PROBABILISTIC DESIGN OF ADVANCED COMPOSITE MATERIALS FOR AEROSPACE STRUCTURES

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Abstract

In the present work the authors outline a procedure to evaluate uncertainity propagation in composite mechanical characteristics and to compute probability of failure of a composite laminate under generic static loads in according to the available literature. A review of probabilistic analysis basic concepts and methods is firstly introduced. A software is developed by authors, based on FORM and SORM, in order to perform analysis on a test case. Preliminary results obtained for a test case, are presented in the paper and compared with Monte Carlo simulation[1].

Nomenclature

$f_\xi(\xi)$: probability density function (pdf) of ξ
$F_\xi(\xi)$: cumulative distribution function (CDF) of ξ
μ_ξ: mean value of ξ
σ_ξ: standard deviation of ξ
ϕ: standard normal probability density function
Φ: standard normal probability density function
$\vec{X}$: set of independent normal random variables
$\vec{U}$: independent standard normal random var.
$\vec{Y}$: independent random variables
G: limit state function
β: reliability index
E_f: fibre longitudinal elastic modulus
E_m: matrix longitudinal elastic modulus
G_f: fibre shear elastic modulus
G_m: matrix shear elastic modulus
$\%V_f$: fibre volume percentage
ν_f: fibre Poisson ratio
ν_m: matrix Poisson ratio
θ: ply-angle
S_{tot}: laminate thickness
$Q_{11},Q_{12},Q_{22},Q_{66}$: ply reduced stiffnesses
A,B,D: laminate extensional, coupling and bending stifness matrices
N_x,N_y,N_{xy}: loads per unit length
$S_{i,c/t}$: compression/tensile strength of lamina in i direction
$\nabla(\bullet)$: gradient of the function $(\bullet)$
$\nabla^2(\bullet)$: Hessian of the function $(\bullet)$
$TR(M)$: trace of the matrix M

1 Introduction

The use of safety factor in structural design leads to an increase of weight with a not-quantified increase of structural reliability. Furthermore the amount of scatter observed in composite material testing tends to be high relative to metals; variability in composite material property data results from a number of sources, including variability in laying up the material, variability of raw materials, high sensitivity to testing environment and material testing methods; deterministic mechanical description of composite material may be too penalizing, leading to an increase in structures weight. For aerospace structures those weight increases could become unacceptable. A way to overcome those two limits of the deterministic approach in design, is the development of a probabilistic design methodology for aerospace advanced composite materials; such methodology should take in account uncertainties relating to quantities involved in design (geometry, loads, material capabilities, operating environment) in order to quantify the structural probability of failure (p_f) and the p_f sensitivities to design variables.

In order to cope with such complex problem, a probabilistic design procedure is being developed by authors. In section 2 a review of First and Second Order Reliability Methods (FORM and SORM) is provided in order to introduce basic concepts in structural reliability and probabilistic analysis. As a first step in probabilistic design methodology development, in sections 3 and 4 FORM and SORM approaches are investigated to evaluate the uncertainty propagation in material mechanical characteristics for a generic laminate and the probability of failure for the same laminate under static loads. In both cases sensitivity analysis is also performed to understand the relative importance of the input variable to the considered response.

2 The Basic Structural Reliability Problem

2.1 Problem Definition

The basic structural reliability problem involves a single load effect S and a single resistance effect R, both expressed by pdfs f_S and f_R. S and R are defined in such a way that the considered structural element will fail if $S>R$. Under those assumptions the basic reliability problem can be written as:

$$p_f = P(R \le S), \tag{1}$$

or more generally:

$$p_f = P\big(G(R,S) \le 0\big), \tag{2}$$

where G is the *limit state function* of the structural element; G divides the space of the random variables in two zones: the *failure domain* (where G<0) and the *safe domain* (where G>0). The probability of failure can be computed solving the integral (3) or the convolution integral (4) for independent random variables:

$$p_f = \iint\limits_{\Omega(G\le 0)} f_{RS}(r,s)\,drds, \tag{3}$$

$$p_f = \int_{-\infty}^{\infty} F_R(x)\, f_S(x)dx. \tag{4}$$

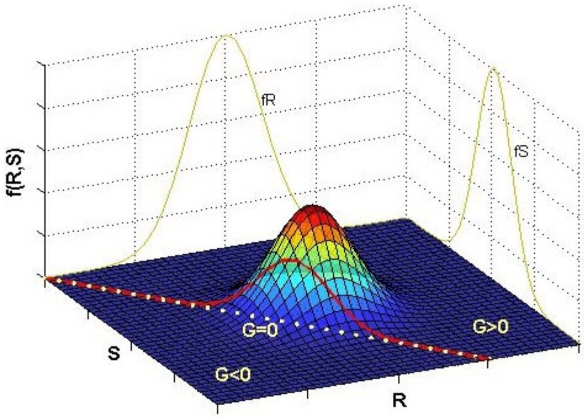


Fig. 1. R, S Joint Probability, Limit State Function and Failure Domain.

Integral (3) can be finally generalized for any number of random variables describing structure behaviour:

$$p_f = \int \ldots \int\limits_{\Omega(G\le 0)} f_{\vec{X}}\left(\vec{x}\right) d\vec{x}. \tag{5}$$

2.2 Computational Aspects – First Order Reliability Methods

Due to the high number of variables involved in (5) for practical cases, an analytic solution can be rarely achieved; the usually adopted numerical methods to perform integration in structural reliability problems are based on repeated simulation of the structural behavior for random values of the input variables (Monte Carlo methods) or on transformation of the joint probability density function in a multi-normal one for which some peculiar properties hold. Despite its simplicity and its robustness, direct Monte Carlo approach requires a great number of trials and a consequent high computational effort, in fact, as suggested by Shooman [1], the number of needed trials in a crude Monte Carlo simulation can be evaluated by:

$$\varepsilon = k\left[\frac{(1-p)}{Np}\right]^{\frac{1}{2}} \tag{6}$$

where ε is the wanted precision, p is the expected value of the probability of failure, N in the number of trials and k the value of the standard normal random variable which the level of confidence corresponds to. The square root of N in (6) implies a slow convergence of

Monte Carlo methods (Fig. 2). Some variance reduction techniques (e.g. *importance sampling*) have been developed in order to obtain a faster convergence by the introduction of a priori information about the problem solution.

