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APPLICABILITY OF THE DOUBLE LINEAR DAMAGE RULE TO DURAL TYPE ALLOYS

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Abstract

Two-stress level rotating bending and axial tensile fatigue tests were run in order to test the applicability of the double linear damage rule - as proposed for high strength steels by MANSON et. al. to two nearly identical dural type alloys. As significant differences from the predicted behavior were observed a more universal definition of the damage concept was attempted. This eliminates some of the differences but for obtaining the practically required numerical accuracy the quality scatter inherent in engineering materials has to be taken into account too. Ways of approach are discussed.

Notation

- n number of load cycles applied
- x dimensionless load cycle number
- D fatigue damage
- N number of load cycles to failure
- N_o max. number of load cycles with zero failure probability
- Q fatigue resistivity index
- δ damage coefficient
- σ normal stress

Subscripts

- a at the first stress level
- b at the second stress level
- σ_a for load level σ_a
- σ_b for load level σ_b
- o initial value
- 1 at stress level σ_1
- 2 at stress level σ_2
- 3 at stress level σ_3
- 10...50...90 of 10...50...90 % probability
- 1-2 $\sigma_a = \sigma_1$, $\sigma_b = \sigma_2$
- 2-1 $\sigma_a = \sigma_2$, $\sigma_b = \sigma_1$

Superscripts

- ' high stress first
- '' low stress first
- * as determined from results of two-stress level tests

1. INTRODUCTION

In aircraft fatigue testing work quite frequent use has to be made of damage calculation procedures based on one of the several damage accumulation theories for the correlation of testing and service conditions. Practical choice between them may be made on the basis of the following requirements:

a/ possible accuracy;
b/ volume of computation;
c/ availability of basic data for the material concerned.

Needless to say, fatigue damage theories strong in complying with one of them used to be objectionable from at least one of the two other viewpoints. Should restrictions on time or on money impose a ban on running simultaneously some ad hoc material tests, then one has quite often no choice but to resort to the well-known Miner linear damage rule with all its inherent limitations in accuracy. This is why the proposal for an improved double linear damage rule as suggested first by GROVER, worked out in detail and demonstrated for some high strength steels by MANSON and his co-workers [1] had much to appeal to us. But Manson's paper lacked - probably not unintentionally - any reference as to dural type alloys and even complex structures might behave otherwise than simple plain rotating bending specimens used to.

It was thought therefore worthwhile to make a series of tests on dural specimens in order to get some first-hand experience on these problems.

As it was expected, test results did not confirm the direct applicability of a double linear damage rule to dural materials. Nevertheless, some interesting phenomena thought to warrant description and perhaps facilitating further investigation were observed.

2. DESCRIPTION OF THE TESTS

2.1. Materials and Specimens

Two dural alloys were selected: Al Cu Mg 2 bar for the rotating bending and Al Cu Mg 1 plate for the tensile specimens. Their nominal composition is seen in Table 1. Tensile strength properties determined by tests on specimens cut from the bars and from the plates are found in Table 2.

Material	Cu	Mg	Mn	Fe, Ni etc.	Al	Condition
AlCuMg2 bar ϕ 16 mm	3.8 -4.9	1.2 -1.8	0.3 -0.9	<1.5	rem in-der	Heat treated, hard
AlCuMg1 plate 5 mm	3.8 -4.8	0.4 -0.8	0.4 -0.8	<1.8	rem in-der	Rolled heat treated hard

Table 1. Material composition and condition

Material	Yield Strength 0.2 % kp mm^{-2}	Ultimate Tensile Strength kp mm^{-2}	Fracture Strength kp mm^{-2}	Reduction of Area %
AlCuMg 2 bar	24.5	45.0	62.6	33.0
AlCuMg1 plate	36.3	44.1	61.7	32.1

Table 2. Tensile properties of test materials

Both may be regarded as fair representatives of their kind of material as employed for light metal sailplane construction in our country.

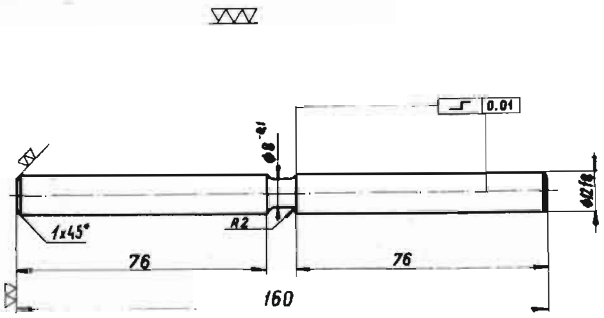

Figure 1. Rotating bending specimen

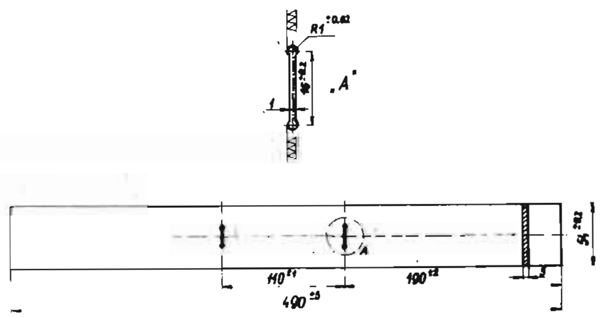

Figure 2. Tensile specimen

Rotating bending specimens /Fig.1 / were of the plain, short type. For tensile tests, a specimen form developed originally by SCHÜTZ [2] /Fig.2 / has been choosen. Advantages claimed for this type include to have fatigue characteristics similar to the Mustang wings investigated by JOHNSTONE and PAYNE in their full-scale tests [3]. We are using Payne's diagram as a basic data source for damage calculation in sailplane fatigue testing, this congruity was therefore much appreciated.

