# ON IMPROVING STABILITY AND EFFICIENCY OF LU-SGS METHOD FOR INCOMPRESSIBLE VISCOUS FLOWS WITH PSEUDO COMPRESSIBILITY

**Ye Junke, Song Bifeng, Song Wenping**
**National Key Laboratory of Aerodynamic Design and Research,**
**Northwesten Polytechnical University, Xi'an, P.R.China, 710072**

**Keywords**: *viscous correction, LU-SGS method, pseudo compressibility*

## Abstract

*In this paper, a viscous correction on LU-SGS method is applied to improve the stability and efficiency for incompressible viscous flows with pseudo compressibility. The viscous correction is implemented by stability analysis considering the contribution of the viscous jacobians. Several illustrative examples of incompressible flows over airfoils are computed with Reynolds number ranging from $10^3$ to $10^6$. The numerical results are in reasonably good agreement with experimental data and other research's computational results. It shows that the present method is stable and efficient for simulations of low Reynolds number flows.*

## 1 Introduction

For a flow solver to be successfully applied in aerodynamic design, two of the most important objectives are obtaining a very stable solution and achieving a high efficiency. This is especially true for Navier-Stokes methods, which require very-fine-resolution grids, particularly for incompressible flows.

Many implicit schemes[1-11] have been developed and applied successfully to steady and unsteady flow simulations. Due to the high efficiency of LU-SGS method[12], it has become very popular, and many improvements to this technique have been proposed in recent years. In reference [6], Yoon and Kwak developed an implicit LU-SGS method based on pseudo compressibility[13][14] for incompressible flows. Following this method, the LU factors are carefully constructed to make the L and U operators' scalar diagonal matrices, and it requires only scalar diagonal inversions. It is derived similar to the implicit method for compressible flows.

Although the incompressible LU-SGS scheme is unconditionally stable in theory, it is proved by practices that stability and efficiency are still affected by viscous jacobians, especially for low Reynolds number flows. According to authors' experiences, as Reynolds number of incompressible flows is reduced, the stability and efficiency are declined more and more if without viscous correction.

In order to improve the stability and efficiency, a viscous correction[3][8] on LU-SGS scheme is performed in this paper. Based on incompressible Navier-Stokes equations with pseudo compressibility, the contribution of viscous jacobians is considered.

Several incompressible viscous flows are simulated by present method to illustrate the effects of our improvement. The computed results are in good agreement with experimental data and reference's results. For simulations of low Reynolds number flows, the CFL number can be increased greatly by stability analysis considering the contributions of the viscous jacobians. Our results demonstrate that, for low Reynolds number, with the contributions of the viscous jacobians, the stability and efficiency of LU-SGS method can be increased greatly compared with the method without viscous correction. It is very useful for the incompressible simulation of low Reynolds number flows.

## 2 Governing Equations

With pseudo compressibility, the incompressible Navier-Stokes equations for a fixed control volume can be expressed in integral form as

$$\frac{d}{dt}\iint_V \boldsymbol{W}\,dV + \int_{\partial V} \boldsymbol{F}\cdot\boldsymbol{n}dS - \int_{\partial V} \boldsymbol{F}_v\cdot\boldsymbol{n}dS = 0 \qquad (1)$$

$$\boldsymbol{W} = \begin{bmatrix} p \\ u \\ v \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} \beta\boldsymbol{q} \\ u\boldsymbol{q}+p\boldsymbol{i}_x \\ v\boldsymbol{q}+p\boldsymbol{i}_y \end{bmatrix} \quad \boldsymbol{F}_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \end{bmatrix}\boldsymbol{i}_x + \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \end{bmatrix}\boldsymbol{i}_y$$

$$\tau_{xx} = 2\nu\frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\nu\frac{\partial v}{\partial y}$$

$$\tau_{xy} = \tau_{yx} = \nu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

Where $\beta$ is the pseudo compressibility parameter, $p$ is the pressure, and $u$, $v$ are components of velocity $\boldsymbol{q}$ in x,y direction, respectively. $V$ is the control volume, $\partial V$ is boundary and $\boldsymbol{n}$ is the unit outward normal vector to the boundary. Here the variables $\boldsymbol{W}$, $\boldsymbol{F}$ and $\boldsymbol{F}_v$ respectively represent the flow variable vector, the corresponding inviscid and viscous flux terms. In this formulation the Reynolds stress has been approximated as a function of the strain rate tensor, and thus $\nu$ represents a sum of the kinematic viscosity and the turbulent eddy viscosity. When Reynolds number is greater than 10,000, the Baldwin-Lomax turbulence model is used.