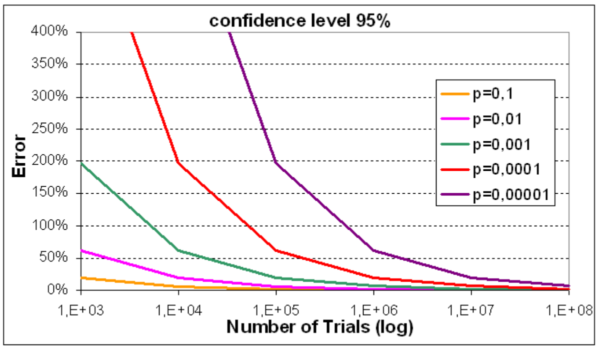


Fig. 2. Number of needed trials in direct Monte Carlo simulation

In this work direct Monte Carlo simulations are used as benchmarks for the numerical results obtained with other probabilistic approaches. As stated above, another way to compute (5) is transformation of the joint *pdf* in a multi-normal one. Let's consider the basic structural reliability problem with independent normal R and S and a limit state function $G=R\text{-}S$. By means of the properties of normal moment generation functions, G is also a gaussian random variables with mean value and variance:

$$\begin{aligned} \mu_G &= \mu_R - \mu_S \\ \sigma_G^2 &= \sigma_R^2 + \sigma_S^2 . \end{aligned} \tag{7}$$

Equation (2), introducing the standard random variable $G^* = (G - \mu_G)/\sigma_G$, becomes:

$$p_f = P(G \le 0) = \Phi\left(\frac{0-\mu_G}{\sigma_G}\right) = \Phi(-\beta) \tag{8}$$

where Φ is the normal standard cumulative distribution function and β is the *reliability index* [2,3]; β can be considered as the distance, in standard deviation unit, between the mean value of the limit state function and the failure domain. Equation (8) holds for any number of independent normal random variables X_i and for any linear limit state function $G = a_0 + a_1 X_1 + \ldots + a_n X_n$ because, under these assumptions and by the means of the axis-symmetry of the multi-normal standard joint *pdf* in standard normal space, it's always possible to compute integral (5) as:

$$\begin{aligned} p_f &= \int_{-\infty}^{+\infty} \phi_1(u_1)\,du_1 \int_{-\infty}^{+\infty} \phi_2(u_2)\,du_2 \ldots \\ &\ldots \int_{-\infty}^{-\beta} \phi_i(u_i)\,du_i \ldots \int_{-\infty}^{+\infty} \phi_n(u_n)\,du_n = \Phi(-\beta), \end{aligned} \tag{9}$$

where $u_i = \left(X_i - \mu_{X_i}\right)/\sigma_{X_i}$.

In order to extend the property expressed in (9) to non-normal random variables and non-linear limit state functions, FORM methods were developed [4]. Non-linear limit state functions can be linearized by first order Taylor's expansion around a point $\vec{U}_0$ of G:

$$G(\vec{U}) \approx G(\vec{U}_0) + \sum_i^n \left.\frac{\partial G}{\partial u_i}\right|_{\vec{U}_0} (u_i - u_{0,i}), \tag{10}$$

mean value and variance of linearized G are:

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[G(\vec{U}_0) + \sum_i^n \left.\frac{\partial G}{\partial u_i}\right|_{\vec{U}_0} (u_i - u_{0,i}) \right] \cdot \\ &\cdot f(\vec{U})\,d\vec{U} = G(\vec{\mu}_U), \end{aligned} \tag{11}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[G(\vec{U}) - G(\vec{\mu}_U) \right]^2 f(\vec{U})\,d\vec{U} = \\ &= \sum_1^n \left| \frac{\partial G(\vec{U})}{\partial u_i} \right|_{\vec{U}_0}^2 VAR[u_i] \end{aligned} \tag{12}$$

Equation (12) shows that evaluation of the probability of failure by (9) depends on the choice of the expansion point $\vec{U}_0$; the most convenient point is the nearest one to the origin of axes of the space of normal standard variables because, among the points of G, it corresponds to the highest value of the joined *pdf* and hence it's the point in which the system will most probably fail [4]; the point is called the *design point* or the *most probable point* (*MPP*) $\vec{U}^*$. If linearized G is used, then structural p_f can be computed using (9); the greatest are the curvatures of the limit state function in the space of normal variables the greatest is the error on the evaluation of the p_f

obtained by linearizing *G*. The CDF of a random variable *Y* can be approximated with normal standard *pdf* through the *normal tail approximation* [6]. The non-normal independent variables are mapped into the standard normal space by:

$$F_Y\left(\hat{y}\right)=\Phi\left(\hat{u}\right)\Rightarrow\hat{u}=\Phi^{-1}\left(F_Y\left(\hat{y}\right)\right), \qquad (13)$$

where $\hat{y}$ is a possible value of *Y*. It's important to notice that (13) doesn't transform the non-normal random variables in a normal ones, but simply map variables in the standard space keeping the cumulative probability content constant. A FORM direct algorithm for independent random variables can be summarized, hence, through the following steps [6]:

- mapping of all the random variables in standard normal random ones;
- transformation of the limit state function according to the transformed variables;
- search for the MPP and its distance β form the origin of the standard normal variables space. This represents a minimization problem in the form:

$$\beta=\min\left\|\vec{u}\right\| \qquad (14)$$

  subject to $G\left(\vec{u}\right)=0$;
- evaluation of the p_f by the means of (9)

