material property data results from a number of sources, including variability in laying up the material, variability of raw materials, high sensitivity to testing environment and material testing methods; deterministic mechanical description of composite material may be too penalizing, leading to an increase in structures weight. For aerospace structures those weight increases could become unacceptable. A way to overcome those two limits of the deterministic approach in design, is the development of a probabilistic design methodology for aerospace advanced composite materials; such methodology should take in account uncertainties relating to quantities involved in design (geometry, loads, material capabilities, operating environment) in order to quantify the structural probability of failure \( (p_f) \) and the \( p_f \) sensitivities to design variables.
In order to cope with such complex problem, a probabilistic design procedure is being developed by authors. In section 2 a review of First and Second Order Reliability Methods (FORM and SORM) is provided in order to introduce basic concepts in structural reliability and probabilistic analysis. As a first step in probabilistic design methodology development, in sections 3 and 4 FORM and SORM approaches are investigated to evaluate the uncertainty propagation in material mechanical characteristics for a generic laminate and the probability of failure for the same laminate under static loads. In both cases sensitivity analysis is also performed to understand the relative importance of the input variable to the considered response.

2 The Basic Structural Reliability Problem

2.1 Problem Definition

The basic structural reliability problem involves a single load effect \( S \) and a single resistance effect \( R \), both expressed by pdfs \( f_S \) and \( f_R \). \( S \) and \( R \) are defined in such a way that the considered structural element will fail if \( S > R \). Under those assumptions the basic reliability problem can be written as:

\[
p_f = P\left( R \leq S \right),
\]

or more generally:

\[
p_f = P\left( G(R,S) \leq 0 \right),
\]

where \( G \) is the limit state function of the structural element; \( G \) divides the space of the random variables in two zones: the failure domain (where \( G < 0 \)) and the safe domain (where \( G > 0 \)). The probability of failure can be computed solving the integral (3) or the convolution integral (4) for independent random variables:

\[
p_f = \int \int_{\Omega(G<0)} f_{RS}(r,s) dr ds,
\]

\[
p_f = \int_{-\infty}^{\infty} F_R(x) f_S(x) dx.
\]

Fig. 1. R, S Joint Probability, Limit State Function and Failure Domain.

Integral (3) can be finally generalized for any number of random variables describing structure behaviour:

\[
p_f = \int \cdots \int f_X^{N}(r) dx.
\]

2.2 Computational Aspects – First Order Reliability Methods

Due to the high number of variables involved in (5) for practical cases, an analytic solution can be rarely achieved; the usually adopted numerical methods to perform integration in structural reliability problems are based on repeated simulation of the structural behavior for random values of the input variables (Monte Carlo methods) or on transformation of the joint probability density function in a multi-normal one for which some peculiar properties hold. Despite its simplicity and its robustness, direct Monte Carlo approach requires a great number of trials and a consequent high computational effort, in fact, as suggested by Shooman [1], the number of needed trials in a crude Monte Carlo simulation can be evaluated by:

\[
\varepsilon = k \left[ \frac{(1-p)}{Np} \right]^{1/2}
\]

where \( \varepsilon \) is the wanted precision, \( p \) is the expected value of the probability of failure, \( N \) in the number of trials and \( k \) the value of the standard normal random variable which the level of confidence corresponds to. The square root of \( N \) in (6) implies a slow convergence of
Monte Carlo methods (Fig. 2). Some variance reduction techniques (e.g. importance sampling) have been developed in order to obtain a faster convergence by the introduction of a priori information about the problem solution.