The governing equation (1) is discretized using a finite volume cell-centered scheme, where the cell-averaged variables are stored at the center of the grid. Combining generalized curvilinear coordinates $(\tau,\xi,\eta)$ and the jacobian of the transformation,

$$\begin{matrix} \tau = t \\ \xi = \xi(t,x,y) \\ \eta = \eta(t,x,y) \end{matrix}, \quad J = \left|\frac{\partial(\tau,\xi,\eta)}{\partial(t,x,y)}\right|$$

The jacobian matrix of this generalized inviscid flux vector can be derived,

$$\hat{\boldsymbol{A}}_i = \begin{bmatrix} k_t & \beta k_x & \beta k_y \\ k_x & k_t + k_x + \boldsymbol{Q} & k_y u \\ k_y & k_x v & k_t + k_y v + \boldsymbol{Q} \end{bmatrix}$$

where $\hat{\boldsymbol{A}}_i = \frac{\partial\hat{\boldsymbol{F}}_i}{\partial\hat{\boldsymbol{W}}} = \frac{1}{J}\frac{\partial\hat{\boldsymbol{F}}_i}{\partial\boldsymbol{W}}$, $\hat{\boldsymbol{A}}_i = \hat{\boldsymbol{A}},\hat{\boldsymbol{B}}$ and $k = \xi,\eta$ for $i$ =1,2 respectively, and $\boldsymbol{Q} = uk_x + vk_y$. The jacobian matrix that based on incompressible Navier-Stokes equations with pseudo compressibility is different from compressible equations'. With the finite volume approximation, the semi-discrete system for a given grid cell can be written as follows:

$$\begin{gathered} \frac{d}{dt}(V_{i,j}\boldsymbol{W}_{i,j}) + \boldsymbol{R}(\boldsymbol{W}_{i,j}) = 0 \\ \boldsymbol{R}(\boldsymbol{W}_{i,j}) = \boldsymbol{Q}_{C_{i,j}} + \boldsymbol{Q}_{D_{i,j}} \end{gathered} \qquad (2)$$

where $\boldsymbol{Q}_{C_{i,j}}$ and $\boldsymbol{Q}_{D_{i,j}}$ are respectively the net convective and viscous flux out of the cell. In the present work local time stepping method are employed.

## 3 Implicit Scheme

To improve stability and efficiency, viscous correction on LU-SGS scheme is applied to incompressible Navier-Stokes Equations. Referring to the ideas of Yoon and Jameson, the implicit LU operator can be obtained.

First, like the LU-SGS scheme for compressible flows, using flux difference concepts, the contribution of the inviscid flux jacobians at each cell face is split into positive and negative part.

$$\begin{aligned} (\boldsymbol{A}\Delta\boldsymbol{W})_{i+\frac{1}{2},j} &= \boldsymbol{A}^+_{i,j}\Delta\boldsymbol{W}_{i,j} + \boldsymbol{A}^-_{i+1,j}\Delta\boldsymbol{W}_{i+1,j} \\ (\boldsymbol{A}\Delta\boldsymbol{W})_{i-\frac{1}{2},j} &= \boldsymbol{A}^+_{i-1,j}\Delta\boldsymbol{W}_{i-1,j} + \boldsymbol{A}^-_{i,j}\Delta\boldsymbol{W}_{i,j} \\ (\boldsymbol{B}\Delta\boldsymbol{W})_{i,j+\frac{1}{2}} &= \boldsymbol{B}^+_{i,j}\Delta\boldsymbol{W}_{i,j} + \boldsymbol{B}^-_{i,j+1}\Delta\boldsymbol{W}_{i,j+1} \\ (\boldsymbol{B}\Delta\boldsymbol{W})_{i,j-\frac{1}{2}} &= \boldsymbol{B}^+_{i,j-1}\Delta\boldsymbol{W}_{i,j-1} + \boldsymbol{B}^-_{i,j}\Delta\boldsymbol{W}_{i,j} \end{aligned} \qquad (3)$$