Another important goal to be achieved in probabilistic analysis is the computation of the sensitivity factors; they can be defined as the variation of the reliability index with respect to the standard normal variables $\dfrac{\partial\beta}{\partial u_i}$ in the neighborhood of the MPP [5]. The variation of the distance from origin of a generic point *P* can be expressed by:

$$\frac{\partial\overline{OP}}{\partial u_i}=\frac{\partial}{\partial u_i}\sqrt{u_1^2+u_2^2+\ldots+u_n^2}=\frac{u_i}{\sqrt{u_1^2+u_2^2+\ldots+u_n^2}}, \qquad (15)$$

in the MPP, where $\vec{U}^*$ and $\nabla G\left(\vec{U}^*\right)$ are parallel, it becomes:

$$\frac{\partial\beta}{\partial u_i}=\frac{u_i}{\beta}=-\frac{\partial G\left(\vec{U}\right)}{\partial u_i}\frac{1}{\left|\nabla G\right|}=\alpha_i, \qquad (16)$$

introducing (16) in equation (12) it can be shown that sensitivities α_i can be also viewed as the contributions of each random variable in the overall variance of the limit state function. Ditlevsen and Madsen [3] have also defined the omission sensitivity factor ζ, as the relative error occurring when some random variables are replaced by fixed values:

$$\zeta\left(U_1=u_1,\ldots,U_q=u_q\right)=\frac{1-\sum_i\dfrac{\alpha_i u_i}{\beta}}{\sqrt{1-\sum_i\alpha_i^2}}, \qquad (17)$$

or, when a variable is replaced with its mean value:

$$\zeta\left(U_1=0,\ldots,U_q=0\right)=\frac{1}{\sqrt{1-\sum_i\alpha_i^2}}. \qquad (18)$$

Since in FORM algorithm the MPP search is usually performed by gradient methods, the sensitivity analysis requires no additional computational effort.

2.3 Second Order Reliablity Methods

When the curvatures of the limit state function around the MPP are not negligible, the linear approximation can leads to significant errors in the evaluations of the p_f. An improvement of the solution can be obtained by a second order approximation. Assuming that $G\left(\vec{U}\right)$ is twice differentiable, it can be written as:

$$\begin{aligned}G\left(\vec{U}\right)\approx\, & G\left(\vec{U}^*\right)+\nabla G\left(\vec{U}^*\right)^T\left(\vec{U}-\vec{U}^*\right)+\\ & +\frac{1}{2}\left(\vec{U}-\vec{U}^*\right)^T\nabla^2 G\left(\vec{U}^*\right)\left(\vec{U}-\vec{U}^*\right),\end{aligned} \qquad (19)$$

dividing by $\left|\nabla G\left(\vec{U}^*\right)\right|$ and introducing $\tilde{H}=\dfrac{\nabla^2 G\left(\vec{U}^*\right)}{\left|\nabla G\left(\vec{U}^*\right)\right|}$, equation (19) can be re-written as:

$$\frac{G\left(\vec{U}\right)}{\left|\nabla G\left(\vec{U}^*\right)\right|} = -\alpha^T\left(\vec{U}-\vec{U}^*\right) + \frac{1}{2}\left(\vec{U}-\vec{U}^*\right)^T \tilde{H}\left(\vec{U}-\vec{U}^*\right) = 0 \qquad (20)$$

It's more convenient to write equation (20) in a new space $\vec{V}$, where $\vec{U}^*$ is parallel to the v_n axis; this can be achieved by an orthogonal matrix T whose last row is $-\frac{\nabla G\left(\vec{U}^*\right)}{\left|\nabla G\left(\vec{U}^*\right)\right|}$, in the new space the linear approximation is simply v_n-β. Equation (20) becomes:

$$\frac{G_V\left(\vec{V}\right)}{\left|\nabla G\left(\vec{U}^*\right)\right|} = -v_n + \beta + \frac{1}{2}\left(\vec{V}-\vec{V}^*\right)^T T\tilde{H}T^T\left(\vec{V}-\vec{V}^*\right) = 0. \qquad (21)$$

Equation (21) can be finally approximated in a quadratic form of the first n-1 variables v:

$$v_n = \beta + \sum_{1}^{n-1} k_i v_i^2, \qquad (22)$$

where k_i are the eigenvalues for the first n-1 rows and columns of the matrix $T\tilde{H}T^T$ and represent the first n-1 principal curvatures of the limit state function at the MPP. The probability of failure is then:

$$p_f = P\left(G \le 0\right) \approx P\left(v_n \ge \beta + \sum_{1}^{n-1} k_i v_i\right). \qquad (23)$$

Based on definition (23), many authors have developed analytical solutions, introducing different assumptions; the first one was Breitung [7], his solution's based on asymptotic analysis as $\beta \to \infty$:

$$p_f \approx \Phi(-\beta)\prod_{i}^{n-1}\left(1+k_i\beta\right)^{-\frac{1}{2}}. \qquad (24)$$

Hohenbicher & Rackwitz [8] proposed the following improvement of the estimate of p_f by importance sampling analysis:

$$p_f \approx \Phi(-\beta)\prod_{1}^{n-1}\left(1+\frac{\phi(\beta)}{\Phi(-\beta)}k_i\right)^{-\frac{1}{2}}, \qquad (25)$$

the estimate (25) reduces to (24) for $\beta \to \infty$ and it's more accurate for lower values of β.

Tvedt [9] developed a three terms approximation by a second order power series expansion of integral (5) written for the quadratic safe set (22); the approximation is:

$$\begin{aligned}
&T_1 = \Phi(-\beta)\prod_{i=1}^{n-1}\left(1+\beta k_i\right)^{-\frac{1}{2}};\\
&T_2 = \left[\beta\Phi(-\beta)-\Phi(\beta)\right]\cdot\\
&\left\{\prod_{i=1}^{n-1}\left(1+\beta k_i\right)^{-\frac{1}{2}} - \prod_{i=1}^{n-1}\left[1+(\beta+1)k_i\right]^{-\frac{1}{2}}\right\};\\
&T_3 = (\beta+1)\left[\beta\Phi(-\beta)-\Phi(\beta)\right]\cdot\\
&\left\{\prod_{i=1}^{n-1}\left(1+\beta k_i\right)^{-\frac{1}{2}} - \mathrm{Re}\prod_{i=1}^{n-1}\left[1+\left(\sqrt{-1}+\beta\right)k_i\right]^{-\frac{1}{2}}\right\};\\
&p_f = T_1+T_2+T_3.
\end{aligned} \qquad (26)$$

The above SORM corrections are defined only for $\beta k_i \ge -1$ and don't give good results for negative curvatures [10].