2.2. Rotating Bending Tests

Tests were run on two identical VEB Werkstoffprüfmaschinen, Leipzig type 520 rotating bending machines. From the 308 specimens tested in all, 76 underwent conventional single level tests on 9 different load levels for determining the single level life data with the required accuracy. For the two-stress level series characteristic of Manson's method (1) the following stress levels have been selected:

1. $\sigma_1 = \pm 3200$ kp cm^{-2},
2. $\sigma_2 = \pm 2750$ kp cm^{-2},
3. $\sigma_3 = \pm 2250$ kp cm^{-2},

giving 3 double combinations. For each combination 5 /on one occasion 4/ two level series of 8 pieces have been run.

Results of single- as well as two-stress level test series have been plotted on Weibull probability paper /Figs. 3,4 /. From the raw data /curve a/ a first approximation to N_o could be guessed. Starting from this a short computer program would give the value of N_o for optimum straightening of the graph. Computer autputs were checked by replotting /curve b/.

2.3. Tensile Tests

For tensile tests there were only 125 specimens at disposal, so a sole double combination of test series between two stress levels has been run. Tests were run on a Carl Schenk Maschinenfabrik GMBH, Darmstadt, type PVTN 20 Mp pulsator.

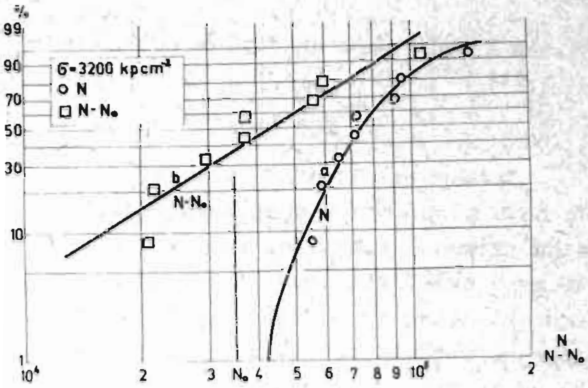

Figure 3. Single-level test series

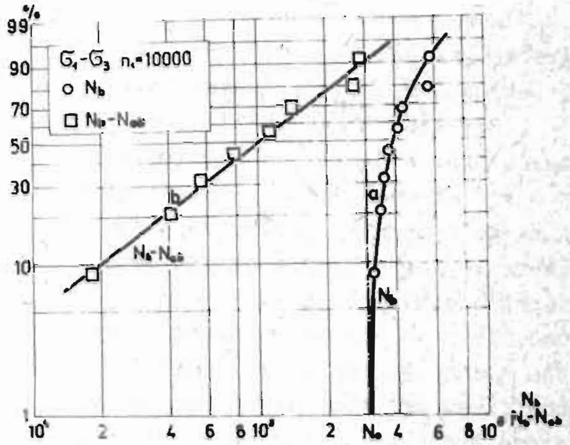

Figure 4. Two-level test series

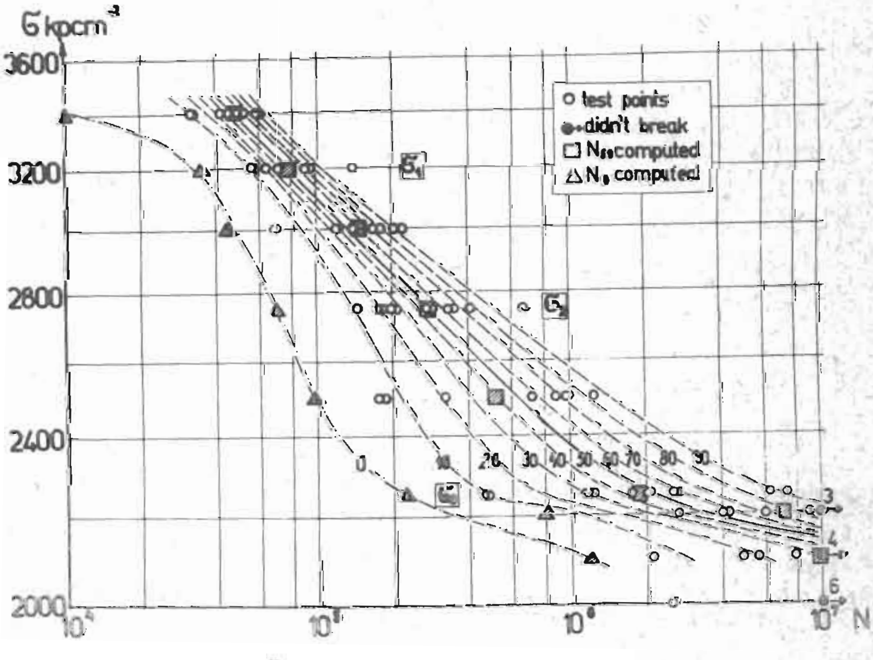

Figure 5. Results of the single-level rotating bending tests

Stress levels were as follows:

1. $\sigma_1 = 465 \pm 410$ kp cm^{-2}
2. $\sigma_2 = 330 \pm 288$ kp cm^{-2}

Stress values were selected so as to have identical stress amplitude to mean stress ratios. For both combinations 5 two level test series have been run. Plotting and fairing of results was done as for the rotating bending tests.

