MATHEMATICAL MODELING OF A LOW-POWER GAS TURBINE ENGINE AND ITS CONTROL SYSTEM

Piroska AILER Ph. D. student
BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
Department of Aircraft and Ships

Keywords: non-linear and linear mathematical model, control loop

Abstract
In this paper a non-linear mathematical model of a low-power gas turbine, which is used in the Budapest University of Technology and Economics, will be presented.

This topic is current, because with this model the manoeuvrability of the engine can increase in special flight situations.

According to the measurements, the parameters of the gas turbine are known, determined.

The method of non-linear modelling is based on the thermodynamical equations, which describe the behaviour of the engine.

With this model after linearization we can design an optimal controller for the gas turbine.

1 State-space representation
The signals are time-dependent functions, which can be scalar-valued or vector-valued, and deterministic or stochastic. In the following only vector-valued and deterministic signals will be used.

The system is any part of the real world surrounded by a well-defined boundary. The system is influenced by its environment via signals (u(t) - input signal), and acts on its environment by other signals (y(t) – output signal).

The state of the system contains all past information on the system up to time t₀ – time instant. If we would like to compute the output signal for t≥t₀ (all future values) we only need u(t), t≥t₀ and the state t=t₀.

A description, which uses the state signal, is called state space description, or state space representation.

The general form of the state space representation of a finite dimensional linear time invariant system is:

\[ \dot{x}(t) = Ax(t) + Bu(t) \] - state equation (1)

\[ y(t) = Cx(t) + Du(t) \] - output equation, (2)

with given initial condition \( x(t_0)=x(0) \), and
\n\[ x(t) \in \mathbb{R}^r, \quad y(t) \in \mathbb{R}^p, \quad u(t) \in \mathbb{R}^n \] (3)

being vectors of finite dimensional spaces, and
\n\[ A \in \mathbb{R}^{r \times r}, \quad B \in \mathbb{R}^{r \times n}, \quad C \in \mathbb{R}^{p \times r}, \quad D \in \mathbb{R}^{p \times n}. \] (4)

The general form of the state space representation of a finite dimensional non-linear time invariant system is:

\[ \dot{x}(t) = f(x(t),u(t)) \] - state equation (5)

\[ y(t) = h(x(t),u(t)) \] - output equation, (6)

with the vector-values state, input and output vectors \( x, u \) and \( y \).

2 Description of the gas turbine
The subject of our analysis is a low-power single-spool gas turbine with single-stage centrifugal compressor and single-stage centripetal turbine. This engine is a special design for technical colleges, universities. According to this attribute the main principles of construction to be observed were simplicity
of design and ease of maintenance. Consequently, the unit was designed with small power-to-weight ratio, limited space requirement, multi-fuel capabilities and vibration-free running.

The most important parameters of the engine (if \( p_0 = 1,0133 \) bar, \( T_0 = 288 \)K, \( n = 50000 \) 1/min):

- **Power:** \( P = 80 \) kW,
- **Mass flow rate of air:** \( m_c = 0,9 \) kg/sec,
- **Compressor pressure ratio:** \( \pi_c^* = 2,8 \),
- **Exhaust gas temperature:** \( T_4^* = 938 \) K.

We are able to measure:
- total pressure and total temperature
  - before and after the compressor \( (p_1^*, \ p_2^* \text{ and } T_1^*, \ T_2^*) \),
  - before and after the turbine \( (p_3^*, \ p_4^* \text{ and } T_3^*, \ T_4^*) \),
- number of revolutions \( n \),
- consumption of fuel \( (m_{\text{fuel}}) \),
- and the moment of load \( (M_{\text{load}}) \).

### 3 Thermodynamic non-linear mathematical model

The thermodynamic non-linear model describes the behaviour of the engine. It has two different parts: the first contains the steady-state equations and the second contains the dynamic equations.

The **steady-state equations** describe the working of the elements of the engine in an operating (equilibrium) point. Our gas turbine has four parts.

1. The first part is the **inlet**. In this component, it is assumed that the flow is adiabatic. Let the pressure loss is \( \sigma_{\text{inlet}} \), the adiabatic exponent is \( \kappa \), the Mach number is \( M \) and the parameters (pressure and temperature) \( p_0 \text{ and } T_0 \)! With them the conditions (total temperature and pressure) at the inlet to the compressor are:

   \[
   T_1^* = T_0^* = T_0 \cdot (1 + \frac{\kappa - 1}{2} M^2 )
   \]  
   \[
   p_1^* = \sigma_{\text{inlet}} \cdot p_0^* = \sigma_{\text{inlet}} \cdot p_0 \cdot (1 + \frac{\kappa - 1}{2} M^2 )^{\frac{\kappa}{\kappa - 1}}
   \]

