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RESEARCH ON THE STRUCTURAL NON-LINEARITY OF AEROELASTICITY IN PROBLEMS OF SUPERSONIC FLIGHT

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EFFECT OF STRUCTURAL NONLINEARITIES IN PANEL FLUTTER RESEARCH

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Abstract

The general equations of nonlinear elasticity are firstly recalled, and transformed in such a way as to be applied to plates and shells of increasing thickness. The importance of shear flexibility, finite curvature, nonlinear inertia is then discussed. The attention is then focused on physical nonlinearities for which it is pointed out a certain lack of scientific information: similar considerations apply to damping as well. Current methods of solutions are briefly reviewed, and the idea of a new method is then proposed.

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1 - Introduction

The problem of panel flutter has recently been considered in very great detail everywhere in the world. The reasons for such a wide interest underly in the technical importance to the problem and in its relatively clear formulation as well: it could be said that there is no nation of some scientific preparation where several panel flutter papers have not been issued.

It is almost impossible to follow the tremendous amount of technical literature. Some attempts have been made, such as in [1], which is a real Cook's tour to the problem; even this tour, however, misses some important, or at least, significant field, such as f.i., the work of the italian school, who has produced a considerable number of works in the last four years.

Therefore, in the discussing the subject under concern, it was my opinion that a complete, comprehensive, comparative study of the state of the art of the problem could not be reached, and that it is much better to discuss ideas more than results, methods and concepts rather than graphs: although results and graphs will be sometimes necessary in order to substantiate such ideas.

Furthermore, despite the title of the lecture, I will sometimes also introduce linear, at least "linearized" effects, which are often ignored, even if their importance is greater than the nonlinear ones.

It is well known to everybody that any aeroelastic problem is governed by the symbolic equation:

$$S(w) - \mathcal{J}(w) - A(w) = q \qquad (1)$$

where $S, \mathcal{J}, A$ are structural, inertial, aerodynamio operators respectively, whereas q is the unit load acting on the body surface. Nonlinear terms appearing in each of the terms are referred to as "nonlinearities". We will mainly discuss nonlinear terms in $S, \mathcal{J}$; as far as A is concerned, the linearized version of the piston theory

$$A = \frac{k p_\infty}{c_\infty}\left[\frac{\partial w}{\partial t} + V\frac{\partial w}{\partial x}\right] \qquad (2)$$

will be accepted.

It has been shown, in fact, [2] that piston theory flutter prediction oorresponds very closely with experimental results, at least for moderate values of q . Furthermore:

(i) there is no real contradiction in using linear aerodynamic theory for nonlinear flutter calculations, because the limiting amplitudes of flutter will still be small from the aerodynamic point of view.

(ii) Very little is gained in considering higher order piston theories for practical cases [3] .

As far as structural nonlinearities are concerned, they can also be ranked into three main groups:

(i) geometric nonlinearities
(ii) structural nonlinearities in the stress-strain relationship
(iii) terms representing internal damping.

We will discuss them separately.

2 - <u>Geometrical nonlinearities</u>

Geometrical nonlinearities arise from two different facts: (i) deformations are finite, so that the equations of equilibrium are to be written in the body after deformation - (ii) the relationships between deformation and displacement are not linear.

In reality, under such conditions, also physical nonlinearities should be introduced, since Hooke's law is valid only for infinitely small strains.

For plates and shells, which are the structural objects of panel flutter, however, the general three-dimensional equations of elasticity must be reduced to two-dimensional.

In Appendix A a general procedure is proposed to obtain such two-dimensional nonlinear counterpart of the general equations, allowing for successive approximations of the order h^2, h^4 (where h is the shell or plate thickness); thus we have the general equations of finite deformations for plates of finite thickness.

Again, in dynamic aeroelasticity of panels certain simplifications are allowed, the most important of which is that strains are negligible, whereas rotations are not. Although both circumstances denoted as (i) (ii) as the begining of this article are still valid, the corresponding nonlinearities are greatly simplified.

Furthermore, in general, the only significant contributions to them arise to the displacements normal to the panel surface.

If such assumptions are retained in a plate whose h^2 is negligible, one obtains the well known Karman's equations for large deflections of a plate. Again in Appendix A the extension of similar equations to plates of arbitrary thickness (and in particular so as to include z^2 terms) is considered.

Karman's equations have been widely used in panel flutter analysis, despite the difficulties to use them in representing situations where in plane inertia forces are to be considered and/or when boundary conditions referring to in-plane displacements are to be considered. It may be of interest, therefore, to have equations in $[u, v, w]$.

In order to check the effects of terms omitted in the common theory of plates, a calculation was performed on the shear flexibility, which as an effect of the order of h^2, (but is not the only one, of course). It is well known that the effect of such flexibility is important essentially on higher frequencies, where the effective span is considerably reduced with respect to the original ones. Since panel flutter is essentially dominated by the second frequency, the effect on the aeroelastic vibrations are expected to be small.

Details of calculation can be found in Appendix B.

Typical results are shown in Fig. 1 which shows the amplitude of the limit cycle with respect to the shear rigid amplitudes vs. the ratio of dynamic pressure to critical (linear) dynamic pressure. The analysis is based on the harmonic balance method.

A typical nonlinearity associated with finite deformation is the one taking into account finite curvature effects. For a plate, the pertinent expressions can be deduced from the general equations of Tab. I; for a plate of infinite span, however, it is well known that we have bending moment = $D \times$ curvature, where the curvature has to be written in the exact form. The general equations accounting for such effects in aeroelastic conditions are written in Appendix C; some typical results are shown in Fig. 2.

A more refined analysis shows however that, since third order terms are retained in the equation, also inertia forces associated with horizontal displacement, must be considered, and thus give rise to a well know phenomenon of nonlinear mechanics, referred to as inertia nonlinearity.

Considering such effect, the curve of Fig. 2 becomes that of Fig.3.

3 - Physical nonlinearities

As far as the subject of physical nonlinearities is concerned, we will confine ourselves to the deviation of stress-strain relationship from the well-known, universally used Hooke's law.

The basic characteristic of such nonlinearities is that they may have a destablizing or a stabilizing effect. To show this, a very simple analysis was conducted on a three-layer symmetric sandwich plate of infinite span consisting of a material for which the stress-strain relationship is of the type:

$$\sigma = E\,[\varepsilon - b\varepsilon^m] \qquad (3)$$

Such a law was proposed by Ambartsumyan [4] in a paper of several years ago, and is based on a former work by Prager [5]. The pertinent panel flutter equations are derived in Appendix D for $m=3$: the physical nonlinearity associated to (3) is depending on a parameter, Ω, that can have either a positive or a negative value, under such circumstances the limit cycle can be either stable or instable. The amplitude of the limit cycle, for the stable case, for several values of the dynamic pressure vs. the value of Ω is represented in Fig. 4.

An important point in physical nonlinearities thus arise; the fact that the values of the physical parameters may have a quite different effect upon aeroelastic vibrations, according to their values, and we know very little about such values. It is therefore, of vital importance, for panel flutter analysis, to have a better knowledge of the material characteristics; and, in my opinion, this cannot be a simple experimental work.

We should know in principle, which are the basic mathematical forms of the stress-strain relationships, and this is a work of theoretical physics, the solid state physics; only after this work will have given some results, a wide exprimental research may provide the informations we have required.

As far as elastic coefficients are considered, another effect which is often ignored is the variations of them with temperature. This is of course a linear effect, but, like shear flexibility, it may have an impact also on nonlinear behaviour.

We consider the very simple case of a plate of thickness a subjected to a heat flux of constant value Ω^* from one side, and (i) insulated on the other side (ii) kept at zero temperature on the other side.