3. TEST RESULTS

3.1. Rotating Bending Tests

Results of the single level tests are summarized in Fig. 5 giving data points and N_0, N_{10} etc. values.

For two-stress level tests, Fig 6 taken from Ref. (1) is showing the pattern of dual life expectancy according to Manson's theory. The pattern shown is for a higher stress level in the first block than in the second one. Should the tests be run in reversed order i.e. the second stress level be higher than the first one then the graph would run above the dotted line AG and it would be a reflection of the line ABG in the line AG.

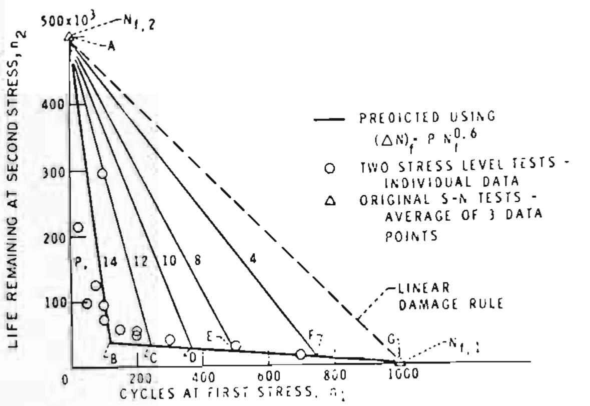

Figure 6. Dual life expectancy pattern after Manson (From Ref. (1))

Results of the dural rotating bending tests /Figs. 7 to 9/ deviated substantially from the ideal double linear pattern. Both graph axes are scaled in dimensionless cycle ratio units derived from the respective cycle numbers by dividing them by the corresponding N_{50} values. Data points are for 50 % survival probability and in most cases with the 90 % confidence band as determined after JOHNSON [4] marked over them. Considering these results the following first conclusions may be drawn:

a/ The double linear damage rule in its original form seems to be unsuitable for dural type alloys.

b/ There is even fairly strong evidence of some amount of working on certain stress levels augmenting the endurance on certain second stress levels.

c/ If a numerical accuracy of say 10 % has to be aimed at, then the number of tests to be done would be in many cases hardly bearable.