Adhikari [11] proposed an asymptotic solution (27) for $n \to \infty$:

$$p_f = \Phi\left(-\frac{\beta+\mathrm{TR}(A)}{\sqrt{1+2\,\mathrm{TR}\left(A^2\right)}}\right), \qquad (27)$$

where A is the matrix containing the first n-1 rows and columns of $\frac{1}{2}T\tilde{H}T^T$.

Tvedt [12] provided also an exact solution the quadratic form (19):

$$p_f = \phi(\beta)\mathrm{Re}\left[i(2/\pi)^{1/2}\int_{t=0}^{\infty}\frac{\exp\left((t+\beta)^2/2\right)}{t}\cdot\left[\prod_{i=1}^{n-1}\left(1-tk_i\right)^{-\frac{1}{2}}\right]dt\right]. \qquad (28)$$

The main computational effort for p_f SORM estimates is the evaluation of the additional $n(n+1)/2$ elements of the Hessian matrix in the MPP. If the limit state function involves time consuming analysis or the problem has an high number of random variables, a direct Hessian computation may become impractical and different approximation methods are required [13].

3 CDF Evaluation and Sensitivity Analysis of Composite Mechanical Characteristics

Statistical description of composite laminate mechanical characteristics can't be directly evaluated through experimental testing for every possible lay-up configuration. One of the task of a composite probabilistic analysis should be to understand how the uncertainty in constituent materials, geometry and ply configuration affects laminate mechanical behaviour. That means probabilistic analysis must provide CDF of the random variables representing the mechanical characteristics and their sensitivities to input variables. FORM and SORM concepts can be used to evaluate the CDF of a function W of random variables, defining a limit state function $G'=W-W'$ where W' is a possible value of W. For such a limit state function, in fact, the probability of failure has the meaning of CDF value in W'. Direct FORM application is, however, not convenient. In fact, since the function W isn't statistically described, it's not possible to make an efficient choice of the range of W' values; it's more effective to choose directly the values of the CDF (i.e. setting a series of distances β between MPP and the origin of the standard normal variables space), and then search for the constant W' corresponding to the limit state function whose MPP is distant β from the origin. Based on this approach, a computer software has been developed in order to perform statistical description of composite laminates and to evaluate p_f for static loading. At the moment, for simplicity, the laminate mechanical model is based on the rule of mixtures and the classic laminate theory [14]. The environmental loads are not taken into account. Further development is in progress in order to include more accurate laminate models.

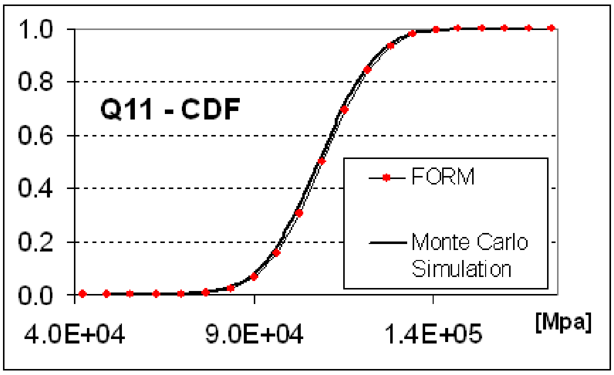
In the software fibre and matrix elastic moduli and Poisson coefficients, fibre volume ratio, thickness and orientations are chosen as the basic independent random variables. For example, in this work, the input random variables are assumed to be described by the pdfs showed in Table 1.

Material Poperty	Distribution	Mean Value	Standard Deviation	M.U.
E_f	Weibull	1.93e+05	1.57e+04	MPa
E_m	Normal	1.00e+05	1.00e+04	MPa
G_f	Lognormal	4.93e+03	4.95e+02	MPa
G_m	Lognormal	4.93e+03	4.95e+02	MPa
$\%V_f$	Normal	0.5	0.05	-
ν_f	Normal	0.3	0.03	-
ν_m	Normal	0.3	0.03	-
θ	Normal	$[0/\pm45/90]_s$	$[10/\pm4.5/9]_s$	°
Stot	Normal	8	0.2828	mm

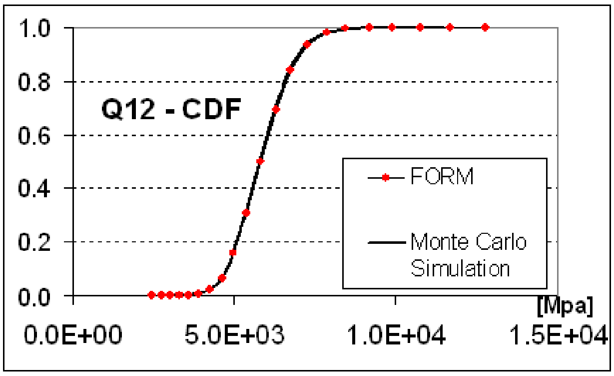
Table 1

For what concerns the FORM inverse application, the software relies on an algorithm proposed by Xiaoping Du and Wei Chen [15]; the algorithm defines a set of hyper-spheres of radius β_i and then iteratively tries to shift the function Q on the spheres until ∇Q (that's the same as $\nabla G'$) becomes parallel to the position vector of the intersection between the sphere and the Q function. The SORM corrections are performed by the means of (24), (25), (26), (27) and (28) equations.

The single ply mechanical behavior is described by the CDFs of reduced stiffness $Q11$, $Q12$, $Q22$ and $Q66$; CDFs obtained by the developed software are reported below and compared with the ones evaluated by Monte Carlo simulation (10^6 cycles). A $[0/-45/45/90]_s$ laminate is then analyzed in order to compute the CDFs of the A, B, D matrix elements; those of A are reported below. In Fig. 5a,b B and D elements CDFs are summarized. Generally FORM CDFs show a good agreement with the ones obtained by Monte Carlo Simulation; for some of the function (e.g. for A22, Fig 4c) FORM approximation can lead to significant errors and a SORM correction may be needed. In Fig. 6 SORM corrections are reported for the three cases in which FORM CDF and Monte Carlo Simulation showed worst agreement, at the moment the software isn't able to automatically evaluate the necessity to apply a SORM corrections without computing the Hessian matrix, but this feature has to be implemented in order to keep computational cost of SORM effective with respect to Monte Carlo Simulation.



