A COMPARISON BETWEEN AN OPTIMAL CONTROL LAW DESIGN AND A POLE-PLACEMENT CONTROL LAW DESIGN WITH RESPECT TO STABILITY CHARACTERISTICS AND GIBSON DROPBACK CRITERION

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ABSTRACT

A design have been carried out for a rate command-attitude hold control law including a proportional plus integral controller acting on pitch rate and angle of attack, both feedbacked to elevator. The proportional feedback enables the rate command characteristics to be tailored as required and the integral feedback drives the error signal to zero obtaining good longer term holding characteristics, also a feedforward acting on the reference input is included in the controller allowing to shape the response as required. The objective of the design is to obtain a flight control system that meets the Gibson dropback criterion and also the stability requirements of MIL-F-8785C. The design have been performed by two methods, pole placement and optimal control (LQR). Both designs have been compared when an actuator dynamics are included and also when the phugoid model is included. The control effort and control rate effort are analyzed and compared, when the flight control system works as a regulator and when works with a pilot input, as required by the dropback criterion. The dynamic characteristics of the aircraft augmented with both designs have been obtained and analyzed. A simplification of the control law design was also studied. A robustness analysis of both designs was also studied when the gains are varied with respect to the nominal designed gains.

1 POLE PLACEMENT CONTROL LAW DESIGN

The aircraft model used was the Boeing B-747, and the appropriate data can be found on Heffley. First the short period longitudinal reduced model is used as,

\[
\begin{bmatrix}
\dot{\omega} \\
q
\end{bmatrix} =
\begin{bmatrix}
a & a_1 & a_2 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega \\
q
\end{bmatrix} +
\begin{bmatrix}
b_1 \\
b_2 \\
b_1 \\
b_2
\end{bmatrix} \eta \quad (1)
\]

with \( x^T = [\omega \ q] \) \quad (2)

it is possible to use

\[
\dot{x} = A x + B \eta \quad (3)
\]

The control law must ensure that the augmented aircraft satisfies the stability requirements contained on MIL-F-8785C and the Gibson dropback criterion. The control anticipation parameter, CAP, contained on MIL-F-8785C will take into account the stability requirements and the dropback parameter, DB, defined in Cook will take into account the dropback criterion. So CAP is defined as,

\[
CAP = \frac{g T \omega^2}{\theta_2 v_{sp}} \quad (4)
\]

and dropback is,
\[ DB = \frac{T \omega_{sp} - 2 \zeta_{sp}}{\omega_{sp}} \]  

(5)

The control law structure is showed on figure (1) and is simply a PI controller with a feedforward gain.

As can be seen it is necessary to add an extra state, that is, \( \varepsilon_q \), defined as,

\[ \varepsilon_q = q - q_d \]  

(6)

in order to take into account the error. With the help of (4) and (5) it is possible to define \( \omega_{sp} \) and \( \zeta_{sp} \) with the constraint that both satisfy the CAP requirement and DB near zero as required by the Gibson criterion. The state space model including \( \varepsilon_q \) can be written as,

\[ x = Ax + B\eta + E q_d \]  

(7)

with now \( x^T = [\omega \ q \ \varepsilon_q] \)  

(8)

and

\[ A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 1 & 0 \end{bmatrix} \]  

(9)

\[ B^T = \begin{bmatrix} b_{11} & b_{21} & 0 \end{bmatrix} \]  

(10)

\[ E^T = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \]  

(11)

The control law is simply given by,

\[ \eta = -G x + G_0 q_d \]  

(12)

with \( G \) a control vector given by,

\[ G = \begin{bmatrix} K_v & K_q & K_{\varepsilon_q} \end{bmatrix} \]  

(13)

So the closed loop system is given by,

\[ x = (A - BG)x + (BG + E)q_d \]  

(14)

where \( G \) is simply obtained by defining the closed loop poles and \( G_0 \) is computed in order to obtain pole zero cancellation as in Friedland. The characteristic equation of the closed loop system is of order three and so it is necessary to define three poles, two closed loop poles are defined with the help of (4) and (5) and the third pole is defined in order that the integral term has a time constant of the same magnitude of the short period natural frequency and so not great changes are introduced in the system. As the short period natural frequency is around 1 rad/sec, the third pole have
been chosen as -1, that is $s = -1$, $\omega_{sp} = 1.20$ rad/sec and $\zeta_{sp} = 0.83$, for 20000 ft mach 0.70 which gives the complex poles $-1.02 \pm i 0.63$. With the use of MATLAB the gains are,

$$G = \begin{bmatrix} 0.0012 & -0.889 & -1.183 \end{bmatrix} \quad (15)$$

$$G_0 = 1.183 \quad (16)$$

and $G_0$ is computed by,

$$G_0 = [C(A-BG)^{-1}B]^T C(A-BG)^{-1}E \quad (17)$$

and so the design is completed. Table (1) shows the closed loop poles choosen for this design, the $\omega_{sp}$ and $\zeta_{sp}$.