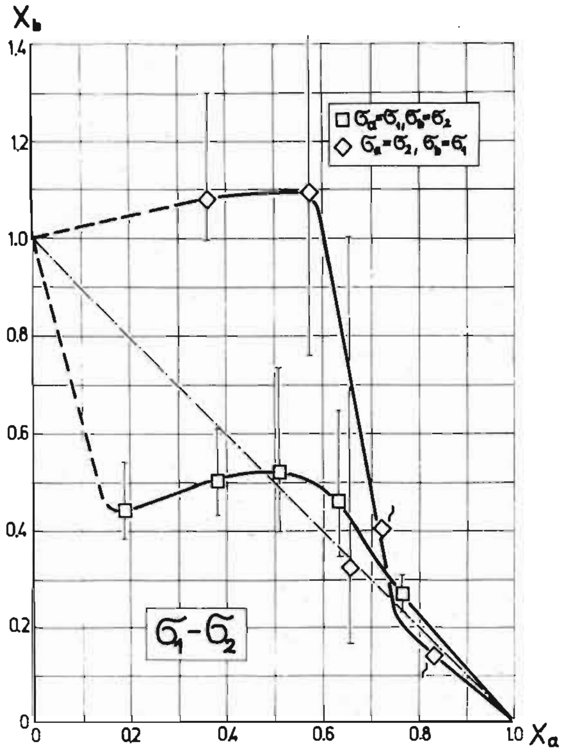

Figure 7. Dual life graph for stress levels σ_1 and σ_2

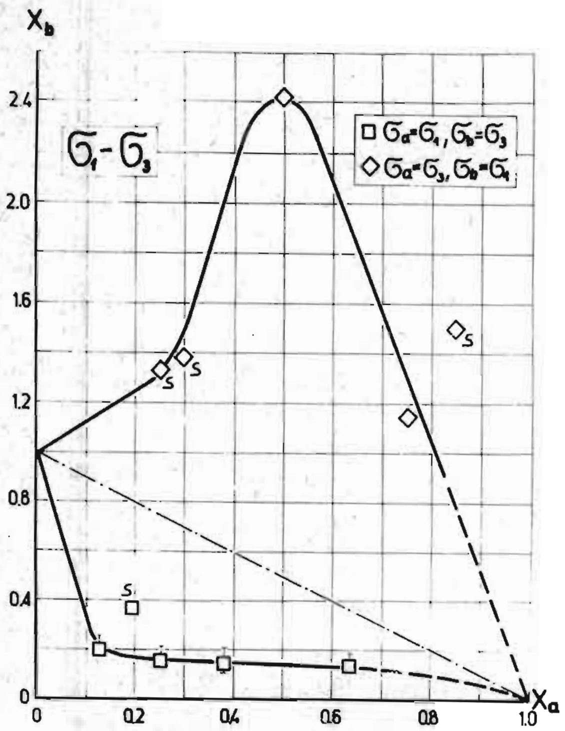

Figure 8. Dual life graph for stress levels σ_1 and σ_3

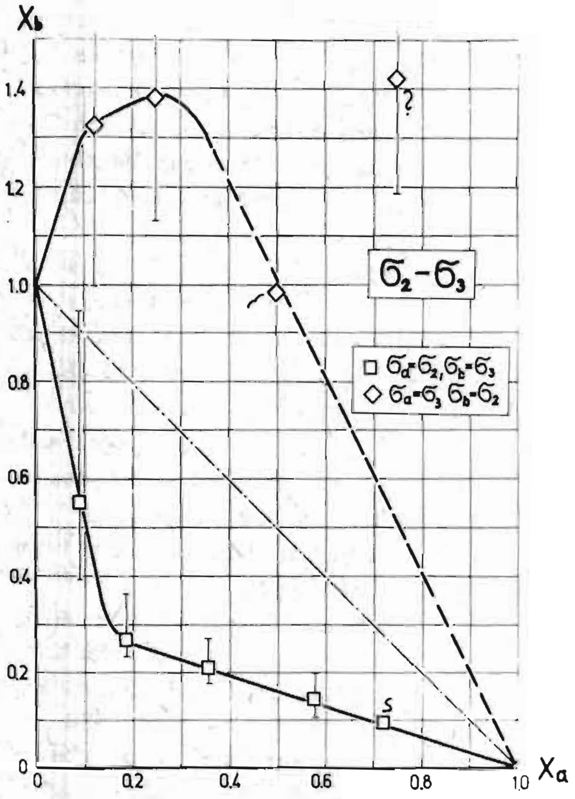

Figure 9. Dual life graph for stress levels σ_2 and σ_3

3.2. Tensile Tests

Part of the tensile tests have shown a scatter too high and too irregular to conclude on numerical results at present.

The fault has been traced down to be partly in specimen fabrication technology and partly in the quadruple test fixture.

Nevertheless, results so far have confirmed the dual life expectancy pattern of Schütz specimens being of the same character as that of rotating bending specimens with slighty more deviation from the linear behavior than for the former.

In the following discussion use is made only of rotating bending results with the implicit assumption of subsequently these proving to be transformable into structural member life expectancy data.

4. DISCUSSION OF RESULTS

4.1. Generalized Damage Concept

As mentioned already, the most significant difference of test results from Manson's theory seemed to be the ability of dural alloys to take sometimes more load cycles on certain stress levels after pre-stressing on certain other load levels than without pre-stressing /Figs. 7 to 9/. For some materials this phenomenon has been known from practice and from literature as well. So e.g. SCHIJVE[5] shows the delaying effect on the crack propagation in Alclad sheet caused by intermittent peak loads. SZERENSZEN[6] has got two level test diagrams like Fig. 8 or 9 on some steel specimens.

As it is of little if any use to speak of interaction phenomena without being able to treat them numerically a necessity of some generalization to the classical damage concept is clearly indicated. In the Appendix an attempt is made at this and the results may be summed up briefly as follows.

As a generalization of the well-known Palmgren-Miner rule, damage may be expressed in the form

$$D=\int_0^X \delta(\sigma,x)\,dx \qquad (1)$$

If fatigue damage /and consequently the damage coefficient $\delta(\sigma,x)$/ is a scalar type quantity and a unique function of σ and x, then the shape of the two-stress level test $x_a - x_b$ diagrams is determined by the respective fatigue coefficient functions /Fig. 10/.

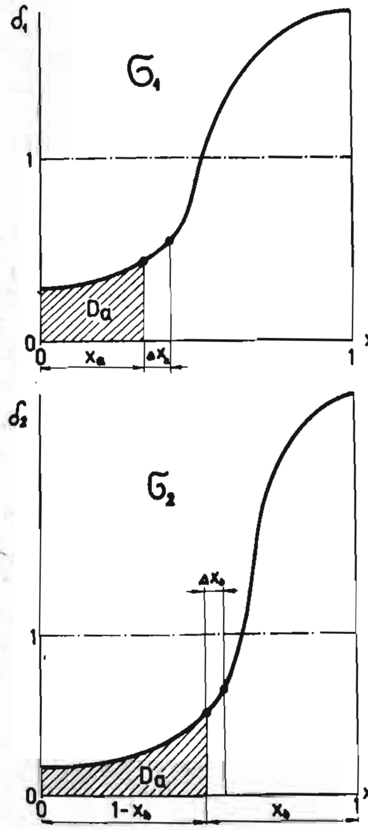

Figure 10. Fatigue coefficient curves

On Fig. 11 for each point P' of the high stress first curve at $x_a' = x_1$

$$\left(\frac{dx_b}{dx_a}\right)' = -\frac{\delta_1(\sigma_1, x_1=x_a')}{\delta_2(\sigma_2, x_2=1-x_b')} \qquad (2)$$