The flux jacobian matrices $\boldsymbol{A}^{\pm}$, $\boldsymbol{B}^{\pm}$ are constructed so that the eigenvalues of $(^+)$ matrices are non-negative and those of $(^-)$ matrices are non-positive. Substituting equation (3) into equation (2), a splitting discretization can be written as follows,

$$\begin{aligned}&\{\boldsymbol{I}+\alpha(\boldsymbol{A}^{+}_{i,j}-\boldsymbol{A}^{-}_{i,j}+\boldsymbol{B}^{+}_{i,j}-\boldsymbol{B}^{-}_{i,j})\}\Delta\boldsymbol{W}_{i,j}\\&+\alpha(-\boldsymbol{A}^{+}_{i-1,j}\Delta\boldsymbol{W}_{i-1,j}-\boldsymbol{B}^{+}_{i,j-1}\Delta\boldsymbol{W}_{i,j-1})\\&+\alpha(\boldsymbol{A}^{-}_{i+1,j}\Delta\boldsymbol{W}_{i+1,j}+\boldsymbol{B}^{-}_{i,j+1}\Delta\boldsymbol{W}_{i,j+1})=-\alpha\boldsymbol{R}^{n}_{i,j}\end{aligned} \quad (4)$$

where $\boldsymbol{I}$ is a unit matrix, $\alpha=\dfrac{\Delta t}{V_{i,j}}$, $\boldsymbol{R}=\boldsymbol{Q}_C+\boldsymbol{Q}_D+\boldsymbol{D}$, and $\boldsymbol{D}$ is the forth-order artificial dissipative terms.

In order to ensure a greater diagonal dominance of the LU factors for a well-conditioned implicit algorithm, the splitting proposed by Yoon and Jameson is used in the present work. The jacobian matrix $\boldsymbol{A}=\boldsymbol{A}^{+}+\boldsymbol{A}^{-}$ is approximated by

$$\boldsymbol{A}^{\pm}=\frac{\boldsymbol{A}\pm r_A\boldsymbol{I}}{2},\quad r_A=\chi\max(|\lambda_A|)$$

and $\lambda_A$ is the eigenvalue of inviscid flux jacobian matrix $\boldsymbol{A}$, $\chi$ is a constant that is greater than or equal to 1. And a similar procedure is applied to the jacobian matrices $\boldsymbol{B}$.

To estimate the contribution of the viscous jacobians, an appropriate expression is used[3][6], and the jacobian matrices are modified with viscous appropriations,

$$\tilde{\boldsymbol{A}}^{\pm}=\boldsymbol{A}^{\pm}\pm2\nu|\nabla\xi|^2\boldsymbol{I}\qquad\tilde{\boldsymbol{B}}^{\pm}=\boldsymbol{B}^{\pm}\pm2\nu|\nabla\eta|^2\boldsymbol{I}$$

where $|\nabla k|=\sqrt{k_x^2+k_y^2}$, $k=\xi,\eta$ respectively. The governing equations are solved by a finite volume cell-centered formulation on structured grids. Refer to the references' methods, the discretized governing equations with LU-SGS scheme can be written as follows:

$$(\boldsymbol{L}+\boldsymbol{D})\boldsymbol{D}^{-1}(\boldsymbol{D}+\boldsymbol{U})\Delta\boldsymbol{W}^{n}=-\alpha\boldsymbol{R}(\boldsymbol{W}^{n}_{i,j}) \quad (5)$$

where

$$\boldsymbol{L}=-\alpha(\tilde{\boldsymbol{A}}^{+}_{i-1,j}+\tilde{\boldsymbol{B}}^{+}_{i,j-1})\qquad\boldsymbol{U}=\alpha(\tilde{\boldsymbol{A}}^{-}_{i+1,j}+\tilde{\boldsymbol{B}}^{-}_{i,j+1})$$

$$\boldsymbol{D}=\boldsymbol{I}+\alpha\left[(r_A+r_B)+2\nu\left(|\nabla\xi|^2+|\nabla\eta|^2\right)\right]\boldsymbol{I}$$

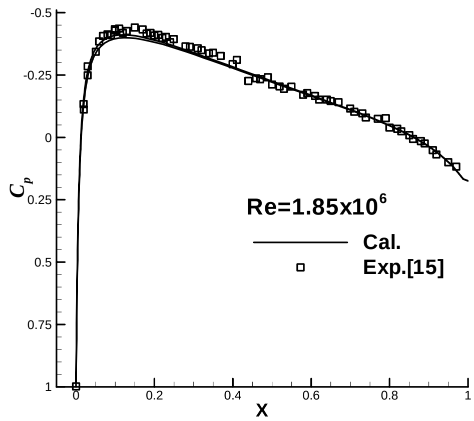
and $\alpha=\dfrac{\Delta t}{V_{i,j}}$.

