NUMERICAL SOLUTIONS OF 3D TRANSONIC VISCOUS FLOWS
BY USING UPWIND-RELAXATION SWEPPING ALGORITHM

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Abstract

Upwind Relaxation-Sweeping (URS) algorithm is employed to solve the 3D Navier-Stoke equations to study the flow field for a three-dimensional non-axisymmetric transonic nozzle. Oblique shock waves and reflections are captured with the interaction with the wall boundary layers. The computational results agree favorably with the experiment.

Introduction

Multiengineled, highly manoeuvrable jet aircraft must operate efficiently over a wide range of power settings and Mach numbers. Such aircraft requires a propulsion exhaust-nozzle system with a variable geometry for high performance. The application of an axisymmetric nozzle system to a typical multiengined jet configuration produces certain aircraft performance penalties, such as high aft-end drag. Investigations of the effects of nozzle design on twin-engine jet aircraft performance show that a high level of nozzle performance, without considerable aft-end drag, results from use of the nonaxisymmetric nozzle concept. The non-axisymmetric nozzle geometry is more efficiently integrated into the airframe eliminating the boat tail gutter interference. Installation of the nonaxisymmetric nozzle allows design options for trustvectoring and thrust reversing, the capabilities, which improve the manoeuvrability and handling of the aircraft[1,2].

The URS[3,4,5] (Upwind Relaxation Sweeping) algorithm suggested by the authors of this paper is employed for its efficiency and robustness to analyze the internal flow field of a nonaxisymmetric Transonic Nozzle.

Governing Equations

The 3D time-dependent, conservation of mass, Navier-Stokes and energy equations are the governing equations solved numerically in this paper. The nondimensional equations for compressible ideal gas in the absence of external forces, in conservation law form and in Cartesian coordinates are given below

\[ \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = \frac{1}{Re} \left[ \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial T}{\partial z} \right] \quad (1) \]

where

\[ U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ p \end{bmatrix}, \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (e+p)u \end{bmatrix}, \quad G = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ (e+p)v \end{bmatrix}, \quad H = \begin{bmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ (e+p)w \end{bmatrix} \quad (2) \]

\[ R = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ R_s \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ S_s \end{bmatrix}, \quad T = \begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ T_s \end{bmatrix} \quad (3) \]
\[ \tau_{xx} = \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right) \] (4)

\[ \tau_{yy} = \frac{2}{3} \mu \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) \] (5)

\[ \tau_{zz} = \frac{2}{3} \mu \left( \frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \] (6)

\[ \tau_{xy} = \tau_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \] (7)

\[ \tau_{xz} = \tau_{zx} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \] (8)

\[ \tau_{yz} = \tau_{zy} = \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \] (9)

\[ R_s = u \tau_{xx} + v \tau_{yy} + w \tau_{zz} + \frac{\mu}{(\gamma - 1)Pr} \frac{\partial a}{\partial x} \] (10)

\[ S_s = u \tau_{xy} + v \tau_{yy} + w \tau_{zz} + \frac{\mu}{(\gamma - 1)Pr} \frac{\partial a}{\partial y} \] (11)

\[ T_s = u \tau_{xz} + v \tau_{yz} + w \tau_{zz} + \frac{\mu}{(\gamma - 1)Pr} \frac{\partial a}{\partial z} \] (12)

\[ p = (\gamma - 1)(e - \rho(u^2 + v^2 + w^2)/2) \] (13)

where \( \gamma \) is the ratio of specific heats, taken as \( \gamma = 1.4 \).