(a)



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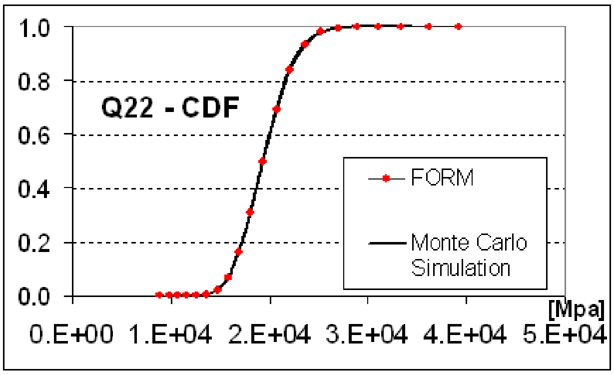

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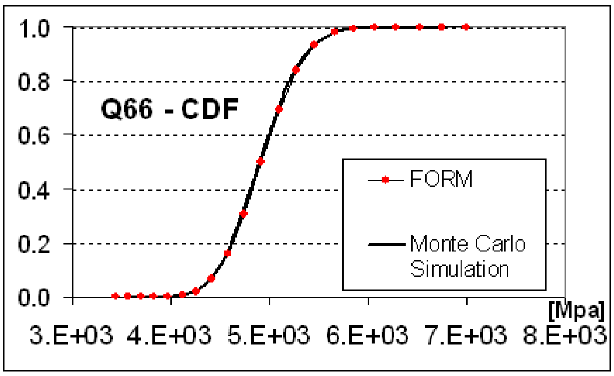
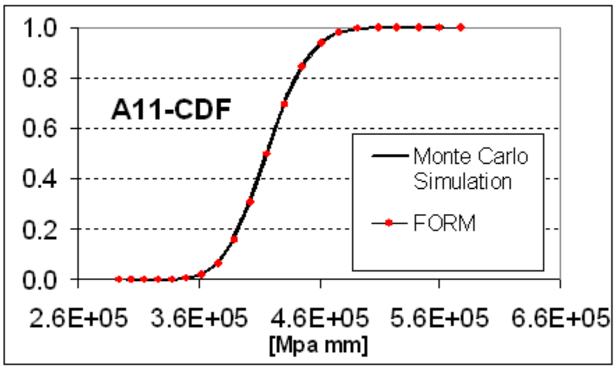
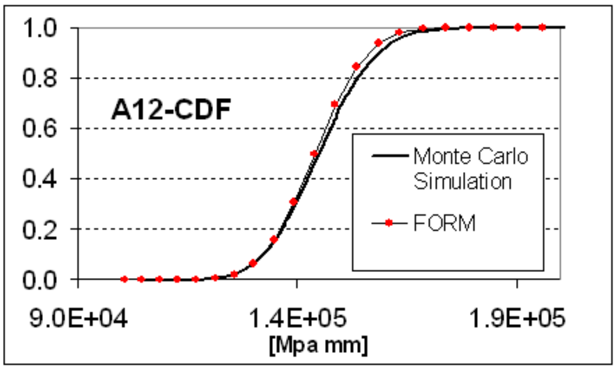


(d)

Fig. 3 a,b,c,d. CDFs of single ply mechanical properties



(a)



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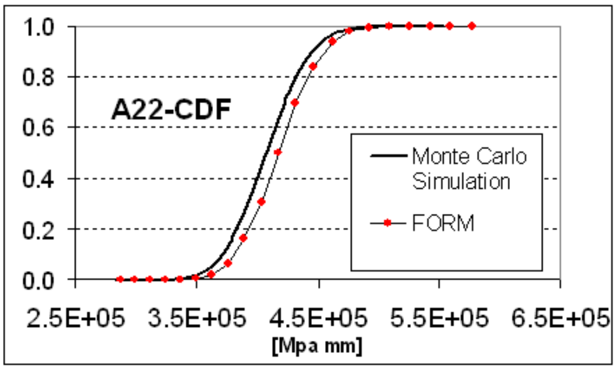
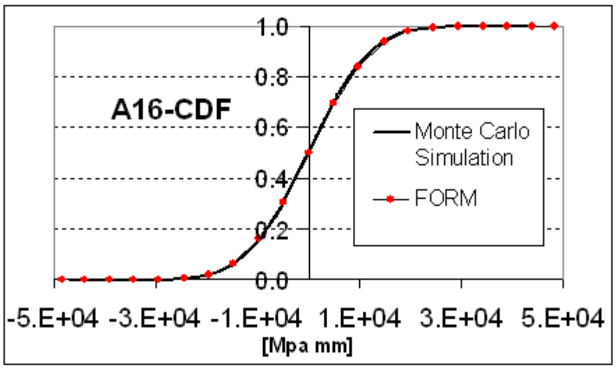


(c)

Fig. 4 a,b,c. CDFs of A matrix elements

A very important information that can be achieved by probabilistic ananlysis is sensitivity of the random variable function to its input variables. Through evaluation of sensitivities as defined in (16), it's possible to compute omission factors by equation (18); an index defined as the difference between the omission sensitivity factor obtained by replacing the same physical quantity in all layers and the unity (i.e the factor obtained by not replacing any random variable) is used to understand wich random variable could be treated as a constant in probabilistic analysis. As shown in Fig. 7a and 7b, for all the A,B and D matrix elements, fibre and matrix shear moduli and Poisson coefficients can be replaced by their mean value without any significant effect on the value of the CDF.