In this work direct Monte Carlo simulations are used as benchmarks for the numerical results obtained with other probabilistic approaches. As stated above, another way to compute (5) is transformation of the joint pdf in a multi-normal one. Let’s consider the basic structural reliability problem with independent normal $R$ and $S$ and a limit state function $G=R-S$. By means of the properties of normal moment generation functions, $G$ is also a gaussian random variable with mean value and variance:

$$
\mu_G = \mu_R - \mu_S
$$
$$
\sigma_G^2 = \sigma_R^2 + \sigma_S^2.
$$

Equation (2), introducing the standard random variable $G^* = (G - \mu_G) / \sigma_G$, becomes:

$$
p_f = P(G \leq 0) = \Phi \left( \frac{0 - \mu_G}{\sigma_G} \right) = \Phi(-\beta)
$$

where $\Phi$ is the standard normal cumulative distribution function and $\beta$ is the reliability index [2,3]; $\beta$ can be considered as the distance, in standard deviation unit, between the mean value of the limit state function and the failure domain. Equation (8) holds for any number of independent normal random variables $X_i$ and for any linear limit state function $G = a_0 + a_1X_1 + \ldots + a_nX_n$ because, under these assumptions and by the means of the axis-symmetry of the multi-normal standard joint pdf in standard normal space, it’s always possible to compute integral (5) as:

$$
p_f = \int_{-\infty}^{\beta} \phi_1(u_1)du_1\int_{-\infty}^{\beta} \phi_2(u_2)du_2\ldots \int_{-\infty}^{\beta} \phi_n(u_n)du_n = \Phi(-\beta),
$$

where $u_i = (X_i - \mu_x_i) / \sigma_x_i$.

In order to extend the property expressed in (9) to non-normal random variables and non-linear limit state functions, FORM methods were developed [4]. Non-linear limit state functions can be linearized by first order Taylor’s expansion around a point $U_0$ of $G$:

$$
G(\bar{U}) = G(U_0) + \sum_{i}^{n} \frac{\partial G}{\partial u_i}(U_0)(u_i - u_{0,i}),
$$

mean value and variance of linearized $G$ are:

$$
\int_{-\infty}^{\infty} [G(\bar{U}) + \sum_{i}^{n} \frac{\partial G}{\partial u_i}(U_0)(u_i - u_{0,i})] f(\bar{U}) d\bar{U} = \int_{-\infty}^{\infty} [G(\bar{U}) - G(\bar{\mu}_U)]^2 f(\bar{U}) d\bar{U} =
$$

$$
= \sum_{i}^{n} \frac{\partial G}{\partial u_i}(U_0)^2 \left[ \frac{\partial u_i}{\partial x_i} \right]^2 VAR[u_i].
$$

Equation (12) shows that evaluation of the probability of failure by (9) depends on the choice of the expansion point $U_0$; the most convenient point is the nearest one to the origin of axes of the space of normal standard variables because, among the points of $G$, it corresponds to the highest value of the joint pdf and hence it’s the point in which the system will most probably fail [4]; the point is called the design point or the most probable point (MPP) $U^*$. If linearized $G$ is used, then structural $p_f$ can be computed using (9); the greatest are the curvatures of the limit state function in the space of normal variables the greatest is the error on the evaluation of the $p_f$.
obtained by linearizing \( G \). The CDF of a random variable \( Y \) can be approximated with normal standard pdf through the normal tail approximation \[6\]. The non-normal independent variables are mapped into the standard normal space by:
\[
F_Y (\hat{y}) = \Phi (\hat{\alpha}) \Rightarrow \hat{\alpha} = \Phi^{-1} (F_Y (\hat{y})), \tag{13}
\]
where \( \hat{y} \) is a possible value of \( Y \). It’s important to notice that (13) doesn’t transform the non-normal random variables in a normal ones, but simply map variables in the standard space keeping the cumulative probability content constant. A FORM direct algorithm for independent random variables can be summarized, hence, through the following steps \[6\]:
- mapping of all the random variables in standard normal random ones;
- transformation of the limit state function according to the transformed variables;
- search for the MPP and its distance \( \beta \) from the origin of the standard normal variables space. This represents a minimization problem in the form:
\[
\beta = \min \|u\| \tag{14}
\]
subject to \( G(\hat{\alpha}) = 0 \);
- evaluation of the pdf by the means of (9)

