WEIGHT OPTIMIZATION OF WING STRUCTURES ACCORDING TO THE GRADIENT METHOD*

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Abstract

For the design of wing structures with optimal weight the gradient method is applied due to the different constraints (stresses, flutter speed). The theory and the computer program are described. As an example, an idealized wing consisting of bending/torsion bar elements is presented for which the stresses as well as the flutter speed are active restrictions.

1. Introduction

The rapid development of large digital computers in the past 20 years has completely changed the structure calculation methods in aircraft construction. While the simple beam theory is applied only for wings without sweep and with high aspect ratios, modern aircraft are largely designed using the finite element method (FEM). As long as the static stresses are to a high degree decisive for dimensioning it is often tried to modify the cross section dimensions by a few FEM calculations carried out subsequently in such a way that the maximum permissible stress is achieved in each element in one of the load cases.

$$F_{i+1} = F_i \frac{\sigma_{i\,act}}{\sigma_{i\,adm}} \quad \text{for stringers} \qquad (1)$$

$$t_{i+1} = t_i \frac{\sigma_{i\,act}}{\sigma_{i\,adm}} \quad \text{for membranes} \qquad (2)$$

The design variables in this case are the stringer cross-sectional areas F and the membrane thicknesses t (in longerons, ribs and the skin). This method is called "fully stressed design" (FSD) and belongs to the class of the optimality criteria. Its advantages are economic calculations and uncomplicated handling. The main disadvantage is based on the fact that for statically undetermined systems an exact weight minimum cannot be forced in spite of numerous iterations. The reason for this is that the request for weight minimum does not directly enter the calculation and the cross section values are only modified by the ratio of actual to admissible stresses.

Due to internal force rearrangements it is very well possible that the statically undetermined structure with the optimum weight is built up in such a way that stresses below $\sigma_{admissible}$ occur in some elements. A second considerably more important disadvantage of the FSD is the unsufficient suitability for additional constraints of a different character, such as maximum deformation and flutter speed.

2. Formulation of the Optimization Task

Let us suppose that the structure of a wing is given by the number and position of the stringers, longerons, ribs as well as the skin. The optimization of the configuration involved in this determination is not discussed here. For the optimization of the structure the cross sections or the thicknesses of the elements of the idealized calculation model are required while certain constraints have to be fulfilled and the weight minimum of the structure has to be found.

For dimensioning the wing we start out from the two most important constraints

- static stresses $\sigma \leq \sigma_{adm}$ (3)
- flutter speed $v_F \geq v_{ll}$ (4)

The index ll means "lower limit". These constraints and the structural weight W are functions of the design variables.

$$\sigma = \sigma\,(x_1, x_2, \ldots, x_m) \qquad (5)$$

$$v_F = v_F(x_1, x_2, \ldots, x_m) \qquad (6)$$

$$W = W\,(x_1, x_2, \ldots, x_m) \qquad (7)$$

The modification of the limited values with partial linearization is expressed as follows:

$$d\sigma_i = \frac{\partial\sigma_i}{\partial x_1} dx_1 + \frac{\partial\sigma_i}{\partial x_2} dx_2 + \ldots + \frac{\partial\sigma_i}{\partial x_m} dx_m \qquad (8)$$

Fig. 1 shows the approximated linearization for the individual stress σ depending on the design variable x.

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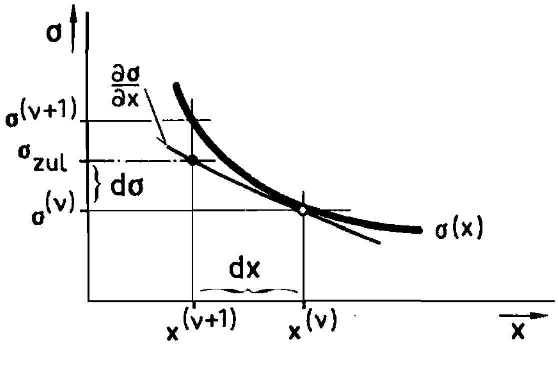

Fig. 1: Linearization of the stress

In the same way the modification of the flutter speed can be expressed

$$dv_F = \frac{\partial v_F}{\partial x_1}dx_1+\frac{\partial v_F}{\partial x_2}dx_2+\ldots+\frac{\partial v_F}{\partial x_m}dx_m \quad (9)$$

The gradients are combined in a matrix.