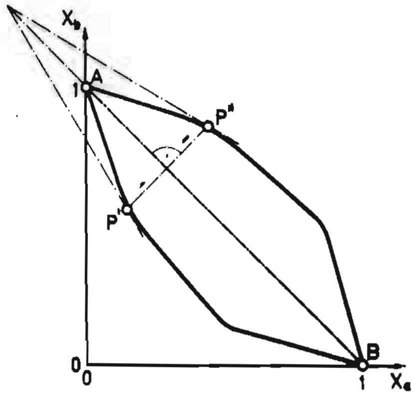

Figure 11. Symmetry of dual life graphs

Likewise, for each corresponding point P'' of the low stress first curve at $x_a'' = x_2 = 1 - x_a'$

$$\left(\frac{dx_b}{dx_a}\right)'' = -\frac{\delta_2(\sigma_2, x_2=x_a'')}{\delta_1(\sigma_1, x_1=1-x_b'')} = -\frac{\delta_2(\sigma_2, x_2=1-x_a')}{\delta_1(\sigma_1, x_1=x_a')} \qquad (3)$$

Following through this train of thought it may be pointed out that:

a/ On two-stress level test diagrams the low-stress-first curve has to be a reflexion of the high-stress-first curve in the line AB and vice-versa, whatever may be the shape of the fatigue coefficient curves;

b/ if the Miner rule should prove to be correct for certain materials or under certain conditions, then this would be only proof of the damage coefficient curves being independent of the load level and would not be automatically proof of the damage per cycle being constant during all the life time of the specimens;

c/ life increase due to prestressing may be possible if, and only if, the damage coefficient function has negative values through a section at last for one of the respective stress levels.

Negative δ values for part of the fatigue life may be interpreted as originate in temporary structural changes augmenting the fatigue capacity of the material e.g.: hardening. The requirement a/ had to be complied with even in these cases resulting in some rather peculiar diagram shapes but, sorry to see, Figs. 7 to 9 are showing clearly this not to be the case, at least not for the 50 % life expectancy values. As this cannot be attributed to the mathematical deduction in the Appendix, the two basic assumptions of the deduction have to be scrutinized.

Strictly speaking, there is no guarantee of the damage being a scalar function of σ and Q only but should even this assumption fail in a way of not giving even mean values, few hope of ever developing a usable cumulative fatigue damage theory would remain.

On the other hand, dropping the assumption $Q = Q_o$ = constant at the start of the tests for all specimens seems to be a feasible although not a pleasant proposition. But before examining this possibility in more detail let the problem of zero failure probability be reviewed briefly.

4.2. Zero Failure Probability

As it is known, the characteristic form of single level fatigue test results plotted on Weibull co-ordinates /see e.g. curve "a" in Fig. 3/ can be straightened by re-plotting on base of $N - N_o$ where the value of N_o is about corresponding to the N co-ordinate of the vertical asymptote of curve "a" /curve "b" in Fig. 3/. The physical meaning of N_o may be interpreted as being the upper limit value of the number of load cycles with zero failure probability.

Following a suggestion by Prof. GILLEMOT N_o values of two-stress level tests have been investigated if some regular pattern could be discovered in them.

Reason for this investigation has been twofold:

a/ Miner's hypothesis had been formulated originally not for total failure but for start of the crack /i.e.: for crack initiation/. Altough N_o may not be and in fact seems not to be identical with the number of cycles to macro crack initiation both may obey similar laws; a statement hardly tenable for the quite different process of crack propagation.

b/ From the engineer's point of view it is not the mean life but the fatigue life of low or of practically zero failure probability we are primarily interested in.

As for two-stress level tests, the plots of second block cycles to failure values on Weibull coordinates showed frequently but not always similar character to the single level diagrams /see e.g. Fig. 4/. N_{ob} values determined this way

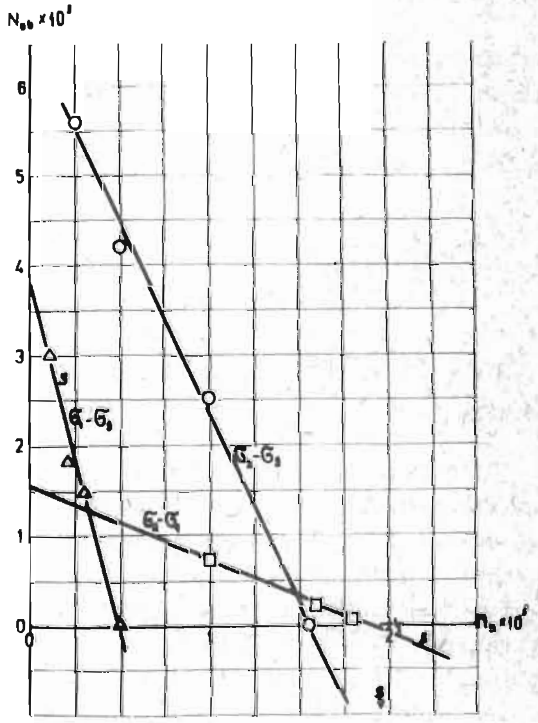

Figure 12. N_{ob} as function of n_a

may even be shown to be a linear decreasing function of n_a, the number of cycles on the first stress level /Fig.12/.