## 4 Results and Discussion

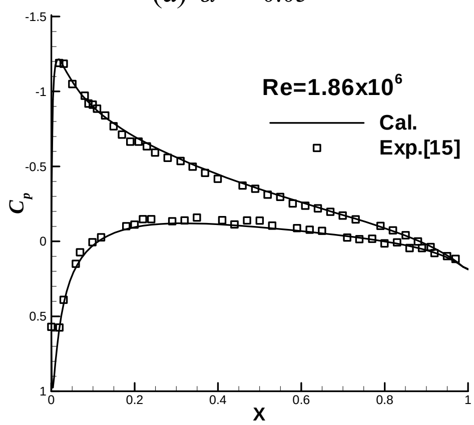
Several incompressible viscous flows are simulated by present method to illustrate the effects of our improvement. The implicit code is first validated by comparison with experimental data and numerical results. Then effect of viscous correction on stability and efficiency is investigated, by comparing performance of present method with one of LU-SGS method without viscous correction.

### *Validation of the method for incompressible viscous problems*

First, to test validation of the method for incompressible viscous problems, the flows over NACA0008 and NACA0012 airfoils have been simulated. For these cases, computations are performed on stretched 192×52 C-meshes, and in the streamwise direction 140 mesh cells have been located on the airfoil surface. And in the normal direction the nondimensional mesh spacing at the wall is about $1.0\times10^{-5}$.




(a) $\alpha=-0.05^{\circ}$




(b) $\alpha=3.59^{\circ}$

Fig.1. Comparison of computed pressure distribution with experimental data over NACA 0012 airfoil at high Reynolds numbers

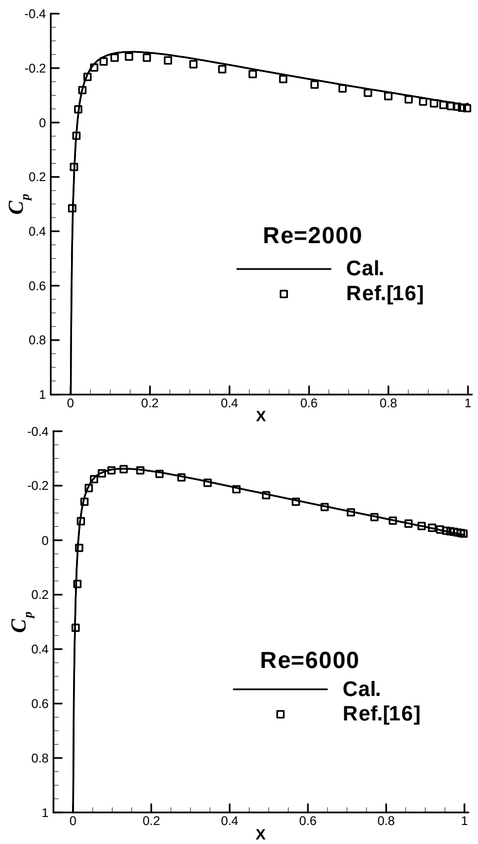



Fig.2. Comparison of computed pressure distribution with references over NACA 0008 airfoil ($\alpha = 0^\circ$) at low Reynolds numbers

These models are designed to study the characteristics of airfoils of incompressible viscous flows, and the pressure distributions are also presented. The first case is for turbulent flows at different high Reynolds numbers and angles of attack respectively, as shown in figure 1. The second case is for laminar flows at different low Reynolds numbers and an angle of attack of $0^o$, as shown in figure 2. In figure 1, 2, we can see that the computed results show good agreement with experimental data[15] and reference's results[16]. For calculations, pseudo compressibility $\beta$ is equal to 3.

### *Effect of viscous correction on stability and efficiency*

For analyzing the effect of viscous correction on stability and efficiency, computations of incompressible flows over NACA0012 airfoil have been performed at different Reynolds numbers. The Reynolds numbers selected for this analysis are $10^3$, $10^4$ and $10^6$, the other parameters is that pseudo compressibility $\beta = 3$ and angle of attack of $0^\circ$. The grid is same as above.