Another important goal to be achieved in probabilistic analysis is the computation of the sensitivity factors; they can be defined as the variation of the reliability index with respect to standard normal variables \( \frac{\partial \beta}{\partial u_i} \) in the neighborhood of the MPP \[5\]. The variation of the distance from origin of a generic point \( P \) can be expressed by:
\[
\frac{\partial \hat{\alpha}}{\partial u_i} = \frac{\partial}{\partial u_i} \sqrt{u_i^2 + u_2^2 + \ldots + u_n^2} = \frac{u_i}{\sqrt{u_i^2 + u_2^2 + \ldots + u_n^2}}, \tag{15}
\]
in the MPP, where \( \hat{\alpha} \) and \( \nabla G(\hat{\alpha}) \) are parallel, it becomes:
\[
\frac{\partial \beta}{\partial u_i} = \frac{u_i}{\sqrt{u_i^2 + u_2^2 + \ldots + u_n^2}}. \tag{16}
\]

Introducing (16) in equation (12) it can be shown that sensitivities \( \alpha_i \) can be also viewed as the contributions of each random variable in the overall variance of the limit state function. Dickhoven and Madsen \[3\] have also defined the omission sensitivity factor \( \xi \), as the relative error occurring when some random variables are replaced by fixed values:
\[
\xi (U_i = u_i, \ldots, U_q = u_q) = \frac{1 - \sum \alpha_i u_i}{\sqrt{1 - \sum \alpha_i^2}}, \tag{17}
\]
or, when a variable is replaced with its mean value:
\[
\xi (U_i = 0, \ldots, U_q = 0) = \frac{1}{\sqrt{1 - \sum \alpha_i^2}}. \tag{18}
\]

Since in FORM algorithm the MPP search is usually performed by gradient methods, the sensitivity analysis requires no additional computational effort.

### 2.3 Second Order Reliability Methods

When the curvatures of the limit state function around the MPP are not negligible, the linear approximation can leads to significant errors in the evaluations of the pdf. An improvement of the solution can be obtained by a second order approximation. Assuming that \( G(\hat{\alpha}) \) is twice differentiable, it can be written as:
\[
G(\hat{\alpha}) \approx G(\hat{\alpha}^*) + \nabla G(\hat{\alpha}^*)^T (\hat{\alpha} - \hat{\alpha}^*) + \frac{1}{2} (\hat{\alpha} - \hat{\alpha}^*)^T \nabla^2 G(\hat{\alpha}^*) (\hat{\alpha} - \hat{\alpha}^*), \tag{19}
\]

in the MPP, where \( \hat{\alpha}^* \) and \( \nabla G(\hat{\alpha}^*) \) are parallel, it becomes:
\[
\frac{\partial \beta}{\partial u_i} = -\frac{u_i}{\sqrt{\nabla^2 G(\hat{\alpha}^*)}^T \nabla^2 G(\hat{\alpha}^*)}, \tag{20}
\]

or, when a variable is replaced with its mean value:
\[
\xi (U_i = 0, \ldots, U_q = 0) = \frac{1}{\sqrt{\nabla^2 G(\hat{\alpha}^*)}}. \tag{21}
\]

Since in FORM algorithm the MPP search is usually performed by gradient methods, the sensitivity analysis requires no additional computational effort.
\[
\frac{G(\bar{U})}{\sqrt{V G(\bar{U})}} = -\alpha^T (\bar{U} - \bar{U}^*) + \frac{1}{2} (\bar{U} - \bar{U}^*)^T \bar{H} (\bar{U} - \bar{U}^*) = 0
\]

It's more convenient to write equation (20) in a new space \( \tilde{V} \), where \( \tilde{U}^* \) is parallel to the \( v_n \) axis; this can be achieved by an orthogonal matrix \( T \) whose last row is \( \sqrt{V G(\bar{U}^*)} \), in the new space the linear approximation is simply \( v_n \beta \). Equation (20) becomes:

\[
\frac{G_v (\tilde{V})}{\sqrt{V G(\bar{U}^*)}} = -v_n + \beta + \frac{1}{2} (\tilde{V} - \tilde{V}^*)^T \bar{H} T \bar{H}^T (\tilde{V} - \tilde{V}^*) = 0.
\]