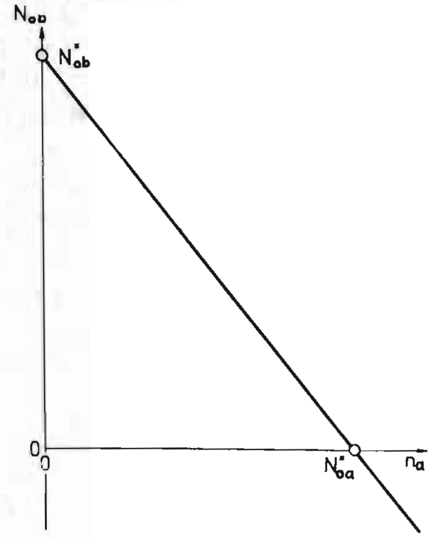

Figure 13. Determination of N^*_{oa} and N^*_{ob}

Unfortunately, N^*_{oa} and N^*_{ob} determined from the intersection points of the regression lines with the co-ordinate axes /Fig. 13/ never equalled N_o values from the corresponding single level test. Nevertheless, this linear relationship may easily be shown by elementary mathematics to give for

$$\sum x_o = \frac{n_a}{(N_o)_{\sigma_a}} + \frac{N_{ob}}{(N_o)_{\sigma_b}} = \frac{N^*_{ob}}{(N_o)_{\sigma_b}} + \left[1 - \frac{(N_o)_{\sigma_a}}{(N_o)_{\sigma_b}} \frac{N^*_{ob}}{N^*_{oa}}\right]\left(\frac{N_{50}}{N_o}\right)_{\sigma_a} \qquad (4)$$

Of course, it would be premature to draw far-reaching conclusions from a few data points but perhaps something could be made on this line. At present there are two fundamental problems precluding any serious theoretical or practical application of these results.

At first, cause and amount of differences between single level and two level test N_o values are to be cleared and investigations are to be extended to three and more stress levels and to stochastic loading as well.

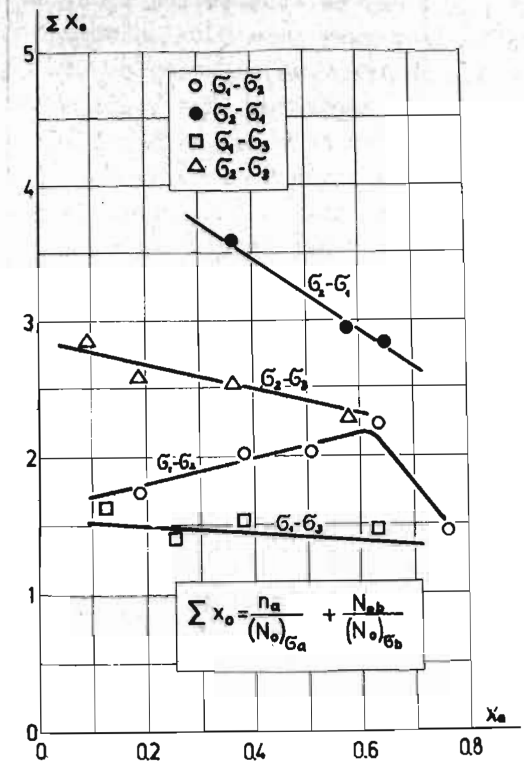

Figure 14. Values of Σx_o

Another difficulty consists in the non-Weibullian character of many of the two-stress level fatigue life distributions. From the 29 test series investigated so far 8, i.e. nearly 28 % belong to this unpleasant category. As there have only been 8 test pieces per series, irregular distributions may be explained by this cause alone but there is a fact contradictory to this simplest explanation.

In Fig. 7 to 9 and 12, non-Weibullian distributions are recognizable by the lack of the confidence band and by being marked with an S, 2 or $/$ symbolizing the shape of the distribution curve on Weibull coordinates.

High-stress-first test series present but few problems: there are only two irregular distributions here with one of them for n_a greater than N^*_{oa}, a partial explanation in itself. Against this, when counting also a doubtfull case to the irregulars, 50 % of the low-stress-first distributions are non-Weibullian.

Considering this it may be admissible to attribute part of the non regular life distributions to other causes than experimental errors or statistical irregularities and to try to find the underlying principles. This leads back to the problem of the quality scatter inherent in the materials.

5.FATIGUE LIFE AND QUALITY SCATTER

Giving up reckoning with the mean material properties and allowing for the statistical quality scatter may necessitate some new procedures. It may be said that every single test piece is now treated individually so the substitution of $\delta(\sigma, x)$ for $\delta(\sigma, Q)$ has to be dropped. As a consequence, use of Q - x diagrams seems advisable. Upholding still the assumption /see 4.1/ of the damage per load cycle being a unique scalar function of σ and Q single level fatigue test results may be treated the following way /Fig. 15/.