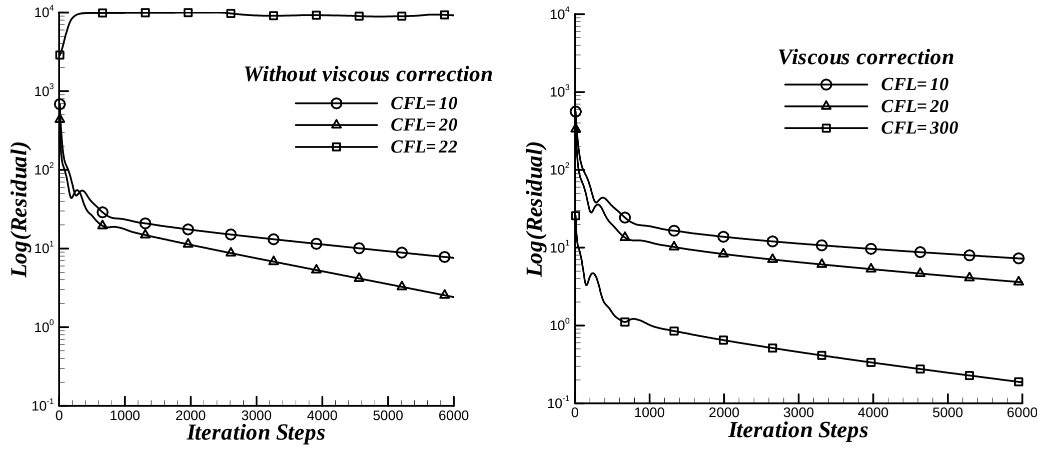
The performance of the present scheme is compared with LU-SGS method without viscous correction in figure 3(a) (b) (c). The figures show the residual histories of divergence of velocity at different Reynolds numbers, with or without viscous correction on LU-SGS method.

Figure 3(a) (b) (c) indicate that for high Reynolds number, the viscous correction takes little effect on the stability, and for low Reynolds number, with the contributions of the viscous jacobians, the CFL number can be increased greatly compared with the implicit method without viscous correction. As a result, the convergence of computation is improved for low Reynolds number. It seems that by considering stability and efficiency, viscous correction is indispensable for impressible simulations of low Reynolds number.

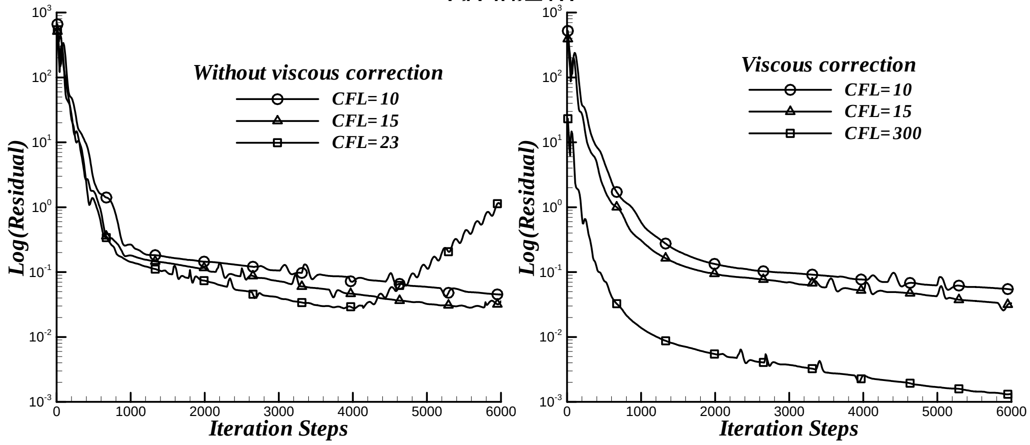
The results of our investigation demonstrate that, with the viscous correction in jacobian matrix, our scheme has the advantage of greatly increasing the CFL numbers at low Reynolds number. Consequently, by increasing the CFL numbers, the computational time is reduced and the efficiency is also improved. According to the computational experience, the present method is nearly unconditional stable both for low Reynolds number and high Reynolds number. This is very useful for the incompressible simulation.