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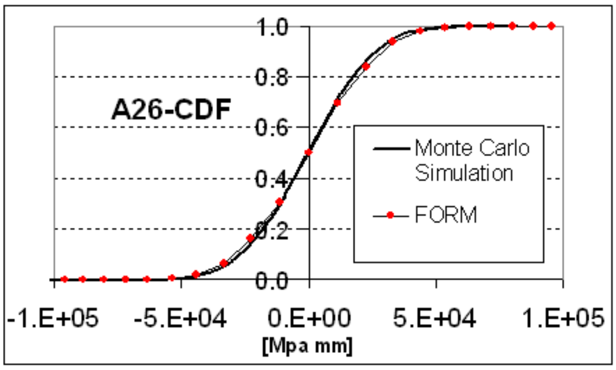
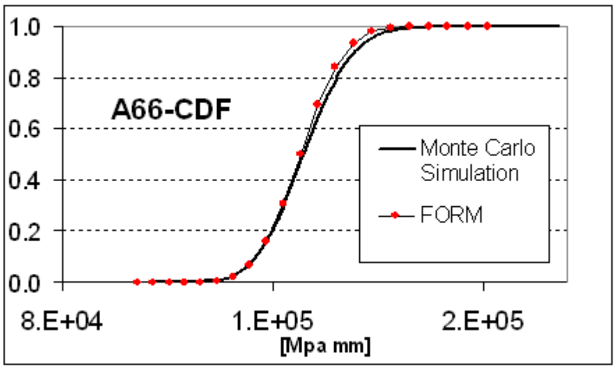


(e)



(f)

Fig. 4 d,e,f. CDFs of A matrix elements

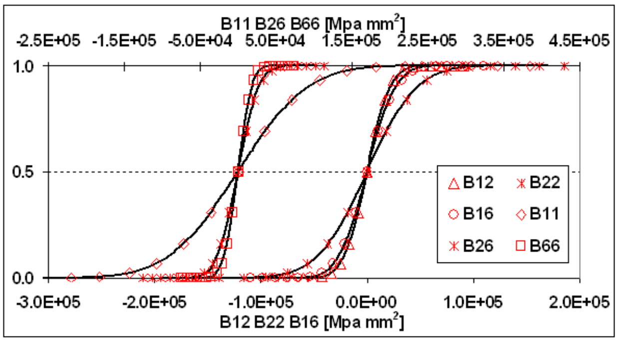
4 Evaluation of Probability of Failure for a Composite Structure: a Test Case

As a very simple example to test p_f computation procedure , let consider the first ply failure of the laminate defined in section 3 under static loads N_x, N_y and N_{xy}. In order to verify if a ply failure occurs, the Tsai-Wu criterion [16] is considered:

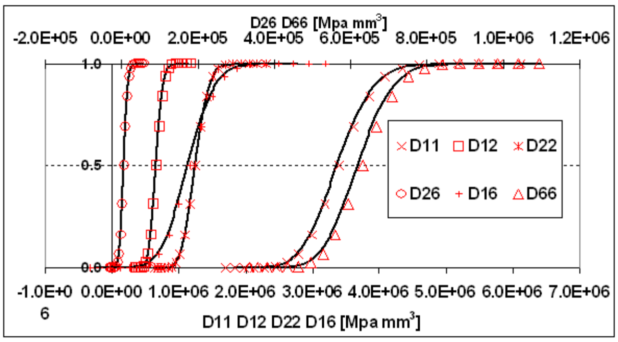
$$\begin{aligned}&\left(\frac{1}{S_{1t}}-\frac{1}{S_{1c}}\right)\sigma_1+\left(\frac{1}{S_{2t}}-\frac{1}{S_{2c}}\right)\sigma_2+\\&+\frac{1}{S_{1t}S_{1c}}\sigma_1^2+\frac{1}{S_{2t}S_{2c}}\sigma_2^2+\frac{1}{S_{12}^2}\tau_{12}^2+\\&-\frac{1}{\sqrt{S_{1t}S_{1c}S_{2t}S_{2c}}}\sigma_1\sigma_2=TW=1\Rightarrow failure,\end{aligned}\tag{29}$$

and consequently the limit state function describing the single ply behavior can be promptly obtained:

$$G_p = 1 - TW = 0. \tag{30}$$



(a)



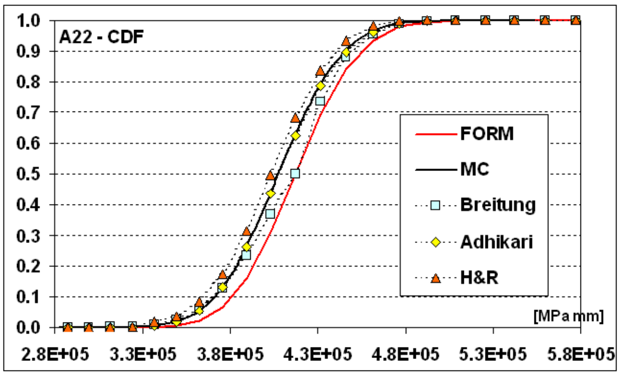
(b)

Fig. 5 a,b. FROM CDFs of B and D matrices elements: black lines correspond to Monte Carlo Simulations

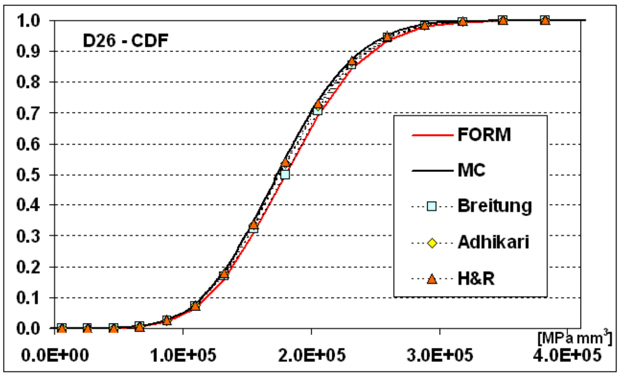
In addition to the variables showed in Table 1, G_P is also function of static loads and strengths that appear in (29); *pdfs* and first two moments for these variables are summarized in Table 2. Moreover, in order to reduce the problem dimension, G_f, G_m, ν_f and ν_m are now considered deterministic variables and their values are fixed to their respective means. Since the first ply failure is considered, the laminate can be described as a series system in which the failure of a single element implies the failure of the system itself. The overall probability of failure can hence be expressed as:

$$P(F_S) = P(F_1) \cup P(F_2 \cap \bar{F}_1) \cup P(F_3 \cap \bar{F}_1 \cap \bar{F}_2) \cup \dots \tag{31}$$