$$[\nabla\sigma]=[\frac{\partial\{\sigma\}}{\partial x_1}\ \frac{\partial\{\sigma\}}{\partial x_2}\cdots\frac{\partial\{\sigma\}}{\partial x_m}]=\begin{bmatrix}\frac{\partial\sigma_1}{\partial x_1} & \frac{\partial\sigma_1}{\partial x_2}\cdot\cdot & \frac{\partial\sigma_1}{\partial x_m}\\ \frac{\partial\sigma_2}{\partial x_1} & \frac{\partial\sigma_2}{\partial x_2}\cdot\cdot & \frac{\partial\sigma_2}{\partial x_m}\\ \vdots & & \\ \frac{\partial\sigma_n}{\partial x_1} & \frac{\partial\sigma_n}{\partial x_2}\cdot\cdot & \frac{\partial\sigma_n}{\partial x_m}\end{bmatrix} \quad (10)$$

$$\{\nabla v_F\} = \{\frac{\partial v_F}{\partial x_1}\ \frac{\partial v_F}{\partial x_2}\ \ldots\ \frac{\partial v_F}{\partial x_m}\} \quad (11)$$

With the modifications of the design variables

$$\{\Delta x\} = \{\Delta x_1\ \Delta x_2\ \ldots\ \Delta x_m\} \quad (12)$$

the restriction system is represented by

$$\{\sigma^{(\nu)}\}+[\nabla\sigma^{(\nu+1)}]\{\Delta x^{(\nu+1)}\} \leqq \{\sigma_{adm}\} \quad (13)$$

$$v_F^{(\nu)}+\{\nabla v_F^{(\nu+1)}\}^t\{\Delta x^{(\nu+1)}\} \geqq v_{Fll} \quad (14)$$

with (ν) respectively $(\nu+1)$ characterizing the iteration step.

The following formulas are valid for the weight function:

$$\{\nabla W\} = \{\frac{\partial W}{\partial x_1}\ \frac{\partial W}{\partial x_2}\ \ldots\ \frac{\partial W}{\partial x_m}\} \quad (15)$$

$$W^{(\nu+1)} = W^{(\nu)}+\Delta W^{(\nu+1)}=W^{(\nu)}+\{\nabla W\}^t\{\Delta x^{(\nu+1)}\} \quad (16)$$

The objective function can be restricted to the following form:

$$\Delta W^{(\nu+1)} = \{\nabla W\}^t\{\Delta x^{(\nu+1)}\} \overset{!}{=} \text{Min} \quad (17)$$

The structural weight can be changed in the positive or negative direction depending on the constraints. The demand for a minimum total weight leads to the highest possible, negative weight modification ΔW of equation (17).

3. The Formation of Gradients

The stress gradient

The relation between the stresses in the elements and the deformations of the total system is given by:

$$\{\sigma\} = [D][T]\{q\} \quad (18)$$

where [T] is the transformation matrix by means of which the deformations in the basic coordinate system are transferred to the element coordinate system. The elements of matrix [D] are corresponding to the strains in the elements.

As the matrices [D] and [T] are independent of the design variables, the stress gradient can be written

$$\frac{\partial\{\sigma\}}{\partial x_j} = [D][T]\ \frac{\partial\{q\}}{\partial x_j} \quad (19)$$

With $$[\nabla q] = [\frac{\partial\{q\}}{\partial x_1}\ \frac{\partial\{q\}}{\partial x_2}\ \ldots\ \frac{\partial\{q\}}{\partial x_m}] \quad (20)$$

and the deviation of the basic equations for statically loaded structures with respect to the design variables

$$\frac{\partial[K]}{\partial x_j}\{q\} + [K]\frac{\partial\{q\}}{\partial x_j} = 0 \quad (21)$$

$$\frac{\partial\{q\}}{\partial x_j} = -\ [K]^{-1}\ \frac{\partial[K]}{\partial x_j}\{q\} \quad (22)$$

the restriction system reads:

$$[\nabla\sigma] = [D][T][\nabla q] \quad (23)$$

[K] is the stiffness matrix of the structure.

σ is generally used for the normal and shear stresses. If the reference stresses in the membranes, e.g. according to an energy hypothesis

$$\sigma_v = \sqrt{\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2} \qquad (24)$$

are limited the following extension is necessary:

$$d\sigma_v = \frac{\partial \sigma_v}{\partial \sigma_x} d\sigma_x + \frac{\partial \sigma_v}{\partial \sigma_y} d\sigma_y + \frac{\partial \sigma_v}{\partial \tau_{xy}} d\tau_{xy} \qquad (25)$$

$$\frac{\partial \sigma_v}{\partial \sigma_x} = \frac{1}{2\sigma_v} (2\sigma_x - \sigma_y) \qquad (26)$$