Equation (21) can be finally approximated in a quadratic form of the first \( n-1 \) variables \( v \):

\[
v_n = \beta + \sum_{i=1}^{n-1} k_i v_i,
\]

where \( k_i \) are the eigenvalues for the first \( n-1 \) rows and columns of the matrix \( T \bar{H} T^T \) and represent the first \( n-1 \) principal curvatures of the limit state function at the MPP. The probability of failure is then:

\[
p_f = P(G \leq 0) \approx P\left(v_n \geq \beta + \sum_{i=1}^{n-1} k_i v_i\right).
\]

Based on definition (23), many authors have developed analytical solutions, introducing different assumptions; the first one was Breitung [7], his solution's based on asymptotic analysis as \( \beta \rightarrow \infty \):

\[
p_f \approx \Phi(-\beta) \prod_{i=1}^{n-1} \left(1 + k_i \beta \right)^{-1/2}.
\]

Hohenbicher & Rackwitz [8] proposed the following improvement of the estimate of \( p_f \) by importance sampling analysis:

\[
p_f = \Phi(-\beta) \prod_{i=1}^{n-1} \left(1 + \frac{\phi(\beta)}{\Phi(-\beta)} k_i \right)^{-1/2},
\]

the estimate (25) reduces to (24) for \( \beta \rightarrow \infty \) and it's more accurate for lower values of \( \beta \).

Tvedt [9] developed a three terms approximation by a second order power series expansion of integral (5) written for the quadratic safe set (22); the approximation is:

\[
T_1 = \Phi(-\beta) \prod_{i=1}^{n-1} \left(1 + \beta k_i \right)^{-1/2};
\]

\[
T_2 = \left[\beta \Phi(-\beta) - \Phi(\beta)\right] \prod_{i=1}^{n-1} \left[1 + (\beta + 1) k_i \right]^{-1/2};
\]

\[
T_3 = (\beta + 1) \left[\beta \Phi(-\beta) - \Phi(\beta)\right] \prod_{i=1}^{n-1} \left[1 + \left(\sqrt{-1 + \beta} k_i \right) \right]^{-1/2};
\]

\[
p_f = T_1 + T_2 + T_3.
\]

The above SORM corrections are defined only for \( \beta k_i \geq -1 \) and don't give good results for negative curvatures [10].

Adhikari [11] proposed an asymptotic solution (27) for \( n \rightarrow \infty \):

\[
p_f = \Phi\left(-\frac{\beta + \text{TR}(A)}{\sqrt{1 + 2 \text{TR}(A^2)}}\right),
\]

where \( A \) is the matrix containing the first \( n-1 \) rows and columns of \( \frac{1}{2} T \bar{H} T^T \).

Tvedt [12] provided also an exact solution the quadratic form (19):

\[
p_f = \phi(\beta) \text{Re} \left[\frac{i (2/\pi)^{1/2}}{\pi} \int_{0}^{\infty} \exp\left(\frac{(t + \beta)^2}{2}\right) t^{-\frac{1}{2}} \prod_{i=1}^{n-1} \left(1 - t k_i \right)^{-1/2} dt\right].
\]

The main computational effort for \( p_f \) SORM estimates is the evaluation of the additional \( n(n+1)/2 \) elements of the Hessian matrix in the MPP. If the limit state function involves time consuming analysis or the problem has an high number of random variables, a direct Hessian computation may become impractical.
and different approximation methods are required [13].