2. The second part is the **compressor**. It is modelled empirically using the compressor characteristic (map), which gives the mass flow across the compressor. The total temperature rise is found by using the isentropic efficiency factor \( \eta_c \), the total pressure is the function of compressor pressure ratio \( \pi_c^* \) and \( n \) is the number of revolution:

   \[
   m_c = f(n, p_2^*, p_1^*, T_1^*)
   \]
   \[
   T_2^* = T_1^* \cdot 1 + \frac{p_2^*}{p_1^*} \left( \frac{\pi_c^*}{\eta_c} \right)^{\frac{\kappa}{\kappa - 1}} - 1
   \]
   \[
   p_2^* = \Pi_c \cdot p_1^*
   \]

3. The third part is the **combustor**. In the combustor the total pressure loss \( (\sigma_f) \) is assumed to be a fixed percentage of its inlet pressure \( p_2^* \) and total temperature rise is given by the steady-state energy equation, in which \( c_p \) is the heat capacity of air in constant pressure, \( \eta_{\text{comb}} \) is the efficiency factor of combustion and \( H_l \) is the lower thermal value of fuel, in the combustor:

   \[
   p_3^* = \sigma_f \cdot p_2^*
   \]
   \[
   T_3^* = T_2^* + q_f \cdot \frac{Q_f \cdot \eta_f}{c_p}
   \]
   \[
   T_4^* = \frac{1 + q_f}{1 + q_f} \cdot \frac{c_p}{q_f}
   \]

where: \( q_f = \frac{m_{\text{fuel}}}{m_c} \)

4. The last part is the **turbine**. It is modelled empirically with the steady-state turbine performance map, which gives the mass...
flow across the turbine. The total temperature drop is found by using the isentropic efficiency factor $\eta_t$, and total pressure is the function of pressure loss $\sigma_g$ in the tube after the turbine:

$$m_t = \frac{\beta \cdot A_t \cdot g(\lambda_t)}{\sqrt{T_3^*}} \cdot p_3^*$$ (15)

$$T_4^* = T_3^* \cdot \left(1 - \eta_t \cdot \left(1 - \frac{1}{\sqrt{\frac{P_3^*}{P_4^*}}}\right)^{\frac{x-1}{1-x}}\right)$$ (16)

$$p_4^* = \sigma_g \cdot p_0$$ (17)

The dynamic equations describe the changes in the gas turbine, when the operating point is varying because of the operator or disturbances.

1. The first dynamic equation comes from the power balance on the compressor/turbine spool, where $P_l$ is from the loading:

$$4 \cdot \Pi^2 \cdot \Theta \cdot n \cdot \frac{dn}{dt} = P_l \cdot \eta_m - P_c - P_l = m_t \cdot c_p \cdot (T_3^* - T_1^*) \cdot \eta_m - m_t \cdot c_p \cdot (T_2^* - T_1^*) - P_l$$ (18)

2. A control volume is needed around the combustor to model the dynamic behaviour. For this control volume a mass balance and an energy balance produced two first-order differential equations:

- mass balance

$$\frac{dp_3^*}{dt} = \frac{R \cdot T_2^*}{V_2} \cdot (m_t + m_{fuel} - m_t)$$ (19)

- energy balance

$$\frac{dT_3^*}{dt} = \frac{1}{m_{comb}} \cdot \left(\left(\dot{m}_t + \dot{m}_{fuel}\right) \cdot T_2^* - \dot{m}_t \cdot T_3^* \right) - \left(\dot{m}_t + \dot{m}_{fuel} - \dot{m}_t\right) \cdot T_3^*$$ (20)

4 Jacobian linearization

It was shown, that a gas turbine can be represented by the following non-linear differential and algebraic equations:

$$\dot{x} = F(x,u)$$

$$y = G(x,u)$$ (21)

In deriving linear models, we assume that functions $F$ and $G$ are continuous and differentiable. If the system described by Eq. (21) is in a steady-state condition, when constant input $u_{ss}$ producing constant state $x_{ss}$, and constant output $y_{ss}$, then the combination $(u_{ss}, x_{ss}, y_{ss})$ satisfies:

$$0 = F(x_{ss}, u_{ss})$$

$$y_{ss} = G(x_{ss}, u_{ss})$$ (22)

The point $(u_{ss}, x_{ss}, y_{ss})$ is an equilibrium point of the gas turbine. Perturbating the control input with $\delta u$ results in state and output perturbation $\delta x$ and $\delta y$, respectively and control input, state and output become $u = u_{ss} + \delta u$, $x = x_{ss} + \delta x$ and $y = y_{ss} + \delta y$, and the Eq. (22) follows:

$$\dot{x}_{ss} + \delta x = F(x_{ss} + \delta x, u_{ss} + \delta u)$$

$$y_{ss} + \delta y = G(x_{ss} + \delta x, u_{ss} + \delta u)$$ (23)