For both cases the temperature distribution is given in Appendis E. There it is also shown that, in case (i), the variation of E is essentially with time $E = E_0[1+\alpha t]$, so th t in Galerkin's equations, terms containing E are to be multiplied by the factor $[1+\alpha t]$. This may change the behaviour of the solution; to see this, consider the variation of the factor $[1+\alpha t]$ during one period of vibration

$$\alpha p = \frac{1}{E_0}\left[\frac{dE}{dt}\right]_0 \frac{\Omega^*}{ca} p \qquad (4)$$

where p is the period of aeroelastic vibration of the plate.

Considering, f.i., a material such as René 41, we have $(1/E_0)(dE/dT)$ of the order of $-1/6000°C^{-1}$

$$c = 0.6 \text{ cal}/°C\,cm^3 \qquad a = 1\text{ cm}$$

$$\alpha p \cong \frac{[\text{heat flux entering in one cycle}]\ \text{cal}/cm^2}{4000}$$

Now, even with a strong heating (say, 100 Btu/sq ft sec) and a period of vibration of 1/10 of second, we have $\alpha p = 25/4000$, which is almost negligible for engineering purposes; therefore we may replace the term $[1+\alpha t]$ with $[1+\alpha t_m]$ where t_m is the medium time about which vibration is taking place. In other words, whereas frequencies of vibrations will be changed in a continuous way on time, the shape of the vibration during one cycle remains essentially unchanged.

The situation is somewhat analogous to the vibrations of a multistage missile, where mass is changing with time, but much more slowly that the vibration phenomenon

As far as case (ii) is concerned, the variation of E is essentially with space, $E = E_0[1+\beta(1-\xi)]$ where $\beta = (1/E_0)(dE/dT)\Omega^* a/k$: this produces a percent reduction in axial and flexural stiffness given by $1+\beta/2$. Again a typical value of β in the example above $\beta = (-1.1/6000)(25/0.032) \simeq -26/180$; in this case, therefore, a reduction of the order of 10% is to be expected.

Again here I would like to emphasize the tremendous lack of information existing in this area.

We do not know very much about variation of E; but almost nothing we know about variation of ν, about anisotropicity effect, about dependency of such values from frequencies, etc.

We shall soon undertake a research program in this area; I do hope that a decisive effort may be conducted in this area.

4- Damping effect

The importance of damping in panel flutter does not need to be emphasized; since for a long time it is known that damping can have destabilizing effect [3]. Again here we have to stress the point that very little information is available. As far as the actual the law governing structural damping are concerned, several laws have been proposed [7] but none of them appears to be satisfactory, because is not based upon any particular physical ground. An excellent attempt is that of obtaining solutions for a set of different laws of damping as proposed by Dugundjy [8]

To show the importance of phenomenon we consider the general law: [9] :

$$G^m |s_{i,j}| = G^n |e_{i,j}| \qquad (5)$$

where G^m and G^n are differential time operators of degree m and n respectively and $S_{i,j}$ $e_{i,j}$ are the deviation tensors of stresses and strains. By considering for simplicity one dimensional structure, the bending moment equation reduces to $F = D\left[\frac{\partial^4 w}{\partial x^4} + A^* \frac{\partial}{\partial t}\left(\frac{\partial^4 w}{\partial x^4}\right)\right]$ (AppendixF) where A^* is a constant.

Again here by employing a two mode Galerkin solution, we obtain the results of Figs. 5, 6, 7.

In Fig. 5 the typical relationship between damping and frequency for various values of A^* (in dimensionless form) shows that for $\varepsilon \neq 0$ the transition is from damped to increasing amplitudes, whereas in the case $\varepsilon = 0$ the transition from undamped to increasing amplitudes.

Figs. 6, 7 give the values of frequencies and damping coefficient vs. ε. This shows in particular that as $\varepsilon \to 0$ the value of the critical dynamic pressure does not tend to the value for $\varepsilon = 0$. This proves that to neglect the damping is very important since even a very slight damping may dangerously anticipate instability.

This point has been discussed for other type of damping in [3].

5 - Methods of solution

The general methods of solution for panel flutter problems are those classical of nonlinear mechanics: boundary of stability and limit cycle.

We will not go into detail of such problems from a mathematical standpoint, since excellent textbooks [10] completely cover the subject.

We simply recall that the most widely used method for eliminating space variables is the well-known Galerkin's method.

A first problem arises here and is that of an adequate choice of describing modes.

It is common to refer to the eigenfunctions of the simply supported panels: it should be noted however, that constraints in the edges normal to wind direction (x) have an important effect whereas constraints in the edges parallel to the wind (y) have pratically no importance. Fig. 8 Ref. [11] provides the variation of critical (linear) dynamic pressure with respect to a parameter defining rotational stiffness of the edges of the panel ($M_f = -\alpha D \theta$ where M_f is the bending moment, and θ is the edge rotation).

In contrast, Fig. 9 (from [12]) shows the variation of the 1st mode in the y-wise direction, for the $\alpha = \infty$ value of the edge restraint coefficient, and completed with the 1st mode in x-wise direction; it should be noted that, as $\alpha = 0$, all curves should merge to $\sin(n\pi y/b)$. Therefore, since the aeroelastic force is essentially proportional to the derivatives along x

and since the function along y is unchanged, very little can be gained or lost in considering conditions other than simply support along y . Again however, the condition $M_f = -D\alpha\theta$ may represent very poorly the actual behavior of a continuous structure. A curious effect for such structures is described in Fig. 10, [13] , where a two-span continuous plate of infinite aspect ratio is considered. It is seen that as the ratio L'/L'' approaches unity, there is a trend to merging of the first critical pressure to the second critical pressure, which greatly reduces instability.

Now, coming back to Galerkin's approach, we note that it may be inadequate to treat nonlinearities if expressed with their complete expressions. For example, a nonlinear factor such as

$$\frac{\partial}{\partial s}\left\{\frac{1}{[1-w'^2]^{1/2}}\frac{\partial}{\partial s}\left[\frac{\partial^2 w/\partial s^2}{[1-w'^2]^{1/2}}\right]\right\}$$

which enters in finite curvature problems should be treated by setting $w = \Sigma w_j(t) f_j(s)$ multiplying by $f_r(s)$ and integrating. But such integrals cannot be, in general, calculated analytically, and this is the reason why we must confirm ourselves to treat expanded expressions of nonlinearity; otherwise one should think of a method of numerical parametric integration, which would be extremely cumbersome.

This point is of some importance since Luigi Morino, who is now working at M.I.T. recently, in a private conversation, told me that he is going to publish a result, that limit cycle with third order nonlinearity is unique, whereas other limit cycles appear with fifth, seventh... etc. order nonlinearities: and also the question of the stability of such limit cycles is open.

This is the reason why we are thinking of a new solution which is based upon a general method of solution of dynamic problems by Luigi Broglio [4] and which pratically abandons the idea of modal approach.