3 CDF Evaluation and Sensitivity Analysis of Composite Mechanical Characteristics

Statistical description of composite laminate mechanical characteristics can’t be directly evaluated through experimental testing for every possible lay-up configuration. One of the task of a composite probabilistic analysis should be to understand how the uncertainty in constituent materials, geometry and ply configuration affects laminate mechanical behaviour. That means probabilistic analysis must provide CDF of the random variables representing the mechanical characteristics and their sensitivities to input variables. FORM and SORM concepts can be used to evaluate the CDF of a function $W$ of random variables, defining a limit state function $G' = W - W'$ where $W'$ is a possible value of $W$. For such a limit state function, in fact, the probability of failure has the meaning of CDF value in $W'$. Direct FORM application is, however, not convenient. In fact, since the function $W$ isn’t statistically described, it’s not possible to make an efficient choice of the range of $W'$ values; it’s more effective to choose directly the values of the CDF in $W'$. Direct FORM application is, however, not convenient. In fact, since the function $W$ isn’t statistically described, it’s not possible to make an efficient choice of the range of $W'$ values; it’s more effective to choose directly the values of the CDF in $W'$. Direct FORM application is, however, not convenient. In fact, since the function $W$ isn’t statistically described, it’s not possible to make an efficient choice of the range of $W'$ values; it’s more effective to choose directly the values of the CDF in $W'$.

For what concerns the FORM inverse application, the software relies on an algorithm proposed by Xiaoping Du and Wei Chen [15]; the algorithm defines a set of hyper-spheres of radius $\beta_i$ and then iteratively tries to shift the function $Q$ on the spheres until $\nabla Q$ (that’s the same as $\nabla G'$) becomes parallel to the position vector of the intersection between the sphere and the $Q$ function. The SORM corrections are performed by the means of (24), (25), (26), (27) and (28) equations.

The single ply mechanical behavior is described by the CDFs of reduced stiffness $Q_{11}, Q_{12}, Q_{22}$ and $Q_{66}$; CDFs obtained by the developed software are reported below and compared with the ones evaluated by Monte Carlo simulation ($10^6$ cycles). A $[0/-45/45/90]_s$ laminate is then analyzed in order to compute the CDFs of the A, B, D matrix elements; those of A are reported below. In Fig. 5a,b B and D elements CDFs are summarized. Generally FORM CDFs show a good agreement with the ones obtained by Monte Carlo Simulation; for some of the function (e.g. for $A_{22}$, Fig 4c) FORM approximation can lead to significant errors and a SORM correction may be needed. In Fig. 6 SORM corrections are reported for the three cases in which FORM CDF and Monte Carlo Simulation showed worst agreement, at the moment the software isn’t able to automatically evaluate the necessity to apply a SORM corrections without computing the Hessian matrix, but this feature has to be

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Distribution</th>
<th>Mean Value</th>
<th>Standard Deviation</th>
<th>M.U.</th>
</tr>
</thead>
<tbody>
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<td>Weibull</td>
<td>1.93e+05</td>
<td>1.57e+04</td>
<td>MPa</td>
</tr>
<tr>
<td>$E_m$</td>
<td>Normal</td>
<td>1.00e+05</td>
<td>1.00e+04</td>
<td>MPa</td>
</tr>
<tr>
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<td>4.93e+03</td>
<td>4.95e+02</td>
<td>MPa</td>
</tr>
<tr>
<td>$G_m$</td>
<td>Lognormal</td>
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<td>4.95e+02</td>
<td>MPa</td>
</tr>
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<td>$%V_f$</td>
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<td>-</td>
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<td>$v_f$</td>
<td>Normal</td>
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<td>0.03</td>
<td>-</td>
</tr>
<tr>
<td>$\nu_m$</td>
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<td>0.3</td>
<td>0.03</td>
<td>-</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Normal</td>
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<td>°</td>
<td></td>
</tr>
<tr>
<td>$S_{tot}$</td>
<td>Normal</td>
<td>8</td>
<td>0.2828 mm</td>
<td></td>
</tr>
</tbody>
</table>

Table 1
implemented in order to keep computational cost of SORM effective with respect to Monte Carlo Simulation.