$$\frac{\partial \sigma_v}{\partial \sigma_y} = \frac{1}{2\sigma_v} (2\sigma_y - \sigma_x) \qquad (27)$$

$$\frac{\partial \sigma_v}{\partial \tau_{xy}} = \frac{3\tau_{xy}}{\sigma_v} \qquad (28)$$

$$[\nabla \sigma_v] = [\frac{\partial\{\sigma_v\}}{\partial x_1} \frac{\partial\{\sigma_v\}}{\partial x_2} \cdots \frac{\partial\{\sigma_v\}}{\partial x_m}] \qquad (29)$$

$$\text{with} \quad \frac{\partial\{\sigma_v\}}{\partial x_j} = \lceil\frac{\partial \sigma_v}{\partial \sigma_x}\rfloor \frac{\partial\{\sigma_x\}}{\partial x_j} + \lceil\frac{\partial \sigma_v}{\partial \sigma_y}\rfloor \frac{\partial\{\sigma_y\}}{\partial x_j} + \lceil\frac{\partial \sigma_v}{\partial \tau_{xy}}\rfloor \frac{\partial\{\tau_{xy}\}}{\partial x_j} \qquad (30)$$

For the calculation of the shear stresses resulting from shearing forces and torsion the matrix [D] includes static moments and stiffness values which are dependent of the design variables x.

$$[\nabla\sigma] = [D][T][\nabla q] + [\nabla D][T][q] \qquad (31)$$

The Flutter Gradient

The characteristic equation of the flutter problem assuming steady-state oscillations and a state of neutral stability may be expressed in the form

$$[[K] - \lambda[M] - \frac{1}{2}\varrho V^2[L]]\{q\} = \{0\} \qquad (32)$$

where [K] = stiffness matrix, [M] = inertia matrix, [L] = air-force matrix. V is the speed and ϱ the air density.

The flutter speed is calculated on the basis of this eigenvalue problem by running through the speed range step by step. The speed at which the real portion of the eigenvalue becomes zero (or positive), for the first time, i.e. the damping disappears or an excitation starts, is the required flutter speed.

The flutter equation can be deviated with respect to the design variables, converted and simplified by the introduction of the left eigenvectors {p} and the reduced frequencies $k = \frac{\omega}{V}$. According to Rudisill and Bhatia [1] the division into a real and an imaginary part (λ and k are real in the steady state oscillation in the flutter point) permits the following solution:

$$\frac{\partial V}{\partial x_j} = \frac{(R_1 - R_2 \frac{\partial\lambda}{\partial x_j})R_3 + (I_1 - I_2 \frac{\partial\lambda}{\partial x_j})I_3}{R_3^2 + I_3^2} \qquad (33)$$

$$\text{with} \quad \frac{\partial\lambda}{\partial x_j} = \frac{R_1 I_3 - R_3 I_1}{R_2 I_3 - R_3 I_2} \qquad (34)$$

$$R_1 + iI_1 = \{p\}^t(\frac{\partial[K]}{\partial x_j} - \lambda\frac{\partial[M]}{\partial x_j})\{q\} \qquad (35)$$

$$R_2 + iI_2 = \{p\}^t([M] + \frac{\varrho}{4k}\frac{\partial[L]}{\partial k})\{q\} \qquad (36)$$

$$R_3 + iI_3 = \omega\{p\}^t(\frac{\varrho}{k}[L] - \frac{\varrho}{2}\frac{\partial[L]}{\partial k})\{q\} \qquad (37)$$

4. Solution Method

The special minimum task to be solved consists of the minimization of the weight function (see equation (17))

$$\underset{(1xm)}{\{\nabla W\}^t} \underset{(mx1)}{\{\Delta x\}} \overset{!}{=} \text{Min} \qquad (38)$$

observing the constraints (see equations (13) and (14))

$$\underset{(nxm)}{[G]} \underset{(mx1)}{\{\Delta x\}} \lesseqgtr \underset{(nx1)}{\{r\}} \qquad (39)$$

The gradient matrix [G] is a measure for the size and direction of the design variable modifications. The vector {r} includes the differences between actual and limit values for the limited stresses and flutter speed.

The simplex algorithm for the solution of this minimum task within one iteration step presupposes that

1) the solutions are greater than or equal to zero

$$\Delta x_j \geq 0, \qquad (40)$$

2) more unknown variables than constraints are available.