The approach is based upon splitting the terms of the actual differential equation into terms describing a linear "tangent" structure and to consider all the other terms (including nonlinearities) as forcing functions. Thus for a plate we would have:

$$D\nabla^4 w + \mu\frac{\partial^2 w}{\partial t^2} = [\text{AERODYNAMICS}] + [\text{DAMPING}] + [\text{STRUCTURAL NONLINEARITIES}] = \Psi\left[w, \text{grad}\, w, \frac{\partial w}{\partial t}\right] \quad (6)$$

Now consider the dynamic Green's function for the left hand side of Eq. (6), $C(P, P'; t-\lambda)$ which is given by:

$$c(P, P'; t-\lambda) = \sum_{0}^{\infty}{}_n \frac{v_n(P)v_n(P')}{\omega_n}\sin\omega_n(t-\lambda) \quad (7)$$

where the ω_n are frequencies, and $v_n(P)$ are normalized modes. We note that C can be calculated as accurately as required with as many terms as necessary, and can even by precomputed and stored in the digital computer. At this point, the solution to Eqs.(6) becomes:

$$w(P,t) = \int_0^t d\lambda \int_\Sigma \Psi(P',\lambda)\, C(P,P';t-\lambda)\, d\Sigma \quad (8)$$

where Σ is the surface of the panel. Thus, by choosing a suitable grid on the panel, for each gridpoint, and for successive time intervals one has to calculate and store the nonlinear term, Φ , and simply apply the convolution integral (8). The work to be done to render the program efficient is essentially a numerical one, to evaluate as accurately as possible, time and space derivatives of the single quantities. A much greater accuracy is expected, since the work involved should increase roughly as the number of grid points, whereas the increase would be much more with the modal approach. In contrast, memory may become a major problem, since you have to store all the nonlinearities for every grid-point and for every time.

Coming back again to Galerkin's equations, they may be written in the form:

$$\dot{\Phi} + A\Phi = F(\Phi) \qquad (9)$$

Here Φ is the vector $(\varphi_1, \varphi_2, \dots \varphi_N; \dot{\varphi}_1, \dot{\varphi}_2, \dots \dot{\varphi}_N)$; the φ_j are the functions describing time-varying modal amplitudes; A is a constant matrix and F is a vector, describing nonlinearities.

It should be noted that F does not depend upon $\dot{\Phi}$.

Methods of solution of Eq. (9) are very numerous, and are of course different according to the available tools. Analog computers [15],[16] are of great help when one has to look for several solutions (such as in the determination of stability boundary) which can be refined later: they are generally inadequate as one has to look for limit cycle characteristics.

With digital computers one can generally use one of the many techniques of numerical integration.

However, in view of the expected oscillatory character of the solution, the method of slowly varying parameters can reduce the danger of too high round off errors. A general approach to such method is given in Appendix G.

It should be noted that the method of slowly varying coefficients may be very useful even if one requires only steady oscillations. In other words, the limit (constant) solution of the slowly varying coefficient c, obtained through integration of the pertinent differential equation $\dot{c} = \Psi(c)$, if equivalent to (9), is the solution of the finite equation $\Psi(c) = 0$; and the former can be much simpler than the latter (Appendix G).

6 - Concluding remarks

Some general ideas of panel flutter research have been investigated. It was firstly checked that existing and actually applied structural theories are generally adequate for today's engineering configurations. A major effort, in the writer's opinion, should be conducted in the field of materials, in order to have better information on the actual material behavior. Also a fresh thought of solution techniques may prove to be useful, eliminating some of the problems arising with the modal approach.

7 - Acknowledgments

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Appendix A

1) We recall here the equations of finite deformation for an elastic body. Let (x,y,z) be the coordinates of a point in the initial undeformed state and $\vec{V}\ (u,v,w)$ be its elastic displacements. We have the following equations [17]:

a) Relationships between displacements and strains:

$$\varepsilon_x=\sqrt{\left[1+\frac{\partial u}{\partial x}\right]^2+\left[\frac{\partial v}{\partial x}\right]^2+\left[\frac{\partial w}{\partial x}\right]^2}-1$$

. . . .

(1)

$$\gamma_{xy}=\sin^{-1}\frac{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}}{(1+\varepsilon_x)(1+\varepsilon_y)}$$

. . . .

b) Equations of equilibrium:

$$\left[\frac{\partial}{\partial x};\frac{\partial}{\partial y};\frac{\partial}{\partial z}\right]D_2^* + \vec{F}_* = 0 \qquad (2)$$

where

$$D_2^*=\left\{D_1^*\left[I+\operatorname{grad}\vec{V}\right]\right\}_T \qquad (3)$$

Here I is the unit tensor, $\operatorname{grad}\vec{V}=[\operatorname{grad}u,\operatorname{grad}v,\operatorname{grad}w]$ and:

$$D_1^*=\left\{\begin{bmatrix}1+\alpha_x & 0 & 0\\ 0 & 1+\alpha_y & 0\\ 0 & 0 & 1+\alpha_z\end{bmatrix} D\right\}\begin{bmatrix}\frac{1}{1+\varepsilon_x} & 0 & 0\\ 0 & \frac{1}{1+\varepsilon_y} & 0\\ 0 & 0 & \frac{1}{1+\varepsilon_z}\end{bmatrix}$$

(4)

where

$$\alpha_x=\left\{\left[\left(1+\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]\left[\left(1+\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]-\right.$$
$$\left.-\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]^2\right\}^{1/2}-1$$

(4')

and D is the stress tensor written in the undeformed state. As is known, the components of D_1^* are symmetric, i.e., $\sigma_{xy}^* = \sigma_{yx}^*$; the same relationship does not apply to the components of D (unless we considered negligible strains and shears).

It is also known that the tensors appearing in (4) take into account the change in angles and length of the elemental volume, parallelepiped before deformation, whereas (3) take into account the rotations of the element. Finally:

$$\vec{F}_* = \vec{F}\ \det\left[I+\operatorname{grad}\vec{V}\right] \qquad (5)$$

where $\vec{F}$ is the external force per unit volume of the undeformed body (including possibly inertia forces).

c) The stress-strain relationship is a complex equation whose explicit (although very theoretical) form can be seen in [9] .

Suffice here to say that the relationship is between generalized stresses σ_{ij}^{*} and the strains:

$$\sigma_{ij}^{*} = f\,[\varepsilon_{ij}] \qquad (6)$$

d) Of course, we have to consider also boundary conditions. In particular at points where the body surface is subjected to specified tractions $\vec{\sigma}_{n*}$, we must have:

$$\vec{D}_2^{*} \times \vec{n} = \vec{\sigma}_{n*} \qquad (7)$$

where $\vec{n}$ is the normal to the body.

2) Let the three components of $\vec{D}_2^{*}$ on the (x,y,z) coordinate lines denoted by $\vec{\sigma}_{x*}, \vec{\sigma}_{y*}, \vec{\sigma}_{z*}$. Then we have:

$$\frac{\partial\vec{\sigma}_{x*}}{\partial x} + \frac{\partial\vec{\sigma}_{y*}}{\partial y} + \frac{\partial\vec{\sigma}_{z*}}{\partial z} + \vec{F}_{*} = 0 \qquad (8)$$

Consider now a plate of thickness h, whose medium plane is the plane $z=0$.

Multiplying by z^j and integrating with respect to z and letting:

$$\vec{N}_{x*}^{(j)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^j \vec{\sigma}_{x*}\,dz$$

$$(9)$$

$$\vec{f}_{*}^{(j)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^j \vec{F}_{*}\,dz + \left[\left(\frac{h}{2}\right)^j \vec{\sigma}_{x*}^{+} + \left(-\frac{h}{2}\right)^j \vec{\sigma}_{x*}^{-}\right]$$

$$\sigma_{x*}^{+} = \left[\sigma_{x*}\right]_{\frac{h}{2}} \qquad \sigma_{x*}^{-} = \left[\sigma_{x*}\right]_{-\frac{h}{2}}$$

we have

$$\frac{\partial\vec{N}_{x*}^{(j)}}{\partial x} + \frac{\partial\vec{N}_{y*}^{(j)}}{\partial y} - j\,\vec{N}_{z*}^{(j-1)} + \vec{f}_{*}^{(j)} = 0 \qquad (10)$$

where the boundary conditions (7) have already been considered.