Let e.g. three test pieces have initial Q_0 values of Q_I, Q_{II} and Q_{III}, respectively. Neglecting the scatter due to test load inaccurancy individual lives would then turn out as x_I, x_{II} and x_{III}. The damage per load cycle depending only on the actual value of Q is synonymous with the gradient of all curves Q/x/ being a unique function of Q. In this case, as on the upper part of Fig. 15, the - let it be called - life consumption curves of all specimens would be congruent, displaced horizontally according to individual Q_0 values. Consequently

$$\overline{OP}_{II} = \overline{B_{II}P_1}$$

$$\overline{OP}_{III} = \overline{B_{III}P_1} \quad \text{etc.}$$

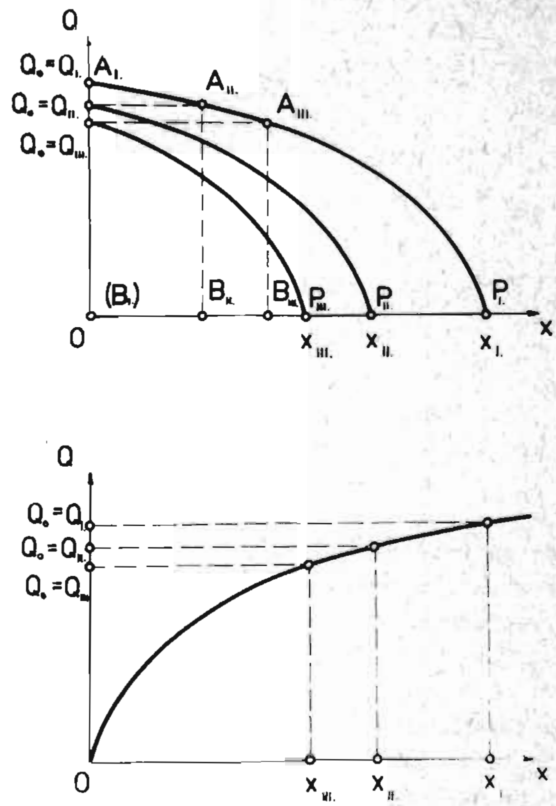

Figure 15. Fatigue life consumption

and individual life values may be obtained by use of a single life consumption curve as shown on the lower diagram.

Theoretically, thereby part of the life consumption curve can be obtained from single level fatigue life distribution data. For a given stress value the low-Q section of the life consumption curve, below the Q_0 value of the weakest specimen, is common to all test pieces. Differences in fatigue life are caused by the higher-Q section of the curve; hence life scatter distributions are attributable to and representative of this part of the curve.

Let us suppose the Q_0 distribution of a series of test pieces to be known /graph on the left, Fig. 16/. In this case, by running a single-level test se-

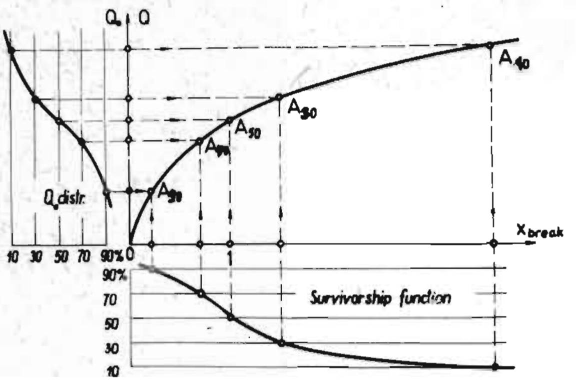

Figure 16. Determination of the life consumption curve

ries on this batch and determining the survivalship function /lower graph/ the corresponding section of the life consumption curve might be plotted /middle part of Fig. 16/. At the present time no reliable measure for capability of resisting to fatigue is known to the author, but perhaps a simple test series aimed at investigating the correlation between dislocation density and fatigue life might shed some light on it. Further sections of the life consumption curve could then be obtained by two-level tests.

6. CONCLUSIONS

Several two-stress level fatigue test series have been run in order to investigate the applicability of the double linear damage rule to dural alloys. In this respect results have been negative but the following peculiarities perhaps worth while for further research were observed:

1. On certain stress levels and for part of the fatigue life - presumably in the nucleation period - temporary augmenting of fatigue resistivity may occur. There is a possibility of accounting numerically for this phenomenon by use of a negative damage coefficient.

2. N_0 values have shown a much more linear trend than N_{50} ones. This may contribute to devising a practical damage accumulation theory.

3. For complex load sequences numerical accurancy may be obtained by accounting for scatter in the material properties too. First step in this direction could be the investigation of correlation between dislocation density and fatigue life.

ACKNOWLEDGEMENT

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APPENDIX

Two-Level Test Life Expectancy Diagram Form for Non-Constant Damage Coefficient

In the Palmgren-Miner theory and in several other fatigue theories the term "damage" is used as being essentially the diminishing of some one dimensional, scalar type property of the material. Without going into physical details, let this material property be named Q with $Q=Q_0$=constant for all specimens at the start of the fatigue tests, i.e. for n=0. This assumption can hardly be supported on the basis of the quality scatter of engineering materials being negligible but may be justified if - and only if - mean or median values of test results prove to conform to the rules derived on this assumption.