Re 1:

This requirement can be fulfilled by a coordinate transformation in the Δx values:

$$0 \leq (\Delta x - \Delta x'_{l1}) = \overline{\Delta x} \leq \Delta x_{u1} - \Delta x_{l1} \qquad (41)$$

Hence follows for the restriction system

$$[G]\{\overline{\Delta x}\} \lessgtr \{\bar{r}\} \quad (42)$$

$$\text{with} \quad \{\bar{r}\} = \{r\} - [G]\{\Delta x_{ll}\} \quad (43)$$

For the objective function the transformation may remain unconsidered since it results only in a deviation by the constant value $\{\nabla W\}^t\{\Delta x_{ll}\}$. The meaning of the indices ll und ul is "lower limit" and "upper limit".

Re 2:

This requirement is always fulfilled simultaneously with the conversion of the inequations of the restriction system into a system of equations. For this purpose an additional variable (slag variable) x_z (i.e. n as a total) is added or subtracted (depending on the direction of the inequality sign in equation (42)) so that the following equality is valid:

$$\underset{(nxm)}{[[G]} \vdots \underset{(nxm)}{[I]]} \underset{(m+n)x1}{\begin{Bmatrix} \{\overline{\Delta x}\} \\ \{x_z\} \end{Bmatrix}} = \underset{(nx1)}{\{r\}} \quad (44)$$

The objective function is also concerned by this extension.

$$\underset{(1xm)}{\{\{\nabla W\}} \vdots \underset{(1xn)}{\{W_z\}\}^t} \underset{(m+n)x1}{\begin{Bmatrix} \{\Delta x\} \\ \{x_z\} \end{Bmatrix}} \overset{!}{=} \text{Min} \quad (45)$$

For convergence reasons it is recommended to select great values for $\{W_z\}$.

For the iterative solution of the system of equation (44)

$$\underset{nx(m+n)}{[A]}\{x\} = \underset{(nx1)}{\{r\}} \quad (46)$$

and the minimum condition (45)

$$\underset{1x(m+n)}{\{C\}^t}\{x\} \overset{!}{=} \text{Min} \quad (47)$$

we select n basic variables $\{x_B\}$ and m slip variables $\{x_N\}$ and build matrix [A] accordingly

$$\underset{(nxn)}{[[A_B]} \vdots \underset{(nxm)}{[A_N]]} \begin{Bmatrix} \{x_B\} \\ \{x_N\} \end{Bmatrix} = \underset{(nx1)}{\{r\}} \quad (48)$$

The elements of vector $\{x_B\}$ can be expressed by the slip variables $\{x_N\}$.

$$[A_B]\{x_B\} + [A_N]\{x_N\} = \{r\} \quad (49)$$

$$\{x_B\} = [A_B]^{-1}\{r\} - [A_B]^{-1}[A_N]\{x_N\} \quad (50)$$

In this step the basic variables are verified (1st requirement). If $x_{Bi} < 0$ repetition of the process with new selection of the basic and slip variables is necessary. If $x_{Bi} \geqq 0$ calculation of the objective function C

$$\{\{c_B\}^t \vdots \{c_N\}^t\} \begin{Bmatrix} \{x_B\} \\ \{x_N\} \end{Bmatrix} = C \quad (51)$$

and replacement of $\{x_B\}$ by $\{x_N\}$ is done. This results in the following form

$$C = c_0 + c_{N1}x_{N1} + c_{N2}x_{N2} + \ldots + c_{Nm}x_{Nm} \overset{!}{=} \text{Min} \quad (52)$$

If all $c_{Ni} \geqslant 0$ the minimum of the special minimum task is achieved. In the case that one or several $c_{Ni} < 0$ the value of C decreases if x_{Ni} increases; and the minimum is thus not achieved. The repetition of the systematically modified basic and slip variables takes place in a so-called inner loop of the program.

The simplex algorithm can be represented geometrically according to fig. 2.

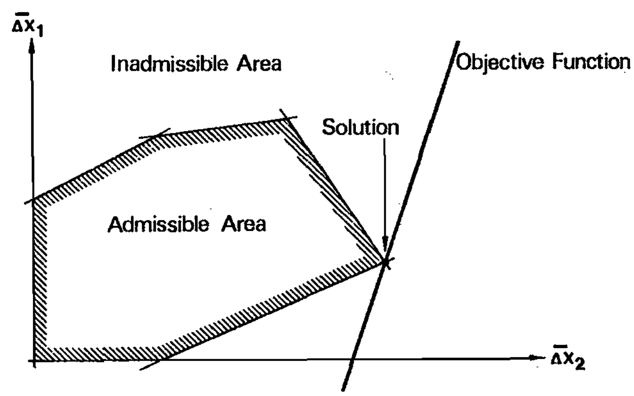

Fig. 2: Geometric description of the simplex algorithm

In the diagram a multidimensional problem is represented in the plane by two of the basic variables. Restrictions form the boundaries of an area of permissible solutions. The objective function is a surface which is displaced towards the area of the permissible solution until it touches it in one corner while its normal and tangent vectors are maintained. The direction from which the objective function is moved towards the solution area depends on whether a minimum or a maximum is required.