3) – Now, if we let

$$\vec{V} = \sum_{0}^{\infty}{}_{j}\; \vec{V}^{(j)}(x,y)\,z^j \qquad (11)$$

we can express, through (1) all strains, and through (6) (9) all stresses and the $\vec{N}_{*}$ as power series of z

Thus Eqs. (10) are equations for $\vec{V}^{(j)}$, from which z has been eliminated.

It should be pointed out that this procedure is a generalizátion of the plates and shell equations, since by retaining a sufficient number of terms of Eqs. (11) and consequently a sufficient number of Eqs. (10) we can allow for approximations of any thickness of the plate.

4) Lets consider a little more in detail the case where the square of thickness is negligible. Thus we have $\vec{V} = \vec{V}^{(0)} + z\,\vec{V}^{(1)}$, and each quantity q (strains, stresses, etc.) can be expressed under the form $q = q^{(0)} + z\,q^{(1)}$.

Then we consider Eqs. (10) written from $j=0$, $j=1$ and their projections on the fixed axes, say (ξ,η,ζ) : we denote the generic equation by $(10)_{\xi}^{(0)}, (10)_{\eta}^{(0)}$ etc. In Eqs. $(10)_{\xi}^{(1)}$, $(10)_{\eta}^{(1)}$, $(10)_{\zeta}^{(1)}$ we have terms of the order of h, and other of the order of h^3 ; as h^2 is negligible, we infer that the former are zero too.

The conditions are written $\sigma_z = 0; \gamma_{yz} = \gamma_{zx} = 0$; this allows to obtain $u^{(1)}, v^{(1)}, w^{(1)}$ in terms of the other three quantities. furthermore, from Eqs. $(10)^{(1)}_{\xi}$, $(10)^{(1)}_{\eta}$, we obtain the expressions for τ_{yz}, τ_{xz} Then the three equations we employ are: $(10)^{(0)}_{\xi}$, $(10)^{(0)}_{\eta}$, $\partial (10)^{(1)}_{\xi}/\partial x + \partial (10)^{(1)}_{\eta}/\partial y - (10)^{(0)}_{\zeta}$

The results are given in Tab. I

5- For panel flutter purposes the simplifications generally admitted are to ignore strains as compared with rotation, and furthermore to consider w much greater than the other displacements. Then we would have, instead of (1):

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\frac{\partial w}{\partial x}\right]^2 \tag{12}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Again we consider the expression (11), but we write Eqs. (10) in the equivalent form:

$$\begin{aligned}
&\frac{\partial N^{(j)}_{xx}}{\partial x} + \frac{\partial N^{(j)}_{xy}}{\partial y} - j N^{(j-1)}_{xz} + \frac{\partial D^{(j)}_{xx}}{\partial x} + \\
&\qquad + \frac{\partial D^{(j)}_{xy}}{\partial y} - j D^{(j-1)}_{xz} + F^{(j)}_x = 0 \\
&\frac{\partial N^{(j)}_{yy}}{\partial y} + \frac{\partial N^{(j)}_{xy}}{\partial x} - j N^{(j-1)}_{yz} + \frac{\partial D^{(j)}_{yy}}{\partial y} + \\
&\qquad + \frac{\partial D^{(j)}_{yx}}{\partial x} - j D^{(j-1)}_{yz} + F^{(j)}_y = 0 \\
&\frac{\partial N^{(j)}_{xz}}{\partial x} + \frac{\partial N^{(j)}_{yz}}{\partial y} - j N^{(j-1)}_{zz} + \frac{\partial D^{(j)}_{zx}}{\partial x} + \\
&\qquad + \frac{\partial D^{(j)}_{zy}}{\partial y} - j D^{(j-1)}_{zz} + F^{(j)}_z = 0
\end{aligned} \tag{13}$$

Here we have:

$$N^{(j)}_{rs} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rs} z^j \, dz \tag{14}$$

$$D^{(j)}_{rs} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^j \left(\text{grad}\, u_r \times \vec{\sigma}_s\right) dz$$

The general equations for the approximation h^3 are written in Tab.II. As h^2 approaches zero, they reduce to the well-know Karman's large deflection Equations.

Appendix B

We write explicitly, in particular, Karman's equations when shear flexibility is taken into account. (It should be noted, however, that such equations are incorrect, since they consider only a portion of the h^2 terms which are included in the equations of Tab.II. Denoting by α_x, α_y the rotations in the $-x, -y$ directions, by w the vertical displacement, by ϕ Airy's function, we have:

$$\frac{\partial w}{\partial x} - \alpha_x + h^2\tau_x\left[\frac{\partial^2 \alpha_x}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 \alpha_x}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 \alpha_y}{\partial x \partial y}\right] = 0$$

$$\frac{\partial w}{\partial y} - \alpha_y + h^2\tau_y\left[\frac{\partial^2 \alpha_y}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 \alpha_y}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 \alpha_x}{\partial x \partial y}\right] = 0$$

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y}\right) + \mu\frac{\partial^2 w}{\partial t^2} =$$
$$\frac{\partial}{\partial x}\left(\alpha_x\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial w}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(\alpha_y\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial w}{\partial x}\frac{\partial^2 \phi}{\partial x \partial y}\right) - p \tag{1}$$

$$\frac{1}{Eh}\nabla^4\phi = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

where

$$\tau_x = \frac{1}{6(1-\nu)}\frac{\chi_x}{a^2}$$

$$\tau_y = \frac{1}{6(1-\nu)}\frac{\chi_y}{a^2}$$

with χ_x, χ_y shear factors, and p is the aeroelastic overpressure:

$$p = \frac{2q}{\beta}\left[\frac{\partial w}{\partial x} + \frac{1}{U}\frac{\beta^2-1}{\beta^2}\frac{\partial w}{\partial t}\right] \qquad (2)$$

with U flight speed, and

$$\beta^2 = M^2 - 1 \qquad (3)$$

Letting, for a simply supported panel:

$$\begin{aligned} \alpha_x(x,y,t) &= \Sigma_m \Sigma_n \frac{m\pi}{a} A_{mn}(t) \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\ \alpha_y(x,y,t) &= \Sigma_m \Sigma_n \frac{n\pi}{b} B_{mn}(t) \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \\ w(x,y,t) &= \Sigma_m \Sigma_n \varphi_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \end{aligned} \qquad (4)$$

from the first two Eqs. (1), we get, at h^2 accuracy:

$$\begin{aligned} A_{mn} &= \varphi_{mn}\left\{1 - h^2\tau_x\pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]\right\} \\ B_{mn} &= \varphi_{mn}\left\{1 - h^2\tau_y\pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]\right\} \end{aligned} \qquad (5)$$

Then, with the aid of the third and fourth of Eqs. (1) one can proceed as described f.i., in [11] . In particular, for the panel of infinite aspect ratio $[b \to \infty]$, we get the following system:

$$\ddot{\varphi}_m + \frac{\lambda_m^4}{1+m^2\theta}\varphi_m + \frac{3(1-\nu^2)}{1+m^2\theta}\left[\sum_{0j}^{\infty} j^2\varphi_j^2\right] m^2\varphi_m + 2\sigma\sum_{0i}^{\infty}\left[\varphi_i c_{im} + \theta^* D_{im}\dot{\varphi}_i\right] = 0 \qquad (6)$$

where:

$$\lambda_m^2 = \frac{\omega_m}{T} \qquad T = \sqrt{\frac{\pi^4 D}{\mu a^4}} \qquad (7)$$

$$\tau = Tt \qquad \frac{d(\,)}{d\tau} = (\,)^{\cdot}$$

$$\sigma = \frac{2q}{aT^2\beta\mu} \qquad \theta^* = \frac{Ta}{U}\frac{\beta^2-1}{\beta^2} \qquad (8)$$

$$\theta = \frac{\pi^2\chi}{6(1-\nu)}\left(\frac{h}{a}\right)^2 \qquad (9)$$

$$D_{im} = \begin{cases} \frac{1}{2} & (i=m) \\ 0 & (i \neq m) \end{cases} \qquad (10)$$

$$c_{im} = i\pi\int_0^1 \sin m\pi\xi \cos i\pi\xi \, d\xi$$

Appendix C

Consider a plate of infinite aspect ratio, fixed at the left and, subjected to the right end to a spring of axial stiffness c_0 , and let the plate be performing aeroelastic vibrations of such magnitude that finite curvature and horizontal inertia effects cannot be neglected.