Because of the continuity requirements imposed on the $F$ and $G$ functions Eq (23) can be expanded in Taylor series about the point $(u_{ss}, x_{ss}, y_{ss})$. Ignoring the higher order items, the results is:

$$\delta x = A \cdot \delta x + B \cdot \delta u$$

$$\delta y = C \cdot \delta x + D \cdot \delta u$$ (24)
The constant matrices A, B, C and D have the dimensions of \( n \times n \), \( n \times m \), \( r \times n \) and \( m \times r \), and given by:

\[
A_{ij} = \frac{\delta F_i}{x_j}, \quad B_{ij} = \frac{\delta F_i}{u_j}, \quad C_{ij} = \frac{\delta G_i}{x_j}, \quad D_{ij} = \frac{\delta G_i}{u_j}
\]

This equation approximates the dynamic behaviour of the non-linear gas turbine in a small region about the operating point.

### 5 State space representation of our gas turbine

In our case the state, input and output vectors will be the follows:

\[
u = \begin{bmatrix} \dot{m}_{\text{fuel}} \\ P_i \\ T_3^* \end{bmatrix}, \quad x = \begin{bmatrix} n \\ p_2^* \\ T_3^* \end{bmatrix}, \quad y = \begin{bmatrix} n \\ p_2^* \\ T_3^* \end{bmatrix}
\]

To construct the A, B, C, D matrices we have to derive the dynamic equations. For example:

\[
\frac{dp_2^*}{dt} = \frac{R \cdot T_2^*}{V_2}
\]

\[
\left( m_c(n, p_2^*) + \dot{m}_{\text{fuel}} - \frac{\beta \cdot A_3 \cdot q(\lambda_3) \cdot p_3^*}{\sqrt{T_3^*}} \right) = F(n, p_2^*, T_3^*, \dot{m}_{\text{fuel}}, P_i)
\]

The \( A_{21} \) parameter (the first element of the second row):

\[
\frac{\partial F}{\partial n} = \frac{R \cdot T_2^*}{V_2} \cdot \frac{\partial \dot{m}_{\text{fuel}}}{\partial n} \quad (p_2^* = \text{const})
\]

The \( A_{22} \) parameter (the second element of the second row):

\[
\frac{\partial F}{\partial p_2^*} = \frac{\kappa - 1 \cdot R \cdot T_0^*}{\kappa} \cdot \frac{\partial \dot{m}_{\text{fuel}}}{\partial p_2^*}
\]

The \( B_{21} \) parameter (the first element of the second row):

\[
\frac{\partial F}{\partial T_3^*} = 0.5 \cdot \frac{R \cdot T_2^*}{V_2} \cdot \frac{\partial \dot{m}_{\text{fuel}}}{\partial T_3^*}
\]

The \( B_{22} \) parameter (the second element of the second row):

\[
\frac{\partial F}{\partial P_i} = 0
\]

### 6 LQ control design

Given a linearised model corresponding to an engine operating point, the standard linear optimal design technique can be used to determine the full state feedback gains by minimising the quadratic performance criteria:

\[
J = \frac{1}{2} \int_0^\infty (\dot{x}^T Q \dot{x} + \dot{u}^T R \dot{u}) dt
\]

and solving the corresponding Control Algebraic Riccati Equation (CARE):

\[
PA + A^T P - PBR^{-1} B^T P + Q = 0
\]

the state feedback gains are:
\[ \delta u = -R^{-1}B^T P \delta x = -K \delta x \]  \hspace{1cm} (35)

This is the optimal control for the gas turbine.

7 Simulation

With the help of the equations above we measured and calculated the parameters, the elements of the A, B, C and D matrices. With these we can simulate the behaviour of the gas turbine, and we can design a controller for the engine.

In the next four diagrams the response-functions can be seen, when the input (or control input) is a Dirac-delta function. In the first two figures the system is without control, these are system-simulations. The next two is with the optimal LQ control.

8 Summary

Comparison:
The LQ control has changed the quantitative features of the gas turbine, the time constants are smaller, so we need less time for the system goes back to the operating point.

Possibility to continue this investigation:

This LQ control design is for an ideal system without any noises, disturbances. So it is important, that in the future the noises, disturbances and uncertainties of the system (because of the measuring and neglected dynamics) will have to be investigated, and with them we will be able to design a new controller.

References

Fig. 1.: Simulation when $m_{\text{fuel}}$ is a Dirac-delta function (without control)

Fig. 2.: Simulation when $P_1$ is a Dirac-delta function (without control)
Fig. 3.: Simulation when $m_{\text{fuel}}$ is a Dirac-delta function (with LQ control)

Fig. 4.: Simulation when $P_1$ is a Dirac-delta function (with LQ control)