Due to the linearization of the problem and the step width limitation (0,1 = Δx/x = 0,4) the solution of the simplex algorithm represents only one step on the way to a structure with the optimal weight. The block diagram (see fig. 3) shows how the final result is calculated by multiple repetition of the external loops.

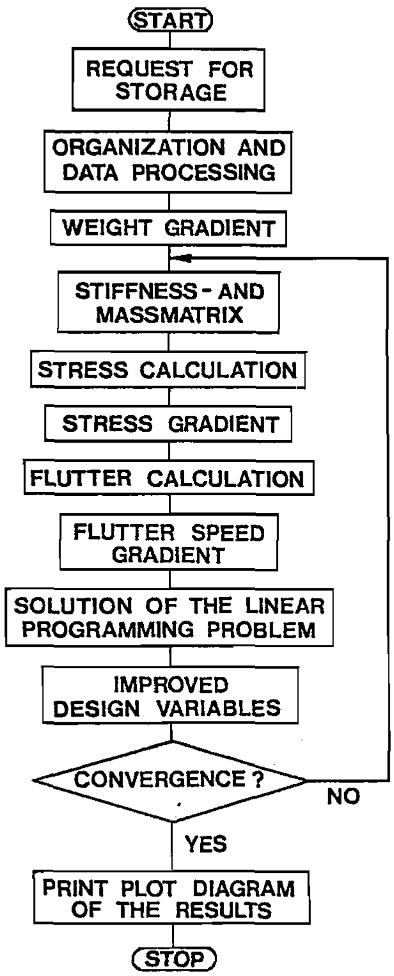

Fig. 3: Flow chart of the DYNOPT calculation

5. Example: Beam model wing

Within the task to modify an aircraft the wing is investigated with respect to a possible weight reduction. The constraints for the weight minimization are maximum stresses and a lower limit for the flutter speed. The buckling is considered by approximation using lower permissible stresses which decrease from the wing root to the wing tip. Due to the changed rib and stringer distances these stresses are higher than in the original design.

As a static load case the positive symmetrical pull-out manoeuvre was selected which due to the wing load belongs to the dimensioning load cases.

The following limits for the stresses and the flutter speed were fixed:

Element	$\sigma_{max} \left[\frac{N}{cm^2}\right]$	$\tau_{max} \left[\frac{N}{cm^2}\right]$
101	$35 \cdot 10^3$	$18 \cdot 10^3$
102	$32 \cdot 10^3$	$15 \cdot 10^3$
103	$29 \cdot 10^3$	$15 \cdot 10^3$
104	$26 \cdot 10^3$	$12 \cdot 10^3$
105	$23 \cdot 10^3$	$12 \cdot 10^3$
106	$20 \cdot 10^3$	$10 \cdot 10^3$

The minimum flutter speed is 308 m/sec. For the modulus of elasticity the values for aluminium are used.

The calculation model is a swept-back system consisting of bending/torsion bars fixed at the wing root (see fig. 4).

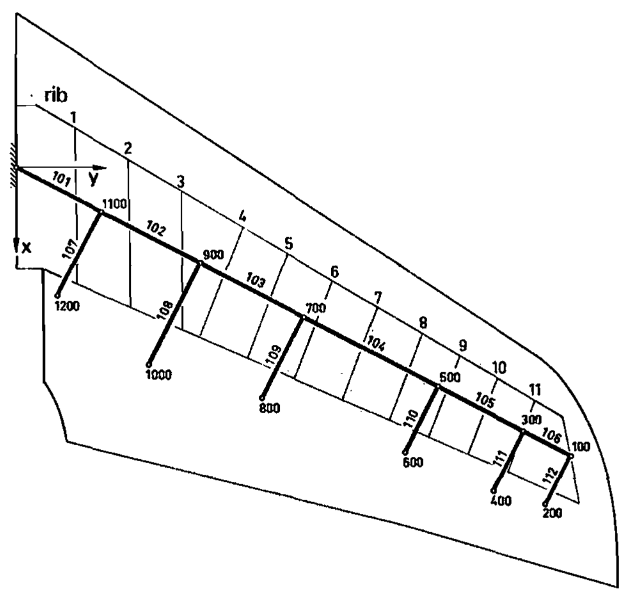

Fig. 4: Idealization of a wing structure with 12 beam elements

The position of the elastic axis of the wing is the 37% line behind the leading edge of the wing.