Under such circumstances it is possible to show that the general equations of the plate reduce to:

$$\begin{cases} \dfrac{\partial}{\partial s}\left[\dfrac{ds}{dx}\dfrac{\partial}{\partial s}\left(\dfrac{D}{\rho}\right)\right]-\dfrac{\partial}{\partial s}\left[H\dfrac{\partial w}{\partial s}\dfrac{ds}{dx}\right]+\mu\dfrac{\partial^2 w}{\partial t^2}=0 \\ \\ \dfrac{\partial H}{\partial s}=\mu\dfrac{\partial^2 u}{\partial t^2} \end{cases} \tag{1}$$

Here, in addition to the symbols of App. B, s is the curvilinear abscissa, w, u are the displacements along the fixed axes before deformation, H is the horizontal thrust, μ is the mass per unit length.

If we consider the plate to be inextensible, we have at second order accuracy:

$$u=-\frac{1}{2}\int_0^a\left(\frac{\partial w}{\partial s}\right)^2 ds \tag{2}$$

If we let:

$$w=\Sigma_j\ \varphi_j(t)\ X_j(s) \tag{3}$$

where, $X_j(s)$ are the fundamental modes of vibration, we obtain the following Galerkin's equations:

$$A_m\left[\ddot{\varphi}_m+\lambda_m^4\varphi_m\right]+\frac{1}{a^2}\Sigma_i\Sigma_j\Sigma_k\alpha_{ijk}^{(m)}\varphi_i\varphi_j\varphi_k - \tag{4}$$
$$-\frac{1}{a^2}\left[\Sigma_i\Sigma_j\Sigma_k\varepsilon_{ijk}^{(m)}(\varphi_i\varphi_j)^{\cdot\cdot}\varphi_k+\nu\ \Sigma_i\Sigma_j\Sigma_k\delta_{ijk}^{(m)}\varphi_i\varphi_j\varphi_k\right]+$$
$$+\ \sigma\ \Sigma_i\left[\varphi_i\ c_{im}+\overset{*}{\theta}D_{im}\dot{\varphi}_i\right]=0$$

where, in addition to the symbols defined in App. C, we have:

$$A_m=\int_0^1 X_m^2\,d\xi$$

$$D_{im}=\int_0^1 X_i X_m\,d\xi$$

$$C_{im}=\int_0^1\frac{dX_i}{d\xi}X_m\,d\xi \qquad\qquad \xi=\frac{s}{a}$$

$$\alpha_{ijk}^{(m)}=\frac{1}{2}\int_0^1 X_m\frac{d}{d\xi}\left(X_j'X_i'X_k'''\right)d\xi+\frac{1}{2}\int_0^1 X_m\frac{d^2}{d\xi^2}\left(X_i'X_j'X_k''\right)d\xi$$

$$\varepsilon_{ijk}^{(m)}=\int_0^1 X_m\frac{d}{d\xi}\left(\chi_{ij}X_k'\right)d\xi$$

$$\delta_{ijk}^{(m)}=\Psi_{ij}(1)\int_0^1 X_m\frac{d}{d\xi}\left(X_kX_j'\right)d\xi \tag{5}$$

$$\chi_{ij}(\xi)=\int_\xi^1\Psi_{ij}(\xi)\,d\xi$$

$$\Psi_{ij}(\xi)=\frac{1}{2}\int_0^\xi X_i'X_j'\,d\xi$$

$$\frac{d(\ )}{d\xi}=(\ )'$$

Appendix D

The following Eqs. are a generalization of those contained in [4]. By assuming again an expansion $w=\Sigma_j X_j(x)\varphi_j(t)$, we have the following panel flutter equations (for a simply supported panel:

$$\ddot{\varphi}_m+\lambda_m^4\varphi_m+\sigma\Sigma_i\left[c_{im}\varphi_i+\theta^* D_{im}\dot{\varphi}_i\right]+$$

$$+8\Omega\Sigma_n\Sigma_p\Sigma_q n^2p^2q^3m\,\varphi_n(\tau)\varphi_p(\tau)\varphi_q(\tau)\,I_{npqm}=0 \quad (1)$$

where:

$$I_{npqm}=\int_0^1 \sin n\pi\xi\,\sin p\pi\xi\,\cos q\pi\xi\,\cos m\pi\xi\,d\xi \quad (2)$$

and:

$$\Omega=\frac{3}{4}\frac{\pi^4}{L^2}\frac{D_1+D_2}{D_e} \quad (3)$$

here:

$$D_1=\frac{1}{45}\left[h_1^5-h_2^5\right]b_1E_1$$

$$D_2=\frac{1}{45}\left[h_2^5b_2E_2\right] \quad (4)$$

$$D_e=\frac{1}{9}\left[E_2h_2^3+E_1(h_1^3-h_2^3)\right]$$

and the subscripts 1,2 refer to the two parts the panel is constituted by (Fig. 1-D). It should by pointed out, however, that the value of Ω may be positive or negative.

Appendix E

The temperature distribution in a slab of thickness a subjected to a heat flux Ω^* from the side $x=0$ insulated from the side $x=a$, with constant thermal coefficient c (specific heat per unit volume) and k (thermal conductivity) is:

$$T=\frac{\Omega^*}{ca}\left[t+\frac{2ca^2}{\pi^2k}\sum_1^\infty{}_n\frac{\cos n\pi\xi}{n^2}\left(1-e^{-n^2\tau}\right)\right] \quad (1)$$

where t =time, and $\xi=\frac{x}{a}$; $\tau=\frac{\pi^2k}{ca^2}t$

For the case of constant zero temperature at $\xi=1$ we have:

$$T=\frac{\Omega^*a}{k}\left[(1-\xi)-\frac{8}{\pi^3}\sum_{n=1,3,\dots}^{\infty}\frac{e^{-n^2\tau}}{n^3}\cos\frac{n\pi\xi}{2}\right] \quad (2)$$

As t approaches infinity (i.e. for sufficiently high value of time) the temperature in the first case is practically uniform and is given by $\frac{\Omega^*}{ca}\left\{t+\frac{ca^2}{k}\left[\frac{3(1-\xi)^2-1}{6}\right]\right\}\simeq\frac{\Omega^*t}{ca}$

Thus, if we confine ourselves to consider small variation of temperature about a reference value T_o, the variation of modulus of elasticity will be $E=E_o\left[1+\frac{1}{E_o}\left(\frac{dE}{dT_o}\right)\frac{\Omega^*t}{ca}\right]=E_o[1+\alpha t]$

where

$$\alpha = \frac{1}{E_0}\frac{dE}{dT}\frac{Q^*}{ca} \qquad (3)$$

In the second case, we have
$E = E_0\left[1 + \beta(1-\xi)\right]$
where

$$\beta = \frac{1}{E_0}\frac{dE}{dT}\frac{Q^* a}{k} \qquad (4)$$

Appendix F

Consider a material for which the generalized stress-strain relationship is:

$$\sigma = E\varepsilon + F\dot{\varepsilon} \qquad (1)$$

For a one-dimensional plate:

$$D\left[\frac{\partial^4 w}{\partial x^4} + A^*\frac{\partial}{\partial t}\frac{\partial^4 w}{\partial x^4}\right] + \mu\frac{\partial^2 w}{\partial t^2} + p = 0 \qquad (2)$$

where A^* is a constant depending on F .