In table (2) the gains obtained are listed.

### TABLE 1

<table>
<thead>
<tr>
<th>$h$ (ft)</th>
<th>Mach (rad)</th>
<th>closed loop poles</th>
<th>CAP ($s^{-2}$)</th>
<th>$\omega_{sp}$ (rad/s)</th>
<th>$\zeta_{sp}$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000</td>
<td>0.60</td>
<td>$-1.08 \pm i 1.11$ , -1</td>
<td>0.117</td>
<td>1.55</td>
<td>0.70</td>
</tr>
<tr>
<td>20000</td>
<td>0.70</td>
<td>$-1.02 \pm i 0.63$ , -1</td>
<td>0.101</td>
<td>1.20</td>
<td>0.85</td>
</tr>
<tr>
<td>40000</td>
<td>0.80</td>
<td>$-1.61 , -0.449$ , -1</td>
<td>0.086</td>
<td>0.85</td>
<td>1.21</td>
</tr>
<tr>
<td>10000</td>
<td>0.40</td>
<td>$-0.58 \pm i 0.59$ , -1</td>
<td>0.092</td>
<td>0.83</td>
<td>0.70</td>
</tr>
<tr>
<td>30000</td>
<td>0.70</td>
<td>$-0.86 \pm i 0.25$ , -1</td>
<td>0.087</td>
<td>0.90</td>
<td>0.96</td>
</tr>
</tbody>
</table>

### TABLE 2

<table>
<thead>
<tr>
<th>$h$ (ft)</th>
<th>Mach (rad)</th>
<th>$K_w$ (-1 ft/s)</th>
<th>$K_q$ s</th>
<th>$K_{\xi_q}$ rad</th>
<th>$G_0$ s</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000</td>
<td>0.60</td>
<td>0.0012</td>
<td>-0.588</td>
<td>-1.219</td>
<td>1.219</td>
</tr>
<tr>
<td>20000</td>
<td>0.70</td>
<td>0.0012</td>
<td>-0.889</td>
<td>-1.183</td>
<td>1.183</td>
</tr>
<tr>
<td>40000</td>
<td>0.80</td>
<td>0.0011</td>
<td>-1.875</td>
<td>-1.697</td>
<td>1.697</td>
</tr>
<tr>
<td>10000</td>
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<td>0.0026</td>
<td>-1.094</td>
<td>-1.270</td>
<td>1.270</td>
</tr>
<tr>
<td>30000</td>
<td>0.70</td>
<td>0.0013</td>
<td>-1.249</td>
<td>-1.252</td>
<td>1.252</td>
</tr>
</tbody>
</table>

2 OPTIMAL CONTROL LAW DESIGN

The same control law structure used in the pole placement design is now used but the design is now performed by optimal control law method, that is, by the LQR method. The state vector is given by (8) and the state space model is given by (7). The appropriate performance index is,

$$V = \int_{0}^{\infty} (x^T Q x + \eta^T R \eta) \, dt \quad (18)$$

$Q$ is a (3x3) matrix and $R$ is a scalar. The $Q$ matrix and $R$ can be choosen by means of a parametrically study and since only $\xi_q$ is of concern in the design than $Q$ will be choosen as a diagonal matrix with the form:
\[ Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (19)

and \( R = \rho \) which will be varied in order to find suitable gains in the way that the augmented aircraft meet CAP and dropback criterion. With \( A, B, Q \) and \( R \) it is possible to find the feedback gains solving the algebraic Riccati equation, that is, the classical LQR problem. To find \( G \) the method given by Friedland is used and the computation is given by,

\[ G_0 = -R^{-1}B^T \left[ A_C^T \right]^{-1}M \]  \hspace{1cm} (20)

where \( A_C = A - BR^{-1}B^T \) and \( M \) is the solution of the algebraic Riccati equation, in the way that \( G \) is given by

\[ G = R^{-1}B^T M \]  \hspace{1cm} (22)

So by performing a parametric study with the variation of \( \rho \) it is possible to find a set of gains that allow the augmented aircraft to meet CAP and dropback criterion. Table 3 shows the resulting closed loop poles, CAP, \( \omega_{sp}, \xi_{sp} \) and final \( \rho \) for this design.