The bending/torsion bars 101 to 106 are situated on this line. In the course of the weight minimization they are varied with respect to the wall thickness t and the additional stringer wall thickness t_s (see fig. 5 and 6).

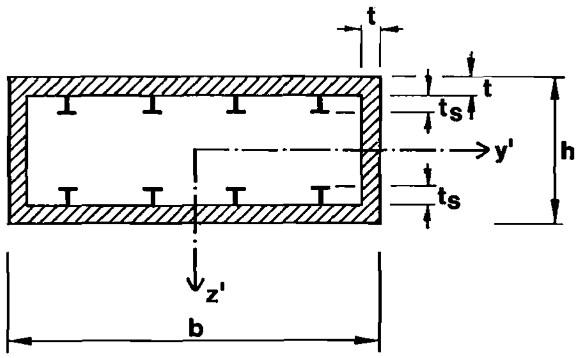

Fig. 5: Cross-section of the beam element

Element	h	b	t	t_S
101	18,3	90	0,52	0,28
102	19,7	82	0,37	0,1
103	15,0	74	0,4	0,12
104	11,0	66	0,45	0,02
105	8,5	58	0,41	0,05
106	10,5	50	0,22	0,13

Fig. 6: Initial values for the thickness of the profiles (in cm)

The beam elements 107 to 112 are used only for the connection of eccentric masses to the system. Their rigidity values remain unchanged during the iteration process. The total system thus has 12 design variables.

The static model for the stress calculations has 36 degrees of freedom. A z-displacement per node as well as torsions around the x- and the y-axis are permitted. The dynamic model for the flutter calculation includes the z-displacements in the nodes on the elastic axis as well as the torsions around the y-axis.

The results of the optimization are the bending and torsion stiffnesses plotted against the wing span (see fig. 7 and 8). They were obtained after eleven iterations which had started from an initial design with rather stiff elements. The weight saving as compared to the initial wing is about 14% with all constraints considered. With the exception of element 106 all the normal stresses achieve their permissible limits. The shear stresses remain below their permissible limits. A further reduction of the flange and web thickness t would increase the shear stresses, but regarding the flutter speed limit the system would become too soft with respect to torsion.

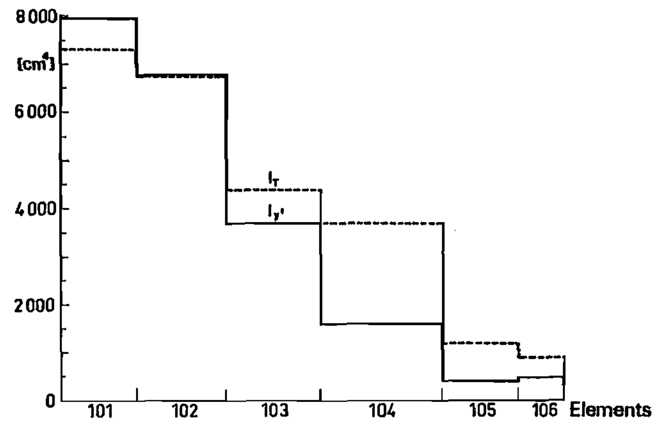

Fig. 7: Final distribution of stiffness due to bending and torsion

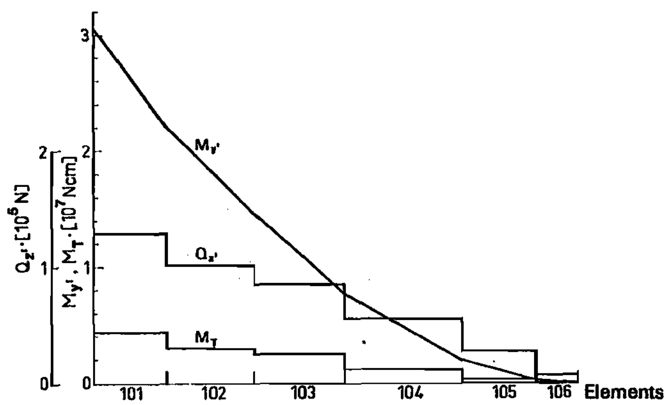

Fig. 8: Bending moment, torsion and shearing force over the wing span

Fig. 9 shows a monotonously decreasing structural weight curve against the iterations. The convergence is good. As desired the weight curve approaches a horizontal line. From the fifth to the sixth iteration step the weight reduction is only 0,3%. The optimization calculation could have been interrupted at this point since, in the following, only the rigidity values t and t_s of each element are rearranged. By indicating a minimum wall thickness t or the ratio t/t_s this rearrangement could be stopped earlier.