By the Galerkin approach:

$$w = \Sigma_j\, \varphi_j(t)\, \sin\frac{j\pi x}{a} \qquad (3)$$

we have:

$$\ddot{\varphi}_m + \lambda_m^4\varphi_m + 2\sigma\Sigma_i\left[c_{im}\varphi_i + \vartheta^* D_{im}\dot{\varphi}_i\right] +$$

$$+\ \omega_m^2\varepsilon\dot{\varphi}_m = 0 \qquad (4)$$

with:

$$\varepsilon = \frac{\pi^2}{a^2}\frac{A^*}{E}\sqrt{\frac{D}{\mu}} \qquad (5)$$

Appendix G

Consider Eq.(1)-D written in the form:

$$\ddot{Y} + AY = \phi(Y,\dot{Y}) \qquad (1)$$

If we are looking for transient response and limit cycle to Eq. (1), we employ the method of slowly varying amplitudes. Lets:

$$Y = \Sigma_k \left[M_k \sin k\omega t + N_k \cos k\omega t\right] \qquad (2)$$

where $M_k(t)$ and $N_k(t)$ vary with time, but we stipulate that the variation is so slow that derivatives of $\dot{M}_k$, $\dot{N}_k$ can be neglected as compared with M_k, N_k.

In such conditions:

$$\frac{\dot{Y}}{\omega} = \Sigma_k k\left[M_k \cos k\omega t - N_k \sin k\omega t\right] \qquad (3)$$

Furthermore we may write:

$$\phi = \Sigma_k\left[\Psi_k(M_j,N_j)\sin k\omega t + \Omega_k(M_j,N_j)\cos k\omega t\right] \qquad (4)$$

where Ω_k, Ψ_k are functions of the unknown values M_j, N_j determined by the Fourier technique. Now we have the system of differential equations.

$$\begin{cases} \omega k \dot{M}_k - \omega^2 k^2 N_k + AN_k = \Psi_k(M_j, N_j) \\ -\omega k \dot{N}_k - \omega^2 k^2 M_k + AM_k = \Omega_k(M_j, N_j) \end{cases} \qquad (5)$$

If Y has, say, q components, and if the expansion (2) is stopped at the k-th term, we have for (5) a set of $qk-1$ scalar equations which is approximately equivalent to (1), since one component can be arbitrarily set equal to zero. The remaining Equation serves to determine ω.

Setting in (5) $\dot{M}_k = \dot{N}_k = 0$, we have the well-known equation of harmonic balance: from a numerical standpoint, however, the steady-state response is better obtained as the limit solution to Eqs. (5), regardless of the initial conditions.

TAB. I

$$\sigma_{x*}^{(0)}=\frac{E}{1-\nu^2}\left[\varepsilon_x^{(0)}+\nu\varepsilon_y^{(0)}\right] \qquad \sigma_{x*}^{(1)}=\frac{E}{1-\nu^2}\left[\varepsilon_x^{(1)}+\nu\varepsilon_y^{(1)}\right]$$

$$\sigma_{y*}^{(0)}=\frac{E}{1-\nu^2}\left[\varepsilon_y^{(0)}+\nu\varepsilon_x^{(0)}\right] \qquad \sigma_{y*}^{(1)}=\frac{E}{1-\nu^2}\left[\varepsilon_y^{(1)}+\nu\varepsilon_x^{(1)}\right]$$

$$\tau_{xy*}^{(0)}=\frac{E}{1-\nu^2}\,\frac{1-\nu}{2}\,\gamma_{xy}^{(0)} \qquad \tau_{xy*}^{(1)}=\frac{E}{1-\nu^2}\,\frac{1-\nu}{2}\,\gamma_{xy}^{(1)}$$

$$\tau_{xz*}^{(1)}=\frac{E}{1-\nu^2}\,\frac{1-\nu}{2}\,\gamma_{xz}^{(1)} \qquad \tau_{yz*}^{(1)}=\frac{E}{1-\nu^2}\,\frac{1-\nu}{2}\,\gamma_{yz}^{(1)}$$

$$\varepsilon_x^{(0)}=\sqrt{\left[1+\frac{\partial u^{(0)}}{\partial x}\right]^2+\left[\frac{\partial v^{(0)}}{\partial x}\right]^2+\left[\frac{\partial w^{(0)}}{\partial x}\right]^2}-1$$

$$\varepsilon_y^{(0)}=\sqrt{\left[1+\frac{\partial v^{(0)}}{\partial y}\right]^2+\left[\frac{\partial u^{(0)}}{\partial y}\right]^2+\left[\frac{\partial w^{(0)}}{\partial y}\right]^2}-1$$

$$\varepsilon_x^{(1)}=\frac{\left[1+\frac{\partial u^{(0)}}{\partial x}\right]\frac{\partial u^{(1)}}{\partial x}+\frac{\partial v^{(0)}}{\partial x}\frac{\partial v^{(1)}}{\partial x}+\frac{\partial w^{(0)}}{\partial x}\frac{\partial w^{(1)}}{\partial x}}{1+\varepsilon_x^{(0)}}$$

$$\varepsilon_y^{(1)}=\frac{\left[1+\frac{\partial v^{(0)}}{\partial y}\right]\frac{\partial v^{(1)}}{\partial y}+\frac{\partial u^{(0)}}{\partial y}\frac{\partial u^{(1)}}{\partial y}+\frac{\partial w^{(0)}}{\partial y}\frac{\partial w^{(1)}}{\partial y}}{1+\varepsilon_y^{(0)}}$$

$$\gamma_{xy}^{(0)}=\sin^{-1}\frac{a_{xy}}{b_{xy}} \qquad \gamma_{xy}^{(1)}=\left[b_{xy}-\frac{a_{xy}d_{xy}}{c_{xy}}\right]\frac{1}{\sqrt{c_{xy}^2-a_{xy}^2}}$$

$$a_{xy}=\frac{\partial u^{(0)}}{\partial x}+\frac{\partial v^{(0)}}{\partial y}+\frac{\partial v^{(0)}}{\partial x}\frac{\partial v^{(0)}}{\partial y}+\frac{\partial w^{(0)}}{\partial x}\frac{\partial w^{(0)}}{\partial y}+\frac{\partial u^{(0)}}{\partial x}\frac{\partial u^{(0)}}{\partial y}$$

$$b_{xy}=\frac{\partial u^{(1)}}{\partial x}+\frac{\partial v^{(1)}}{\partial y}+\frac{\partial u^{(0)}}{\partial x}\frac{\partial u^{(1)}}{\partial y}+\frac{\partial u^{(1)}}{\partial x}\frac{\partial u^{(0)}}{\partial y}+\frac{\partial v^{(0)}}{\partial x}\frac{\partial v^{(1)}}{\partial y}+$$

$$+\frac{\partial v^{(1)}}{\partial x}\frac{\partial v^{(0)}}{\partial y}+\frac{\partial w^{(0)}}{\partial x}\frac{\partial w^{(1)}}{\partial y}+\frac{\partial w^{(0)}}{\partial y}\frac{\partial w^{(1)}}{\partial x}$$