A very important information that can be achieved by probabilistic analysis is sensitivity of the random variable function to its input variables. Through evaluation of sensitivities as defined in (16), it’s possible to compute omission factors by equation (18); an index defined as the difference between the omission sensitivity factor obtained by replacing the same physical quantity in all layers and the unity (i.e the factor obtained by not replacing any random variable) is used to understand which random variable could be treated as a constant in probabilistic analysis. As shown in Fig. 7a and 7b, for all the A,B and D matrix elements, fibre
and matrix shear moduli and Poisson coefficients can be replaced by their mean value without any significant effect on the value of the CDF.

\[
\begin{align*}
\left( \frac{1}{S_{11}} - \frac{1}{S_{11}} \right) \sigma_1 + \left( \frac{1}{S_{22}} - \frac{1}{S_{22}} \right) \sigma_2 + \\
+ \frac{1}{S_{11} S_{12}} \sigma_1 \tau_{12} + \frac{1}{S_{22} S_{12}} \sigma_2 \tau_{12} + \\
- \frac{1}{\sqrt{S_{11} S_{12} S_{22} S_{12}}} \sigma_1 \sigma_2 &= TW = 1 \Rightarrow \text{failure},
\end{align*}
\]

and consequently the limit state function describing the single ply behavior can be promptly obtained:

\[ G_p = 1 - TW = 0. \]  

4 Evaluation of Probability of Failure for a Composite Structure: a Test Case

As a very simple example to test \( p_f \) computation procedure, let consider the first ply failure of the laminate defined in section 3 under static loads \( N_x, N_y \) and \( N_{xy} \). In order to verify if a ply failure occurs, the Tsai-Wu criterion [16] is considered:

\[
\begin{align*}
& \left( \frac{1}{S_{11}} - \frac{1}{S_{11}} \right) \sigma_1 + \left( \frac{1}{S_{22}} - \frac{1}{S_{22}} \right) \sigma_2 + \\
&+ \frac{1}{S_{11} S_{12}} \sigma_1 \tau_{12} + \frac{1}{S_{22} S_{12}} \sigma_2 \tau_{12} + \\
&- \frac{1}{\sqrt{S_{11} S_{12} S_{22} S_{12}}} \sigma_1 \sigma_2 = TW = 1 \Rightarrow \text{failure},
\end{align*}
\]

In addition to the variables showed in Table 1, \( G_p \) is also function of static loads and strengths that appear in (29); \( pdfs \) and first two moments for these variables are summarized in Table 2. Moreover, in order to reduce the problem dimension, \( G_f, G_m, \nu_f \) and \( \nu_m \) are now considered deterministic variables and their values are fixed to their respective means. Since the first ply failure is considered, the laminate can be described as a series system in which the failure of a single element implies the failure of
the system itself. The overall probability of failure can hence be expressed as:

$$P(F_S) = P(F_1) \cup P(F_2 \cap \bar{F_1}) \cup P(F_1 \cap \bar{F}_1 \cap \bar{F}_2) \cup \ldots$$

(31)

where $F_x$ denotes the event “failure of the element $x$” and $\bar{F}_x$ its negation; $P(F_i \cap \bar{F}_2)$ can

be written as $P(F_2) - P(F_2 \cap F_i)$ as well as all the other intersection terms of (31), that becomes:

$$P(F_S) = \sum_{i=1}^{n} P(F_i) - \sum_{i=2}^{n} \sum_{j=1}^{i-1} P(F_i \cap F_j) + \sum_{i=3}^{n} \sum_{j=2}^{i-1} \sum_{k=1}^{j-1} P(F_i \cap F_j \cap F_k) - \ldots$$

(32)

Computing the system probability of failure means being able to compute the probability content of each intersection that appears in (32) i.e. evaluating integral (5) over overlapping areas of two or more failure domains. Since this can rarely be achieved, usually $P(F_S)$ is
expressed in terms of upper and lower bounds. The bounds used for this example are the ones suggested by Ditlevsen [17] for which only the two terms intersections have to be evaluated:

$$P(F_s) \geq P(F_i) + \sum_{i=2}^{n} \max \left\{ P(F_i) - \sum_{j=1}^{i-1} P(F_i \cap F_j); 0 \right\}$$

(33)

$$P(F_s) \leq \sum_{i=2}^{n} P(F_i) - \sum_{i=2,j<i}^{n} \max \left\{ P(F_i \cap F_j); 0 \right\}.$$  

(34)