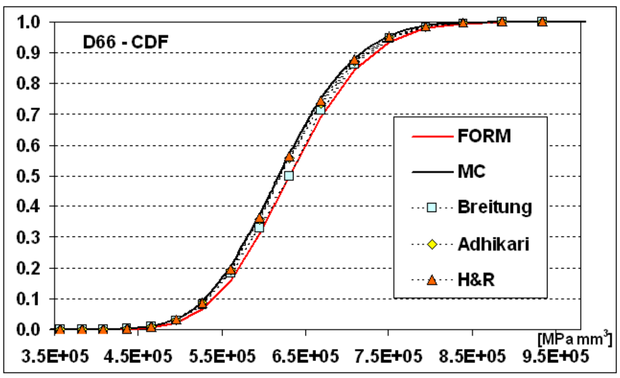
where F_x denotes the event "failure of the element x" and $\bar{F}_x$ its negation; $P(F_1 \cap \bar{F}_2)$ can



(a)



(b)

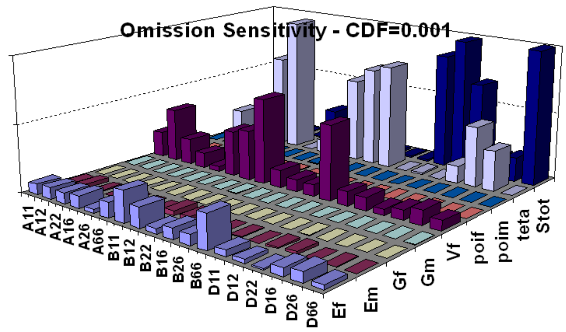


(c)

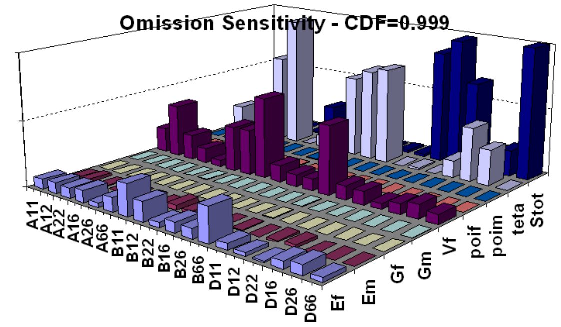
Fig. 6 a,b,c. SORM corrections for A22 and D66 CDFs

be written as $P(F_2) - P(F_2 \cap F_1)$ as well as all the other intersection terms of (31), that becomes:

$$P(F_S) = \sum_{i=1}^{n} P(F_i) - \sum_{i=2}^{n} \sum_{j=1}^{i-1} P(F_i \cap F_j) + \sum_{i=3}^{n} \sum_{j=2}^{i-1} \sum_{k=1}^{j-1} P(F_i \cap F_j \cap F_k) - \dots \tag{32}$$



(a)



(b)

Fig. 7 a,b. Omissivity indexes at 0.001 level probaility (a) and 0.999 level probability (b)

Variable	Distribution	Mean Value	Standard Deviation	M.U.
S_{1t}	Weibull	2279	199	MPa
S_{1c}	Weibull	61	7	MPa
S_{2t}	Weibull	1455	148	MPa
S_{2c}	Weibull	209	34	MPa
S_{12}	Weibull	96	7	MPa
N_x	Normal	2000	400	N/mm
N_y	Normal	2000	400	N/mm
N_{xy}	Normal	500	100	N/mm

Table 2.

Computing the system probability of failure means being able to compute the probability content of each intersection that appears in (32) i.e. evaluating integral (5) over overlapping areas of two or more failure domains. Since this can rarely be achieved, usually $P(F_S)$ is expressed in terms of upper and lower bounds. The bounds used for this example are the ones suggested by Ditlevsen [17] for which only the two terms intersections have to be evaluated:

$$P(F_S) \geq P(F_1) + \\ +\sum_{i=2}^{n} \max\left\{ P(F_i) - \sum_{j=1}^{i-1} P(F_i \cap F_j); 0 \right\} \quad (33)$$

$$P(F_S) \leq \sum_{i=1}^{n} P(F_i) - \sum_{i=2,\, j<i}^{n} \max\left\{ P(F_i \cap F_j) \right\}; \quad (34)$$

the terms $P(F_i \cap F_j)$ can be finally bounded considering the intersection between the respective linearized failure domains already defined in FORM; if β_i and β_j are the reliability index associated with the $P(F_i)$ and $P(F_j)$, it can be written [17]:

$$\text{if } \rho_{ij} \leq 0$$

$$0 \leq P(F_i \cap F_j) \leq \min\left\{ \begin{array}{l} \Phi(-\beta_i)\Phi(-\beta_{j|i}), \\ \Phi(-\beta_j)\Phi(-\beta_{i|j}) \end{array} \right\}; \quad (35)$$

$$\text{if } \rho_{ij} \geq 0;$$

$$\max\left\{ \begin{array}{l} \Phi(-\beta_i)\Phi(-\beta_{j|i}), \\ \Phi(-\beta_j)\Phi(-\beta_{i|j}) \end{array} \right\} \leq P(F_i \cap F_j) \quad (36)$$
$$\leq \Phi(-\beta_i)\Phi(-\beta_{j|i}) + \Phi(-\beta_j)\Phi(-\beta_{i|j}),$$

where:

$$\beta_{i|j} = \frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}}, \quad \beta_{j|i} = \frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}} \quad (37)$$

and ρ_{ij} is correlation coefficient between the two linearized function that, only for the linear case, is equal to the scalar product of the director cosines at MPP [5]. Bounds (33), (34), (35) and (36) must be combined in order to obtain an interval as wide as possible. It's also important to notice that obtained bounds depend on the way the failures are ordered; as reported in [5] it seems convenient, for obtaining narrower bounds, to order the failure modes according to decreasing values of probability $P(F_i)$.