In table 4 the gains obtained by this method are listed,

In figure 2 the performance of both designs with respect to the dropback criterion are plotted and as can be seen both satisfy the criterion quite well. Also can be seen that both satisfy CAP very well, as the CAP requirement is given by,

\[ 0.085 \leq \text{CAP} \leq 3.6 \]

in the MIL-F8776C flying qualities requirements.

### Table 3

<table>
<thead>
<tr>
<th>( h )</th>
<th>Mach</th>
<th>( \omega_{sp} )</th>
<th>( \xi_{sp} )</th>
<th>( \rho )</th>
<th>( \text{CAP} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{ft} )</td>
<td>( \text{rad} )</td>
<td>( \text{rad} / \text{s} )</td>
<td>( \text{rad} )</td>
<td>( \text{rad} )</td>
<td>( \text{rad} / \text{s}^2 )</td>
</tr>
<tr>
<td>1000</td>
<td>0.60</td>
<td>1.64</td>
<td>-0.23 , -1.03 ± i 1.27</td>
<td>10</td>
<td>0.130</td>
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<tr>
<td>20000</td>
<td>0.70</td>
<td>1.42</td>
<td>-0.27 , -0.75 ± i 1.20</td>
<td>5</td>
<td>0.141</td>
</tr>
<tr>
<td>40000</td>
<td>0.80</td>
<td>1.19</td>
<td>-0.24 , -0.61 ± i 1.03</td>
<td>1.5</td>
<td>0.168</td>
</tr>
<tr>
<td>10000</td>
<td>0.40</td>
<td>1.11</td>
<td>-0.19 , -0.60 ± i 0.93</td>
<td>5</td>
<td>0.165</td>
</tr>
<tr>
<td>30000</td>
<td>0.70</td>
<td>1.18</td>
<td>-0.21 , -0.56 ± i 1.03</td>
<td>5</td>
<td>0.150</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th>( h )</th>
<th>Mach</th>
<th>( K_w )</th>
<th>( K_q )</th>
<th>( K_{\xi_q} )</th>
<th>( G_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{ft} )</td>
<td>( \text{rad} )</td>
<td>( \text{rad} / \text{s} )</td>
<td>( \text{s} )</td>
<td>( \text{rad} )</td>
<td>( \text{s} )</td>
</tr>
<tr>
<td>1000</td>
<td>0.60</td>
<td>0.0002</td>
<td>-0.135</td>
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<td>1.290</td>
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<tr>
<td>20000</td>
<td>0.70</td>
<td>0.0003</td>
<td>-0.216</td>
<td>-0.447</td>
<td>1.286</td>
</tr>
<tr>
<td>40000</td>
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<td>0.0005</td>
<td>-0.543</td>
<td>-0.816</td>
<td>1.724</td>
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<tr>
<td>10000</td>
<td>0.40</td>
<td>0.0006</td>
<td>-0.2803</td>
<td>-0.447</td>
<td>1.906</td>
</tr>
<tr>
<td>30000</td>
<td>0.70</td>
<td>0.0004</td>
<td>-0.257</td>
<td>-0.447</td>
<td>1.541</td>
</tr>
</tbody>
</table>
3 COMPARISON OF BOTH DESIGNS

Having designed both control law an assessment of both have been carried out with respect to dynamic parameters with the help of the transfer function $q/q_d$. The findings of the study are summarized as:

i The feedforward gain of the optimal control law design is always greater than the feedforward gain of the pole placement control law design, that means higher control effort.

ii In the pole placement control law design $K$ and $G$ are the same, i.e. that fact simplifies the implementation.

iii The optimal control law design offers a greater settling time than the pole placement control law design.

iv The optimal control law design has a better performance with respect to CAP than the pole placement control law design.

v The pole placement control law design has a lower bandwidth compared to the optimal control law design.

The pole placement control law design offers also a lower resonant peak than the optimal control law design, and as specified in D'Souza the optimum range is between 0.83 dB and 3.52 dB, so the pole placement control law design meets this requirement while the optimal control law design does not meet.

In figure (3) there is a time response comparison of both designs for flight condition 2.