To discretize the equations using finite volume method, the equations should be written in the integral form. Let

\[ R' = (F - \frac{\alpha}{Re}) \frac{\partial u}{\partial x} i_x + (G - \frac{\alpha}{Re}) \frac{\partial v}{\partial y} i_y + (H - \frac{\alpha}{Re}) \frac{\partial w}{\partial z} i_z \] (14)

Using the Gauss theorem, the integral form of Eq. (1) is

\[ \int_Q \frac{\partial U}{\partial t} dQ + \int_S R' ndS = 0 \] (15)

where \( Q \) is the volume bounded by the surface \( S \) and \( n \) is the outward pointing unit vector normal to the surface expressed as:

\[ n = n_x i_x + n_y i_y + n_z i_z \] (16)

The equations are discretized in the physical domain on the arbitrary body-fitted grid. The flux crossing an interface of two adjacent cells is the normal component of vector \( R' \) given in Eq. (14). Let \( P_1 \) be the inviscid normal component of \( R_{inv} \) passing through unit interface.

where

\[ R_{inv} = F_{ix} + G_{iy} + H_{iz} \] (17)

Thus

\[ P_1 = R_{inv} n = F n_x + G n_y + H n_z \]

\[ \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \end{bmatrix} = U_n + \begin{bmatrix} 0 \\ p n_x \\ p n_y \\ p n_z \end{bmatrix} \] (18)

where \( U_n \) is the normal component of the velocity.
expressed as:

\[ U_n = u_n x + v_n y + w_n z \]  \hspace{1cm}  (19)

Van Leer's Flux Vector Splitting scheme is used to evaluate \( P_i \) at the volume interface\(^{[6,7]}\).

**URS Procedure**

The concept of Upwind Relaxation-Sweeping is to select a direction with relatively smaller variable gradients as the global sweeping direction and to implement the local relaxation iteration on the control volume block in the main flow direction. The flow field is calculated by a series of global alternating outward/inward sweeps in the sweeping direction with the local forward/backward Gauss-Seidel iteration on each stream wise plane, one global sweep per time step. The global sweeping is also the time marching procedure.

Upwind differencing (Van Leer’s FVS) is used for the convective and pressure terms and central differencing for the shear stress and heat flux terms. Third order MUSCL-type differencing\(^{[8]}\) is used to evaluate the interface flux.

Suppose the sweeping direction is in z-direction with the index k increasing. Discretize the governing equations (15) nearly fully implicitly for the inviscid terms and explicitly for the viscous terms, we have:

\[
\frac{U_{ij,k}^{n+1} - U_{ij,k}^n}{\Delta t} Q_{ij,k} + (P_{i,j-1/2,k}^{+1} + P_{i,j+1/2,k}^{-1}) S_{i,j-1/2}^{1/2}
\]

\[+ (P_{i-1/2,j,k}^{+1} + P_{i+1/2,j,k}^{-1}) S_{i-1/2,j}^{1/2} + (P_{i,j+1/2,k}^{+1} + P_{i,j-1/2,k}^{-1}) S_{i,j+1/2}^{1/2}
\]

\[+ (P_{i-1/2,j,k}^{+1} + P_{i+1/2,j,k}^{-1}) S_{i+1/2,j}^{1/2} + (P_{i,j+1/2,k}^{+1} + P_{i,j-1/2,k}^{-1}) S_{i,j-1/2}^{1/2}
\]

\[+ (P_{i-1/2,j,k}^{+1} + P_{i+1/2,j,k}^{-1}) S_{i,j}^{1/2} = RHS_{viscous}^n \]  \hspace{1cm}  (20)

It is noted that \( P_{i,j,k}^{+1/2} \) is discretized explicitly. To make the solution independent of the time step size, the implicit terms should be changed to Delta-form. To construct the implicit operator for Gauss-Seidel iteration, the Delta-form is only implemented for the terms with the same k index.

One implicit term left on the LHS is moved to RHS. Eq. (20) is then changed to:

\[
\frac{1}{\Delta t} Q + \sum \left( \left( \frac{dP}{dU} \right)_k^+ - \left( \frac{dP}{dU} \right)_k^- \right) \delta^+ U_i^{n+1} - \delta^- U_i^n = RHS_{invicid}^n + RHS_{viscous}^n
\]  \hspace{1cm}  (21)

where:

\[ \delta U_i^n = U_i^{n+1} - U_i^n, \Delta t^n = t^{n+1} - t^n \] and \( n \) is the iteration index.