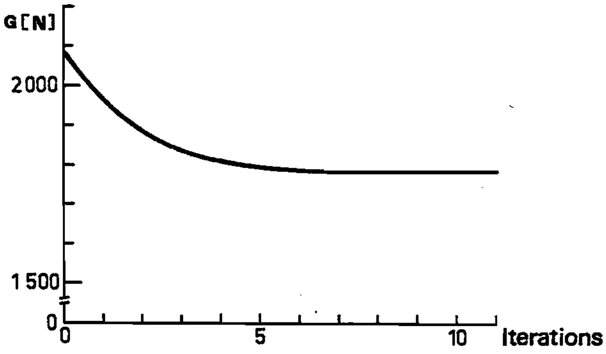

Fig. 9: Structural weight against iterations

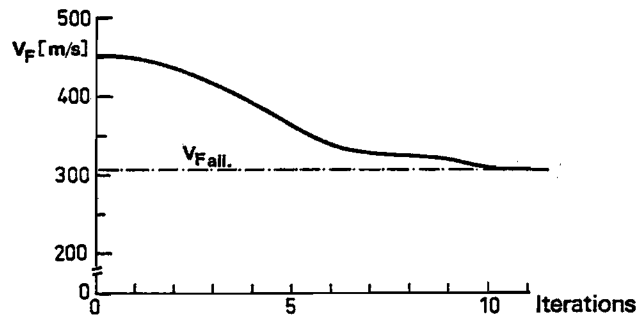

Fig. 10: Flutter speed against iterations

The figures 10 to 14 show that, as a consequence of the decreasing wall thickness t or the reduction of the torsional rigidity, also the flutter speed approaches the lowest permissible value.

During the entire optimization process the structure is in the permissible range with respect to all constraints.

7. Final consideration

The particular features of the structure optimization according to the gradient method are as follows:

- all constraints are considered simultaneously and with the same priority, whereby a real weight minimum can be achieved;
- a great number of calculation steps and a detailed knowledge of the procedures are required.

With the increasing extension of the computers and reduction of the calculation costs this procedure becomes more and more important.

Further development in view of element types, constraints, and linked variables as well as efforts to increase the profitability are necessary. In spite of the optimization calculation it will not be possible, even in the far future, to do without the experienced engineer!