4 INFLUENCE OF AN ACTUATOR

As the control law designs will be working with an actuator it is useful to analyze what happens if an actuator is included in the model. The actuator model is given by,

$$\begin{bmatrix}
\dot{\eta} \\
\dot{\eta}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-100 & -14
\end{bmatrix} \begin{bmatrix}
\eta \\
\eta
\end{bmatrix} + \begin{bmatrix}
0 \\
100
\end{bmatrix} \eta_c$$ \hspace{1cm} (23)

with a damping ratio of 0.70 and a natural frequency of 10 rad/sec. So with the inclusion of this actuator into both designs an evaluation was carried out and the results can be summarized as: When the actuator is included the pole placement control law design still satisfying the
The results have shown that with phugoid model included as also actuator model the optimal control law design still satisfying the CAP for all flight cases, but not the dropout criterion, by the other side the pole placement control law design does not satisfy the CAP requirement more for practically all flight cases, as also does not satisfy more the dropout criterion for some flight cases. Figure (4) shows the degradation that occurs in both control laws. It have been also noticed that the steady state error, that is the relationship of

\[ \frac{q}{q_d} \right|_{ss} = 1 \]

is practically maintained in the pole placement control law design, but is lost in the optimal control law design. Certainly this can be attributed to the design method of the feedforward gain in each control law.

6 EFFECT OF A SIMPLIFIED CONTROL LAW

Looking for the obtained gains in each design it have been noticed that the magnitude of \( K_w \) is small compared
to the others, so if $K_w$ can be made as zero only two feedback gains will be used, and also it will be not necessary to use an angle of attack sensor, so the flight control system will be simpler and cheaper. A study have been performed with both control law designs using $K_w = 0$, and the results have showed that in this case both designs do not meet the dropback criterion, but both still satisfying the CAP requirement very well. This can be attributed to the fact that the steady state error be dependent much more of $K_w$ and of $K_w$. Figure (5) shows what happens to both control law designs with $K_w = 0$.

In order to get some idea of how robust are each control law design when the gains are varied from the nominal computed gains a study have been performed with both designs, varying the gains to $+20\%$ and to $-20\%$. In this study all the gains are varied at the same time. The study have shown that with respect to meet the dropback criterion the pole placement control law is more robust than the optimal control law when the gains are varied, that is, the pole

**Figure 4 - Phugoid model and actuator influence**

Pitch rate time history comparison for a step input in $q_d$ at 20000 ft, Mach 0.50

**Figure 5 - Pitch rate time history of the augmented aircraft with both designs following a step input of $q_d$ when $K_w = 0$ in both designs.**
placement control law does not meet the criterion with this amount of variation, but the deterioration is small compared with the deterioration that occurs in the optimal control law design. However with respect to CAP the optimal control law still satisfying CAP for all flight cases with both variations, that is + 20 % and - 20 %, instead the pole placement control law does not meet CAP for some flight cases with both variations. Figure 6 illustrates what happens for the pole placement control law design with gain variations and figure 7 shows the same for the optimal control law design.

\[\text{Figure 6 - EFFECT OF GAIN VARIATION ON PPCL DESIGN} \]
pitch-rate time history following a step input in \( q_d \) at 1000 ft Mach 0.30

\[\text{Figure 7 - EFFECT OF GAIN VARIATION ON THE OCL DESIGN} \]
pitch-rate time history following a step input in \( q_d \) at 1000 ft Mach 0.30
optimal control law design is requiring more control rate effort than the pole placement control law design. Table 5 shows a comparative result obtained for a step input of $q_d = 5 \text{ deg/sec}$ for both control law designs.

<table>
<thead>
<tr>
<th>$h$ (ft)</th>
<th>Mach (rad)</th>
<th>$\eta$ (deg/sec)</th>
<th>$\eta$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>PPCL</td>
<td>OCL</td>
</tr>
<tr>
<td>1000</td>
<td>0.30</td>
<td>42.8</td>
<td>54.6</td>
</tr>
<tr>
<td>20000</td>
<td>0.50</td>
<td>33.7</td>
<td>45.3</td>
</tr>
<tr>
<td>40000</td>
<td>0.70</td>
<td>43.6</td>
<td>45.7</td>
</tr>
<tr>
<td>10000</td>
<td>0.30</td>
<td>42.1</td>
<td>57.7</td>
</tr>
<tr>
<td>30000</td>
<td>0.50</td>
<td>38.6</td>
<td>47.7</td>
</tr>
</tbody>
</table>