\[ RHS_{invicid}^n = - \left[ \left( P_{i,j,k}^{+1} + P_{i,j,k}^{-1} \right) S_{i,j}^{1/2} \right.
\]

\[+ \left( P_{i+1/2,j,k}^{+1} + P_{i-1/2,j,k}^{-1} \right) S_{i,j-1/2}^{1/2} + \left( P_{i,j+1/2,k}^{+1} + P_{i,j-1/2,k}^{-1} \right) S_{i,j+1/2}^{1/2}
\]

\[+ \left( P_{i+1/2,j,k}^{+1} + P_{i-1/2,j,k}^{-1} \right) S_{i+1/2,j}^{1/2} + \left( P_{i,j+1/2,k}^{+1} + P_{i,j-1/2,k}^{-1} \right) S_{i,j-1/2}^{1/2}
\]

\[+ \left( P_{i+1/2,j,k}^{+1} + P_{i-1/2,j,k}^{-1} \right) S_{i,j}^{1/2} \left( P_{i,j,k}^{+1} + P_{i,j,k}^{-1} \right) S_{i,j,k}^{1/2}
\]

\[+ \left( P_{i+1/2,j,k}^{+1} + P_{i-1/2,j,k}^{-1} \right) S_{i+1/2,j,k}^{1/2} + \left( P_{i,j+1/2,k}^{+1} + P_{i,j-1/2,k}^{-1} \right) S_{i,j+1/2,k}^{1/2}
\]

\[+ \left( P_{i+1/2,j,k}^{+1} + P_{i-1/2,j,k}^{-1} \right) S_{i+1/2,j,k}^{1/2} \]  \hspace{1cm}  (22)

To keep the diagonal dominance and save computational work, first order differencing is used for the implicit terms and therefore the matrix \( M \) is penta-diagonal. The matrix equation for the cell \((i,j,k)\) therefore can be written as:

\[ B \delta U_{i,j,k}^{n+1} + A \delta U_{i,j,k}^n + C \delta U_{i-1,j,k}^{n+1} + E \delta U_{i,j-1,k}^{n+1} \]

\[+ D \delta U_{i,j+1,k}^{n+1} = RHS_{invicid}^n + RHS_{viscous}^n \]  \hspace{1cm}  (23)

where the coefficients \( A, B, C, D \) and \( E \) are 5x5 block
matrices. The line Gauss-Seidel iteration is employed to
inverses the matrix at the block composed of the cells with
the same k index. Two sweeps are implemented at the
local block k, one forward and the other backward.
Implementation of only two sweep iterations is an
approximation. After two local sweeps, the global
iteration proceeds to the next block at k+1. The global
(iteration) starts with the inner solid wall with increasing
k index. The variables are updated at each block soon
after the two local sweeps for Gauss-Seidel iteration are
completed. When the global iteration sweeps up to the
outside solid wall, all the variables at time level n+1 are
obtained and then the sweeping direction is reversed with
decreasing k index to continue the iteration for the next
time step. It is believed that the iteration with forward
and backward sweeps at the local block, and outward and
inward globally is beneficial for the information travelling
through the entire flow field three-dimensionally and
therefore rapidly. It is noted that $RHS^{n+1}_{k+1}$ in Eq. (22)
is not evaluated completely by using the variables at time
level n and contains one term, $P^{n+1}_{i,j,k-\frac{1}{2}}$, which is
available due to the completion of the iteration at the k-1
block. When the global iteration sweeps in the direction
with the decreasing k index, the explicit term in Eq. (20)
is $P^{n}_{i,j,k-\frac{1}{2}}$ instead of $P^{n}_{i,j,k+\frac{1}{2}}$; the implicit term at
time level n+1 in Eq. (22) is $P^{n+1}_{i,j,k+\frac{1}{2}}$ instead of
$P^{n+1}_{i,j,k-\frac{1}{2}}$. Supposing each interface sub-flux has the
same weight, the URS method is thus 11/12 implicit and
1/12 explicit. The URS algorithm is proven to be
unconditionally stable. The gain from the 1/12 explicit
discretization is significant: 1) the Jacobians and $\delta U^{n+1}$
are only stored in one plane on which Gauss-Seidel iteration
is implemented and therefore the whole storage is greatly
reduced; 2) only one global sweep per time step is needed
to solve all the unknowns in a time level and therefore the
CPU time per time step is saved.