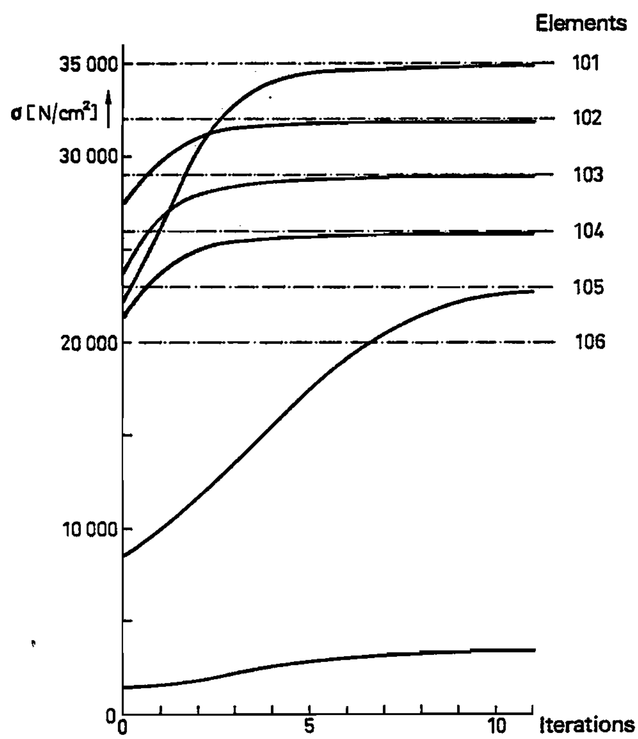

Fig. 11: Normal stresses in the elements

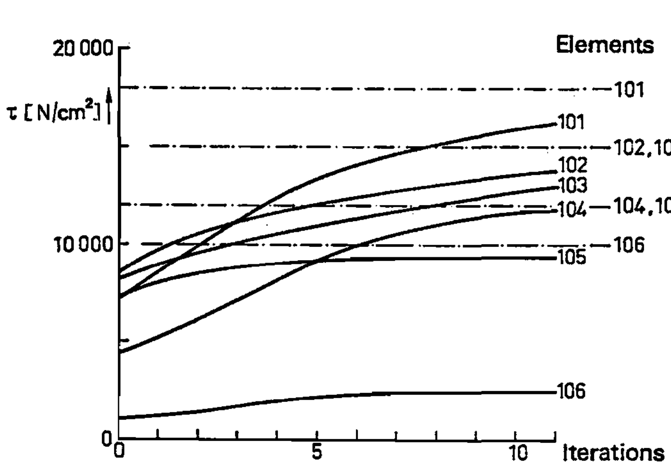

Fig. 12: Shear stresses in the elements

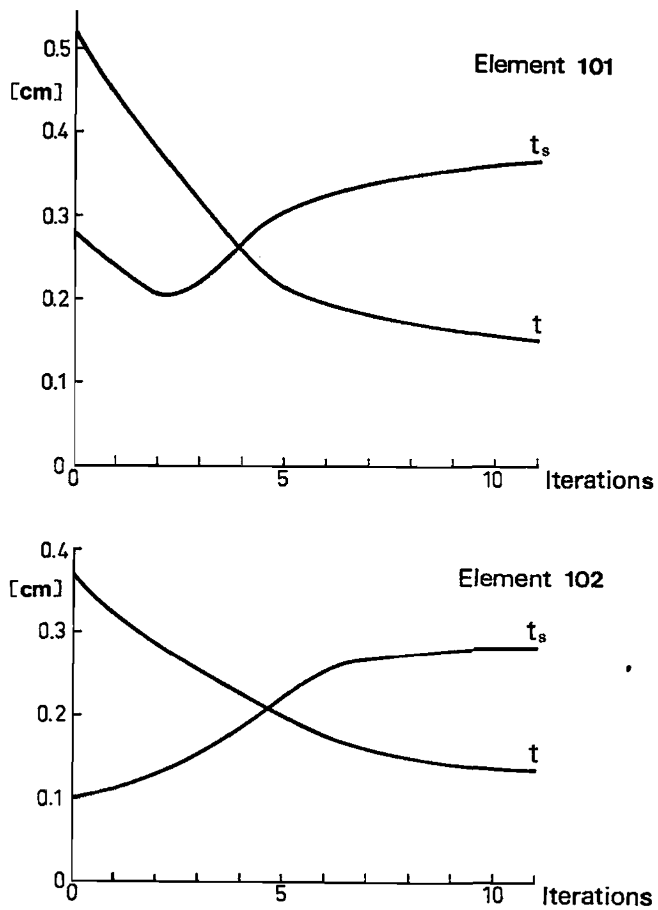

Fig. 13: Thickness of the flanges in the elements

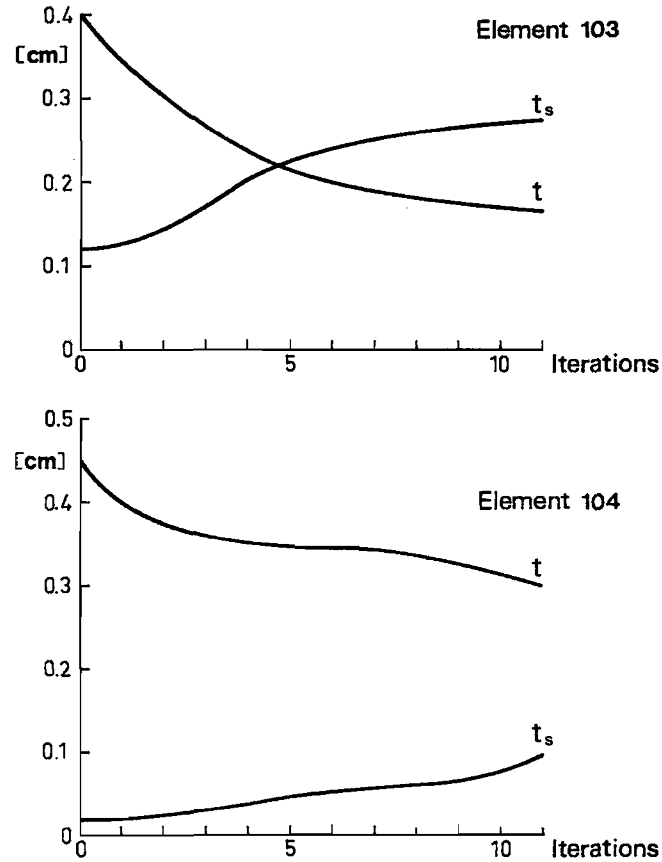

Fig. 14: Thickness of the flanges in the elements

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