Let us suppose that the damage due to each load cycle is a function of σ and Q. Then the damage due to Δn load cycles may be expressed by the following equation:

$$\Delta D = -\frac{1}{Q_0}\frac{\Delta Q(\sigma, Q)}{\Delta n}\Delta n \qquad (5)$$

and in a series of constant or different load level blocks failure is to be expected at 50 % probability when

$$D = \sum_i \Delta D_i = -\sum_{n=1}^{n}\frac{1}{Q_0}\frac{\Delta Q(\sigma, Q)}{\Delta n}\Delta n = 1 \qquad (6)$$

For more convenient handling let both Δn be divided by $N_{50}/\sigma/$ and formally allow for a physically meaningless 0-th load cycle.

$$D = \sum_{n=1}^{n} - \frac{\frac{\Delta Q(\sigma,Q)}{Q_0}}{\frac{\Delta n}{N_{50}(\sigma)}}\,\frac{\Delta n}{N_{50}(\sigma)} =$$

$$= \sum_{n=0}^{n} - \frac{\frac{\Delta Q(\sigma,Q)}{Q_0}}{\frac{\Delta n}{N_{50}(\sigma)}}\,\frac{\Delta n}{N_{50}(\sigma)} \qquad (7)$$

The following shorthand notation may be introduced:

$$\frac{n}{N_{50}(\sigma)} = x \qquad \frac{\Delta n}{N_{50}(\sigma)} = \Delta x$$

and for $\Delta n=1$: $\Delta x=dx$, $\Delta Q=dQ$.

For each load level Q is assumed to be a unique function of n, therefore infinitesimal damage as function of x may be substituted for function of Q.

$$D = \int_{x=0}^{x} -\frac{dQ}{Q_0}(\sigma, x)\,dx \qquad (8)$$

Introducing what might be called the damage coefficient:

$$-\frac{\frac{dQ}{Q_0}(\sigma,x)}{dx} = \delta(\sigma,x)$$

$$D = \int_{x=0}^{x} \delta(\sigma,x)\,dx \qquad (9)$$

with

$$\int_{x=0}^{1} \delta(\sigma,x) = 1 \qquad (10)$$

The Palmgren - Miner theory is equivalent of having

$$\delta(\sigma,x) \equiv 1$$

Manson's concept of two separate linear damage phases /in the crack initiation and in the crack propagation period respectively/ is equivalent to have two δ values for each load level, one each before and after initiation of the "effective" crack.

If on two load levels δ_1/σ_1, $x_1/$ and δ_2/σ_2, $x_2/$ were as shown on Fig.10

the pattern of results from a two-level test series would be as follows /Fig.17/.

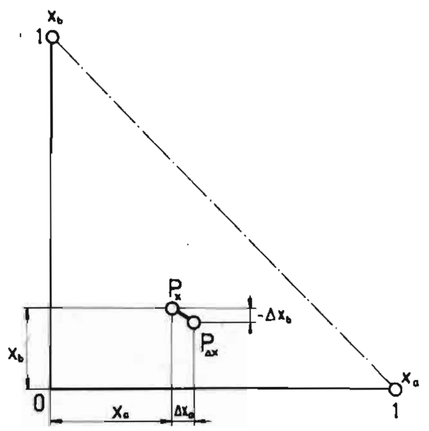

Figure 17. Two-level "Gedankenexperiment"

Let the first of the tests be run with σ_1 up to n_a then with σ_2 to failure. For the second test σ_1 should be applied up to $n_a + \Delta n_a$ and then σ_2 to failure. The condition of the comulative damage D=1 at failure may be expressed by

$$\int_0^{x_a} \delta_1(\sigma_1,x_1)\,dx_1 + \int_{1-x_b}^{1} \delta_2(\sigma_2,x_2)\,dx_2 = 1 \qquad (11a)$$

$$\int_0^{x_a} \delta_1(\sigma_1,x_1)\,dx_1 + \int_{x_a}^{x_a+\Delta x_a} \delta_1(\sigma_1,x_1)\,dx_1 + \int_{1-x_b}^{1} \delta_2(\sigma_2,x_2)\,dx_2 -$$

$$- \int_{1-x_b+\Delta x_b}^{1-x_b} \delta_2(\sigma_2,x_2)\,dx_2 = 1 \qquad (11b)$$

Forming the difference of the two equations gives

$$\int_{x_a}^{x_a+\Delta x_a} \delta_1(\sigma_1,x_1)\,dx_1 - \int_{1-x_b+\Delta x_b}^{1-x_b} \delta_2(\sigma_2,x_2)\,dx_2 = 0 \qquad (12)$$

If $\Delta n_1 \rightarrow 1$, then $\Delta x_1 \rightarrow dx_1$, $\Delta x_2 \rightarrow dx_2$; hence

$$\delta_1(\sigma_1,x_a)\,dx_1 + \delta_2(\sigma_2,1-x_b)\,dx_2 = 0 \qquad (13)$$

hence

$$\frac{dx_2}{dx_1} = \left(\frac{dx_b}{dx_a}\right)_{1-2} = -\frac{\delta_1(\sigma_1,x_a)}{\delta_2(\sigma_2,1-x_b)} \qquad (14)$$

Should the two stress levels be applied in reversed order, i.e. σ_2 first and σ_1 in the second block, then a deduction identical to the previous one would give:

$$\frac{dx_1}{dx_2} = \left(\frac{dx_b}{dx_a}\right)_{2-1} = -\frac{\delta_2(\sigma_2,x_a)}{\delta_1(\sigma_1,1-x_a)} \qquad (15)$$

For $x_a'' = 1 - x_a'$

$$\left(\frac{dx_b}{dx_a}\right)_{x_a'=x_a'} = \frac{1}{\left(\frac{dx_b}{dx_a}\right)''_{x_a''=1-x_a'}} \qquad (16)$$