$$c_{xy}=\left[1+\varepsilon_x^{(0)}\right]\left[1+\varepsilon_y^{(0)}\right]$$

$$d_{xy}=\varepsilon_x^{(0)}+\varepsilon_y^{(0)}+\varepsilon_x^{(1)}\varepsilon_x^{(0)}+\varepsilon_y^{(1)}\varepsilon_y^{(0)}$$

$$\frac{N_{y\eta}^{(0)}}{h}=\sigma_{y*}^{(0)}\left[1+\frac{\partial v^{(0)}}{\partial y}\right]+\tau_{xy*}^{(0)}\frac{\partial v^{(0)}}{\partial x}+\tau_{yz*}^{(0)}v^{(1)}$$

$$\frac{N_{y\eta}^{(1)}}{\frac{h^3}{12}}=\sigma_{y*}^{(1)}\left[1+\frac{\partial v^{(0)}}{\partial y}\right]+\sigma_{y*}^{(0)}\frac{\partial v^{(1)}}{\partial y}+\frac{\partial v^{(1)}}{\partial x}\tau_{xy*}^{(0)}+$$

$$+\frac{\partial v^{(0)}}{\partial x}\tau_{xy*}^{(1)}+\tau_{yz*}^{(1)}v^{(1)}$$

$$\begin{bmatrix}\tau_{xz*}^{(0)}\\ \tau_{yz*}^{(0)}\end{bmatrix}=\frac{1}{h}\begin{bmatrix}1+\frac{\partial u^{(0)}}{\partial x} & \frac{\partial u^{(0)}}{\partial y}\\ \frac{\partial v^{(0)}}{\partial x} & 1+\frac{\partial v^{(0)}}{\partial y}\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial N_{x\xi}^{(1)}}{\partial x}+\frac{\partial N_{y\xi}^{(1)}}{\partial y}-f_{\xi*}^{(1)}\\ \frac{\partial N_{x\eta}^{(1)}}{\partial x}+\frac{\partial N_{y\eta}^{(1)}}{\partial y}-f_{\eta*}^{(1)}\end{bmatrix}$$

solving equations

$$\frac{\partial N_{x\xi*}^{(0)}}{\partial x}+\frac{\partial N_{y\xi*}^{(0)}}{\partial y}+f_{\xi*}^{(0)}=0$$

$$\frac{\partial N_{x\eta*}}{\partial x}+\frac{\partial N_{y\eta*}}{\partial y}+f_{\eta*}=0$$

$$\frac{\partial^2 N_{x\xi*}}{\partial x^2}+\frac{\partial N_{y\eta*}}{\partial y^2}+\frac{\partial^2}{\partial x\partial y}\left[N_{y\xi*}^{(1)}+N_{x\eta*}^{(1)}\right]+$$

$$+\frac{\partial f_{\xi*}^{(1)}}{\partial x}+\frac{\partial f_{\eta*}^{(1)}}{\partial y}+f_{\xi*}^{(0)}+h\frac{\partial}{\partial x}\left\{\left[1+\frac{\partial u^{(0)}}{\partial x}\right]\tau_{zx*}^{(0)}+\right.$$

$$\left.+\frac{\partial u^{(0)}}{\partial y}\tau_{zy*}^{(0)}-\frac{\partial u^{(0)}}{\partial x}\tau_{zx*}^{(0)}-\frac{\partial u^{(0)}}{\partial y}\tau_{zy*}^{(0)}\right\}+$$

$$+h\frac{\partial}{\partial y}\left\{\frac{\partial v^{(0)}}{\partial x}\tau_{xz*}^{(0)}+\left[1+\frac{\partial v^{(0)}}{\partial y}\right]\tau_{zy*}^{(0)}-\right.$$

$$\left.-\frac{\partial v^{(0)}}{\partial x}\tau_{zx*}^{(0)}+\frac{\partial v^{(0)}}{\partial y}\tau_{zy*}^{(0)}\right\}=0$$

$$\gamma_{xz}^{(1)}=\left[b_{xz}-\frac{a_{xz}d_{xz}}{c_{xz}}\right]\frac{1}{\sqrt{c_{xz}^2-a_{xz}^2}}$$

$$a_{xz}=u^{(1)}+\frac{\partial w^{(0)}}{\partial x}+\left[u^{(1)}\frac{\partial u^{(0)}}{\partial x}+v^{(1)}\frac{\partial v^{(0)}}{\partial x}+w^{(1)}\frac{\partial w^{(0)}}{\partial x}\right]$$

$$b_{xz}=\frac{\partial w^{(1)}}{\partial x}+\left[u^{(1)}\frac{\partial u^{(1)}}{\partial x}+v^{(1)}\frac{\partial v^{(1)}}{\partial x}+w^{(1)}\frac{\partial w^{(1)}}{\partial x}\right]$$

$$c_{xz}=\left[1+\varepsilon_x^{(0)}\right]\left[1+\varepsilon_z^{(0)}\right] \qquad d_{xz}=\varepsilon_x^{(1)}\left[1+\varepsilon_z^{(0)}\right]$$

$$\varepsilon_z^{(0)} = \sqrt{[1+w^{(1)}]^2 + u^{(1)^2} + v^{(1)^2}} - 1$$

$$u^{(1)} = u'^{(1)}[1+w^{(1)}]$$

$$v^{(1)} = v'^{(1)}[1+w^{(1)}]$$

$$w^{(1)} = \frac{1 - \frac{\nu}{1-\nu}\left[\varepsilon_x^{(0)} + \varepsilon_y^{(0)}\right]}{\sqrt{1 + [u'^{(1)}]^2 + [v'^{(1)}]^2}} - 1$$

$$u'^{(1)} = -\frac{\begin{vmatrix} \frac{\partial w^{(0)}}{\partial x} & \frac{\partial v^{(0)}}{\partial x} \\ \frac{\partial w^{(0)}}{\partial y} & 1 + \frac{\partial v^{(0)}}{\partial y} \end{vmatrix}}{\begin{vmatrix} 1 + \frac{\partial u^{(0)}}{\partial x} & \frac{\partial v^{(0)}}{\partial x} \\ \frac{\partial u^{(0)}}{\partial y} & \frac{\partial v^{(0)}}{\partial y} \end{vmatrix}}$$

$$v'^{(1)} = -\frac{\begin{vmatrix} 1 + \frac{\partial u^{(0)}}{\partial x} & \frac{\partial u^{(0)}}{\partial x} \\ \frac{\partial w^{(0)}}{\partial y} & \frac{\partial w^{(0)}}{\partial y} \end{vmatrix}}{\begin{vmatrix} 1 + \frac{\partial u^{(0)}}{\partial x} & \frac{\partial v^{(0)}}{\partial x} \\ \frac{\partial u^{(0)}}{\partial y} & \frac{\partial v^{(0)}}{\partial y} \end{vmatrix}}$$

$$\frac{N_{x\xi *}^{(0)}}{h} = \sigma_{x*}^{(0)}\left[1 + \frac{\partial u^{(0)}}{\partial x}\right] + \tau_{xy*}^{(0)}\frac{\partial u^{(0)}}{\partial y} + \tau_{xz*}^{(0)} u^{(1)}$$

$$\frac{N_{x\xi *}^{(1)}}{\frac{h^3}{12}} = \sigma_{x*}^{(1)}\left[1 + \frac{\partial u^{(0)}}{\partial x}\right] + \tau_{xy*}^{(0)}\frac{\partial u^{(1)}}{\partial x} + \tau_{xy*}^{(0)}\frac{\partial u^{(1)}}{\partial y} + \tau_{xy*}^{(1)}\frac{\partial u^{(0)}}{\partial y} + \tau_{xz*}^{(1)} u^{(1)}$$