The accuracy of the converged solution is controlled by
RHS. Even though the inviscid terms are evaluated by
using third order accuracy, the general accuracy of the
solution for the 3D Navier-Stokes equations is second
order since central differencing is used for the viscous
terms.

Results

To test the accuracy of the computer code, the
flat plate Blasius problem is solved first with Reynolds
number 10000 at Mach number equal to 0.3. The grid is
101x31x51 and the cell Reynolds number 0.18 for the first
grid on the wall and 2.0 for the second grid. The velocity
profile and surface friction coefficient agree well with the
analytical solution as shown in Fig. 1.

Figure 2 is the half geometry of the nozzle with the grid
size of 101x31x51. Only the half nozzle is calculated due
to the symmetry of the geometry. Figure 3 presents the
Mach number contours of the flow field at different
locations in z-direction. In the middle plane (Z/L=0.0),
there is an oblique shock wave right after the throat. The
shock wave are weak because the Mach number is only
slightly greater than one after the throat. The two shock
waves extending from the lower and upper wall intersect
at the center-line of the nozzle and then reach the wall of
the other side. After approaching the wall, the oblique
shock waves reflect. These reflections intersect and reflect
again until the flow approaches the exit of the nozzle. It
can be seen from Fig. 3 that the intensity of the shock
waves becomes weaker with repeated intersections and
reflections. This can be also quantitatively seen from the
pressure distributions given in Fig. 4. Fig. 4 (a), (b), (c)
are at the bottom wall with different locations from the
center-line (Z/L=0.0) to the one near the side wall (Z/L=0.875). Fig.4 (d) is the pressure distribution along
the side wall center-line. Most of the computational points
agree very well with the experiment except the point at
the first shock reflection. The first reflection calculated
seems to be not strong enough. Finer grid may be needed
to obtain the better resolution of the reflection. The
flow fields are highly three-dimensional in the vicinities
of the corners due to the boundary layer interaction
between the side walls and the lower or upper walls,
particularly after the throat, where the main flow is
supersonic. Fig. 5 shows the spanwise section velocity
vector fields for nozzle A1. Fig. 5 indicates that there is
very weak secondary flow before the throat where the
flow is subsonic and the secondary flow becomes much
stronger after the throat where the flow is supersonic.
Fig. 5 (b), (c), (d) and (e) show that the cross velocity
vector fields are different at different locations. The cross
flow is at the corners for Fig. 5 (b) where the oblique
shock wave after the throat just grows up. Fig. 5 (c)

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shows that there exist cross flow near the center plane in y-direction. It is made by the shock wave and side wall boundary layers interactions, where the shocks from lower and upper wall intersect. After the shock intersection, the cross flow is again concentrated in the corners as shown in Fig. 5 (d).

The maximum CFL number used reaches the order of $10^7$. Rapid convergence rate has been obtained. Four magnitude order of the residual is reduced by using less than 200 iterations.

**Conclusion**

The viscous flow field of a three-dimensional non-axisymmetric transonic nozzle is numerically analyzed. URS algorithm is employed to solve the 3D Navier-Stoke equations. Oblique shock waves and reflections are captured with the interaction with the wall boundary layers. There is strong secondary flow after the throat in the supersonic region in the corner of the side and bottom walls due to the shock wave/boundary layer interaction. The computed pressure distributions agree well with the experiment.

**References**


Fig. 1 Results for Blasius Flow

Fig. 2 The Geometry and Grid of the Nozzle

Fig. 3 Mach Number Contours of the Nozzle
Fig. 4 Pressure Distributions of the Nozzle
- Experiment, – Calculation

Fig. 5 Spanwise Velocity Fields at Different Locations