The "real" system probability of failure is considered to be 2.04E-04; this value is predicted by Monte Carlo simulation with 10^6 trials. FORM probabilities of failure for each ply and probabilities of simultaneous failures occurance evaluated by the means of (33)÷(36) are reported in Table 3 and Table 4 respectively.

Ply	P_f	β	Ply	P_f	β
1	4.48E-05	3.92	5	6.71E-05	3.82
2	7.25E-11	6.41	6	1.44E-05	4.18
3	1.44E-05	4.18	7	7.25E-11	6.41
4	6.71E-05	3.82	8	4.48E-05	3.92

Table 3.

Bounds of $P(F_i \cap F_j)$					
i,j	**Lower**	**Upper**	**i,j**	**Lower**	**Upper**
1,2	1.79E-11	2.48E-11	3,5	1.68E-07	3.21E-07
1,3	7.69E-08	1.49E-07	3,6	3.27E-06	6.53E-06
1,4	0.00E+00	8.21E-21	3,7	1.10E-12	1.85E-12
1,5	0.00E+00	3.72E-22	3,8	1.19E-07	2.30E-07
1,6	1.19E-07	2.30E-07	4,5	1.02E-05	2.04E-05
1,7	1.30E-11	1.86E-11	4,6	1.68E-07	3.21E-07
1,8	1.18E-05	2.36E-05	4,7	0.00E+00	7.35E-16
2,3	4.75E-13	8.27E-13	4,8	0.00E+00	3.72E-22
2,4	0.00E+00	5.75E-17	5,6	1.60E-07	3.05E-07
2,5	0.00E+00	7.35E-16	5,7	0.00E+00	5.75E-17
2,6	1.10E-12	1.85E-12	5,8	0.00E+00	8.21E-21
2,7	2.94E-12	5.87E-12	6,7	4.75E-13	8.27E-13
2,8	1.30E-11	1.86E-11	6,8	7.69E-08	1.49E-07
3,4	1.60E-07	3.05E-07	7,8	6.19E-07	1.24E-06

Table 4.

These probabilities, rearranged accordingly to decreasing order of the value in Table 3, give the following bounds for the overall probability of failure (Table 5).

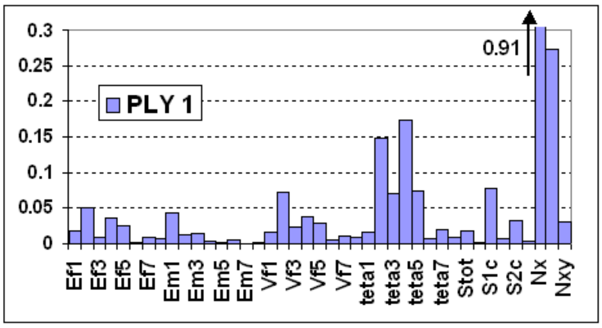
System Probability of Failure Bounds	
Lower	**Upper**
2.03E-04	2.29E-04

Table 5

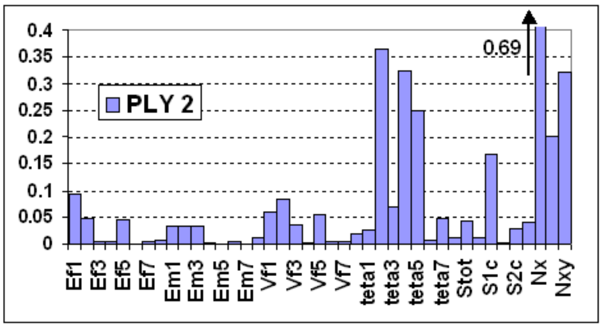
For the single ply p_f the sensitivities are also evaluated accordingly with (16); the results for the first four plies are reported in Fig. 8 a,b,c,d.

5 Conclusion

In this paper the preliminary developing phase of a probabilistic analysis program is presented. Based on the probabilistic design concepts available in the open literature, FORM and SORM approaches are implemented.



(a)



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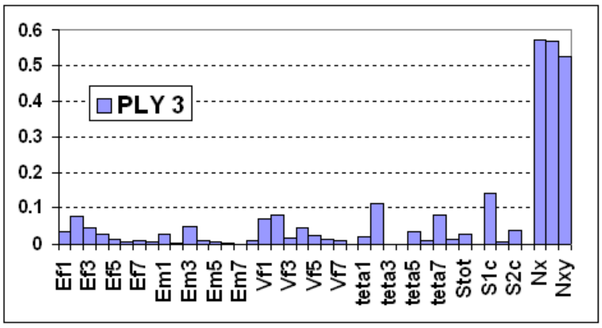

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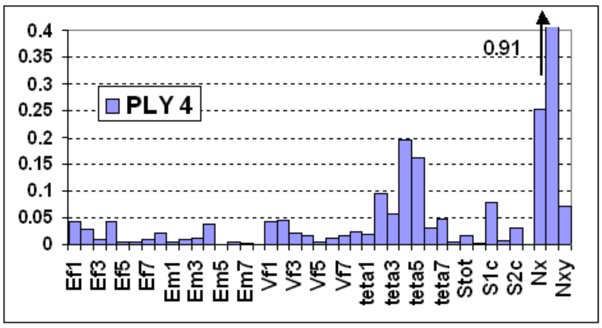

(d)

Fig 8 a,b,c,d. Sensitivities to the input random variables of the single ply probability of failure.

The presented procedure is requested to cope with the variability and scatter observed in composite materials and manufacturing procedures. A simple application to the stiffness evaluation of a composite laminate with assumed properties is presented and compared to the Monte Carlo simulation. A good correlation is obtained confirming the well behaved program. Sensitivity analysis concludes this first part of the procedure giving some interesting design indications about some of the main involved variables. A simple test case is also presented. The analysis of a simple laminate according to Tsai-Wu criterion is considered. First ply failure is assumed as reference failure condition. The obtained results confirm the expected Monte Carlo result. The activity is under development for a complete assessment of the program and in order to define a more complex applications to real composite structures coupling the software with FEM analysis.

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