## 5 Conclusion

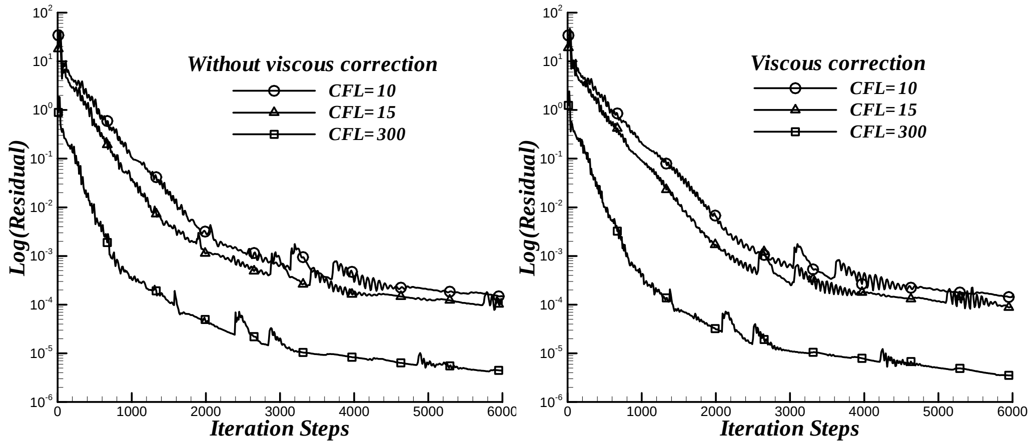
The present method has been applied to solve a variety of incompressible viscous problems on structured grids. The numerical results obtained in this study demonstrate that the LU-SGS method with viscous correction is more stable and efficient than the method only including the inviscid flux jacobians. The numerical solution can quickly reach the steady state with large CFL numbers owing to the improving stability. It is very useful for the incompressible simulation of low Reynolds number flows.




(a) $Re=10^3$




(b) $Re=10^4$




(c) Re=106

Fig.3. Stability and efficiency of LU-SGS method for NACA0012 airfoil at different Reynolds numbers with $\beta = 3$, $\alpha = 0^\circ$

## 6 References


[1] Kroll N, Radespiel R and Rossow C C. Accurate and Efficient Flow Solvers for 3D Applications on Structured Meshes. *AGARD Report R-807*, pp. 4.1-4.59, 1995.

[2] Jameson A, Tukel E. Implicit Schemes and LU Decompositions. *Mathematics of Computation*, Vol.37, No.156, 1981.

[3] Obayashi S, Kuwahara K. LU Factorization of Implicit Scheme for the Compressible Navier-Stokes Equations. *AIAA-84-1670*, 1984.

[4] Buratynski, E K Caughey and D A. An Implicit LU Scheme for the Euler Equations Applied to Arbitrary Cascades. *AIAA Journal*, Vol.24, No.1, pp 39-46, 1986.

[5] Jameson A., Yoon S. Lower-Upper Implicit Schemes with Multiple Grids for the Euler Equations. *AIAA Journal*, Vol.25, No.7, 1987.

[6] Yoon S, Jameson A. Lower-upper symmetric-Gauss-Seidel Method for the Euler and Navier-Stokes equations. *AIAA Journal*, Vol. 26, No. 9, pp 1025-1026, 1988.

[7] Rieger H, Jameson A. Solution of Steady Three Dimensional Compressible Euler and Navier Stokes Equations by an Implicit LU Scheme. *AIAA Paper 88-0619*, 1988.

[8] Grasso F, Marini M. LU Implicit TVD Scheme for the Solution of Viscous Two Dimensional High Speed Flows. *AIAA Paper 91-1573-Cp*, 1991.

[9] Yoon S, Kwak D. Three-dimensional incompressible Navier-Stokes solver using lower-upper symmetric-Gauss-Seidel algorithm. *AIAA Journal,* Vol. 29, No. 6, pp 874-875, 1991.

[10] Yoon S, Kwak D. Implicit Navier-Stokes Solver for Three Dimensional Compressible Flow. *AIAA Journal*, Vol.30, No.11, pp 2653-2659, 1992.

[11] Klopfer, G H and Yoon S. Multizone Navier-Stokes Code with the LU-SGS Scheme. *AIAA Paper 93-2965*, 1993.

[12] Bai Peng, Cui Erjie. An Implicit Method for the Pseudo-Compressibility Incompressible Navier-stokes Equation. *Chinese Journal of computational physics*, Vol.22, No.5, pp386-392, 2005.

[13] A J Chorin. A Numerical Method for Solving Incompressible Viscous Flow Problems. *Journal of Computational Physics 2*, pp.12-26, 1967.

[14] Stuart E Rogers, Dochan Kwak. An Upwind Differencing Scheme for the Incompressible Navier-Stokes Equations. *Applied Numerical Mathematics*, Vol.8, pp 43-46, 1991.

[15] Experimental Data Base for Computer Program Assessment, AGARD-AR-138, 1979.

[16] Peter J Kunz, Ilan Kroo. Analysis Design of Airfoil for Use at Ultra-Low Reynolds Numbers. *Fixed and Flapping Wing Aerodynamics for Micro Air Vehicle Applications*, USA, AIAA, pp 35-60, 2001.