The terms \(P(F_i \cap F_j)\) can be finally bounded considering the intersection between the respective linearized failure domains already defined in FORM; if \(\beta_i\) and \(\beta_j\) are the reliability index associated with \(P(F_i)\) and \(P(F_j)\), it can be written [17]:

if \(\rho_{ij} \leq 0\)

$$0 \leq P(F_i \cap F_j) \leq \min \left\{ \Phi(-\beta_i)\Phi(-\beta_j) \right\}$$

(35)

if \(\rho_{ij} \geq 0\);

$$\max \left\{ \Phi(-\beta_i)\Phi(-\beta_j) \right\} \leq P(F_i \cap F_j) \leq \Phi(-\beta_i)\Phi(-\beta_j) + \Phi(-\beta_j)\Phi(-\beta_i)$$

(36)

where:

$$\beta_{ij} = \frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1 - \rho_{ij}^2}}, \quad \beta_{ji} = \frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1 - \rho_{ij}^2}}$$

(37)

and \(\rho_{ij}\) is correlation coefficient between the two linearized function that, only for the linear case, is equal to the scalar product of the director cosines at MPP [5]. Bounds (33), (34), (35) and (36) must be combined in order to obtain an interval as wide as possible. It’s also important to notice that obtained bounds depend on the way the failures are ordered; as reported in [5] it seems convenient, for obtaining narrower bounds, to order the failure modes according to decreasing values of probability \(P(F_i)\).

The “real” system probability of failure is considered to be 2.04E-04; this value is predicted by Monte Carlo simulation with 10^6 trials. FORM probabilities of failure for each ply and probabilities of simultaneous failures occurrence evaluated by the means of (33)-(36) are reported in Table 3 and Table 4 respectively.

<table>
<thead>
<tr>
<th>Ply</th>
<th>(P_i)</th>
<th>(\beta_i)</th>
<th>Ply</th>
<th>(P_j)</th>
<th>(\beta_j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.48E-05</td>
<td>3.92</td>
<td>5</td>
<td>6.71E-05</td>
<td>3.82</td>
</tr>
<tr>
<td>2</td>
<td>7.25E-11</td>
<td>6.41</td>
<td>6</td>
<td>1.44E-05</td>
<td>4.18</td>
</tr>
<tr>
<td>3</td>
<td>1.44E-05</td>
<td>4.18</td>
<td>7</td>
<td>7.25E-11</td>
<td>6.41</td>
</tr>
<tr>
<td>4</td>
<td>6.71E-05</td>
<td>3.82</td>
<td>8</td>
<td>4.48E-05</td>
<td>3.92</td>
</tr>
</tbody>
</table>

Table 3.

These probabilities, rearranged accordingly to decreasing order of the value in Table 3, give the following bounds for the overall probability of failure (Table 5).

<table>
<thead>
<tr>
<th>System Probability of Failure Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower</td>
</tr>
<tr>
<td>2.03E-04</td>
</tr>
</tbody>
</table>

Table 5

For the single ply \(p_i\) the sensitivities are also evaluated accordingly with (16); the results for the first four plies are reported in Fig. 8 a,b,c,d.

5 Conclusion

In this paper the preliminary developing phase of a probabilistic analysis program is presented. Based on the probabilistic design
concepts available in the open literature, FORM and SORM approaches are implemented.

![Graphs showing sensitivities to input random variables for different plies.](image)

The presented procedure is requested to cope with the variability and scatter observed in composite materials and manufacturing procedures. A simple application to the stiffness evaluation of a composite laminate with assumed properties is presented and compared to the Monte Carlo simulation. A good correlation is obtained confirming the well behaved program. Sensitivity analysis concludes this first part of the procedure giving some interesting design indications about some of the main involved variables. A simple test case is also presented. The analysis of a simple laminate according to Tsai-Wu criterion is considered. First ply failure is assumed as reference failure condition. The obtained results confirm the expected Monte Carlo result. The activity is under development for a complete assessment of the program and in order to define a more complex applications to real composite structures coupling the software with FEM analysis.

**Acknowledgement**

Special thanks Prof. E. Antona for fruitful discussions and advices.

Part of the presented activity is included in the PRIN2004(MURST) research project.

**References**