PPCL = pole placement control law design
OCL = optimal control law design

9 THE REGULATOR PERFORMANCE OF BOTH DESIGNS

It is now useful to analyze the regulator performance of both designs, that is, the ability of each design to eliminate the effect of disturbances. The study was performed by simulating an angle of attack initial disturbance, that is a $\omega(0) \neq 0$, what is commonly called in the aeronautical industry as an alpha release simulation. The study was performed with the short period reduced order model, and the findings are: In this case as there is no pilot input, that is, the reference input $q_d = 0$ here, the pole placement control law design has required higher control effort and control rate effort than the optimal control law design, this was expected, since the feedback gains of the pole placement design are higher than the feedback gains of the optimal control law design. It have been also noticed that the pole placement control law design takes more time

From the study carried out in this work it have been noticed that the optimal control law design is more robust with respect to meet the CAP requirement, as also is better with respect to regulator performance, that is, disturbance rejection. The optimal control law design also requires the same control effort when there is a pilot input, and lower control effort and control rate effort when working as a regulator. The disadvantage of the optimal control law is relative to control rate effort when working with pilot input, that requires higher control rate effort than the pole placement control law design. The pole placement control law design offers a better performance with respect to dropback criterion when the actuator is included or the phugoid model is included, however does not offer a reasonable level of robustness with respect to meet the CAP requirement (stability). It appears that the
optimal control law design is more flexible to be redesigned in order to meet the dropback criterion again when the phugoid model is included or the actuator, in other words, is a design that can accept changes more easily than the pole placement control law design, and continuing to satisfy the requirements.

11 AIRCRAFT DATA

The aircraft data used in this work is summarized here and can be found on Heffley¹.

<table>
<thead>
<tr>
<th>flight condition</th>
<th>20000 ft - Mach 0.70</th>
<th>30000 ft - Mach 0.70</th>
<th>40000 ft - Mach 0.80</th>
</tr>
</thead>
<tbody>
<tr>
<td>data for the complete model</td>
<td>data for the reduced short period model</td>
<td>data for the complete model</td>
<td>data for the reduced short period model</td>
</tr>
<tr>
<td>$A =$</td>
<td>$A =$</td>
<td>$A =$</td>
<td>$A =$</td>
</tr>
<tr>
<td>\begin{bmatrix} -0.0048 &amp; 0.0596 &amp; -21.528 &amp; -32.18 \ -0.1243 &amp; -0.6660 &amp; 732.76 &amp; -0.9717 \ 0.0001 &amp; -0.0018 &amp; -0.7070 &amp; 0.0002 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{bmatrix}</td>
<td>\begin{bmatrix} -0.6660 &amp; 732.76 \ -0.0018 &amp; -0.7070 \end{bmatrix}</td>
<td>\begin{bmatrix} -0.0035 &amp; 0.0480 &amp; -55.220 &amp; -32.09 \ -0.1140 &amp; -0.4800 &amp; 696.70 &amp; -2.5840 \ 0.0001 &amp; -0.0014 &amp; -0.5060 &amp; 0.0003 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{bmatrix}</td>
<td>\begin{bmatrix} -0.4800 &amp; 696.70 \ -0.0014 &amp; -0.5060 \end{bmatrix}</td>
</tr>
<tr>
<td>$B^T =$</td>
<td>$B^T =$</td>
<td>$B^T =$</td>
<td>$B^T =$</td>
</tr>
<tr>
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<td>\begin{bmatrix} -24.400 &amp; -1.4190 \end{bmatrix}</td>
</tr>
</tbody>
</table>
elements of the aircraft longitudinal control matrix B

control effort

control rate effort

aircraft longitudinal state vector

control input to the aircraft

a aircraft longitudinal state space matrix

B aircraft longitudinal control matrix

vector of the feedback control gains

feedforward gain

short period natural frequency

short period damping ratio

g gravity acceleration

numerator term of the transfer function q/η obtained with the reduced order short period model.

steady state velocity of the aircraft

DB dropback

feedback gain relative to w feedback path

feedback gain relative to q feedback path

feedback gain relative to εq feedback path

pitch rate demand to be tracked

control anticipation parameter

integral of the pitch rate error

control rate effort

aircraft closed loop state matrix

Riccati matrix

performance index

state weight matrix

control weight matrix

control weight parameter

Laplace operator

seconds

seconds

radians

13 REFERENCES