$$\frac{N_{y\xi *}^{(0)}}{h} = \tau_{xy*}^{(0)}\left[1 + \frac{\partial u^{(0)}}{\partial x}\right] + \sigma_{y*}^{(0)}\frac{\partial u^{(0)}}{\partial y} + \tau_{yz*}^{(0)} u^{(1)}$$

$$\frac{N_{y\xi *}^{(1)}}{\frac{h^3}{12}} = \tau_{xy*}^{(1)}\left[1 + \frac{\partial u^{(0)}}{\partial x}\right] + \tau_{xy*}^{(0)}\frac{\partial u^{(1)}}{\partial x} + \sigma_{y*}^{(1)}\frac{\partial u^{(0)}}{\partial y} + \sigma_{y*}^{(0)}\frac{\partial u^{(1)}}{\partial y} + \tau_{yz*}^{(1)} u^{(1)}$$

$$\frac{N_{x\eta *}^{(0)}}{h} = \sigma_{x*}^{(0)}\frac{\partial v^{(0)}}{\partial x} + \tau_{xy*}^{(0)}\left[1 + \frac{\partial v^{(0)}}{\partial y}\right] + \tau_{xz*}^{(0)} v^{(1)}$$

$$\frac{N_{x\eta *}^{(1)}}{\frac{h^3}{12}} = \sigma_{x*}^{(1)}\frac{\partial v^{(0)}}{\partial x} + \sigma_{x*}^{(0)}\frac{\partial v^{(1)}}{\partial x} + \tau_{xy*}^{(1)}\left[1 + \frac{\partial v^{(0)}}{\partial y}\right] + \tau_{xy*}^{(0)}\frac{\partial v^{(1)}}{\partial y} + \tau_{xz*}^{(0)} v^{(1)} + \tau_{xz*}^{(1)} v^{(0)}$$

TAB. II

$$D_{rs}^{(0)} = h\left[\vec{\sigma}_s^{(0)} \times \operatorname{grad} u_r^{(0)} + \frac{h^2}{12}\sum_{k=0}^{2} \vec{\sigma}_s^{(k)} \times \operatorname{grad} u_r^{(2-k)}\right]$$

$$D_{rs}^{(1)} = \frac{h^3}{12}\left[\sum_{k=0}^{1} \vec{\sigma}_s^{(k)} \times \operatorname{grad} u_r^{(1-k)} + \frac{3h^2}{20}\sum_{k=1}^{2} \vec{\sigma}_s^{(k)} \times \operatorname{grad} u_r^{(3-k)}\right]$$

$$D_{rs}^{(2)} = \frac{h^3}{12}\left[\vec{\sigma}_s^{(0)} \times \operatorname{grad} u_r^{(0)} + \frac{3h^2}{20}\sum_{k=0}^{2} \vec{\sigma}_s^{(k)} \times \operatorname{grad} u_r^{(2-k)}\right]$$

$$N_{rs}^{(0)} = h\left[\tau_{rs}^{(0)} + \frac{h^3}{12}\tau_{rs}^{(2)}\right]$$

$$N_{rs}^{(1)} = \frac{h^3}{12}\tau_{rs}^{(1)}$$

$$N_{rs}^{(2)} = \frac{h^3}{12}\left[\tau_{rs}^{(0)} + \frac{3h^2}{20}\tau_{rs}^{(2)}\right]$$

$$\vec{\sigma}_x^{(j)} = \begin{bmatrix} \sigma_{xx}^{(j)} \\ \sigma_{xy}^{(j)} \\ \sigma_{xz}^{(j)} \end{bmatrix} \qquad \vec{\sigma}_y^{(j)} = \begin{bmatrix} \sigma_{yx}^{(j)} \\ \sigma_{yy}^{(j)} \\ \sigma_{yz}^{(j)} \end{bmatrix} \qquad \vec{\sigma}_z^{(j)} = \begin{bmatrix} \sigma_{zx}^{(j)} \\ \sigma_{zy}^{(j)} \\ \sigma_{zz}^{(j)} \end{bmatrix}$$

$$u_x^{(j)} = u^{(j)} \qquad u_y^{(j)} = v^{(j)} \qquad u_z^{(j)} = w^{(j)}$$

$$\sigma_{rr}^{(j)} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_r^{(j)} + \nu(\varepsilon_p^{(j)} + \varepsilon_q^{(j)})\right] \qquad r \neq p \neq q$$

$$\sigma_{rs}^{(j)} = G\,\gamma_{rs}^{(j)}$$

$$\varepsilon_r^{(0)} = \frac{\partial u_r^{(0)}}{\partial r} + \frac{1}{2}\left[\frac{\partial w^{(0)}}{\partial r}\right]^2 \qquad r \neq z$$

$$\varepsilon_r^{(1)} = \frac{\partial u_r^{(1)}}{\partial r} + \frac{\partial w^{(1)}}{\partial r}\frac{\partial w^{(0)}}{\partial r} \qquad r \neq z$$

$$\varepsilon_r^{(2)} = \frac{\partial u_r^{(2)}}{\partial r} + \frac{1}{2}\left\{\left[\frac{\partial w^{(1)}}{\partial r}\right]^2 + 2\frac{\partial w^{(0)}}{\partial r}\frac{\partial w^{(2)}}{\partial r}\right\} \qquad r \neq z$$

$$\varepsilon_z^{(0)} = w^{(1)} + \frac{1}{2}\left\{\left[u^{(1)}\right]^2 + \left[v^{(1)}\right]^2\right\}$$

$$\varepsilon_z^{(1)} = 2\left[w^{(2)} + u^{(1)}u^{(2)} + v^{(1)}v^{(2)}\right]$$

$$\varepsilon_z^{(2)} = 2\left\{\left[u^{(2)}\right]^2 + \left[v^{(2)}\right]^2\right\}$$

$$\gamma_{xy}^{(0)} = \frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} + \frac{\partial w^{(0)}}{\partial x}\frac{\partial w^{(0)}}{\partial y}$$

$$\gamma_{xy}^{(1)} = \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} + \frac{\partial w^{(0)}}{\partial x}\frac{\partial w^{(1)}}{\partial y} + \frac{\partial w^{(1)}}{\partial x}\frac{\partial w^{(0)}}{\partial y}$$

$$\gamma_{xy}^{(2)} = \frac{\partial u^{(2)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} + \frac{\partial w^{(0)}}{\partial x}\frac{\partial w^{(2)}}{\partial y} + \frac{\partial w^{(1)}}{\partial x}\frac{\partial w^{(1)}}{\partial y} + \frac{\partial w^{(2)}}{\partial x}\frac{\partial w^{(0)}}{\partial y}$$

$$\gamma_{rz}^{(0)} = u_r^{(1)} + \frac{\partial w^{(0)}}{\partial r} - w^{(1)}\frac{\partial u_r^{(0)}}{\partial r} - u_r^{(1)}\frac{\partial w^{(0)}}{\partial r}$$

$$\gamma_{rz}^{(1)} = 2u_r^{(2)} + \frac{\partial w^{(1)}}{\partial r} - \left[w^{(1)}\frac{\partial u_r^{(1)}}{\partial r} + 2w^{(2)}\frac{\partial u_r^{(0)}}{\partial r}\right] - \left[u_r^{(1)}\frac{\partial w^{(1)}}{\partial r} + 2u_r^{(2)}\frac{\partial w^{(0)}}{\partial r}\right]$$

$$\gamma_{rz}^{(2)} = \frac{\partial w^{(2)}}{\partial r} - \left[2w^{(2)}\frac{\partial u_r^{(1)}}{\partial r} + w^{(1)}\frac{\partial u_r^{(2)}}{\partial r}\right] - \left[2u_r^{(2)}\frac{\partial w^{(1)}}{\partial r} + u_r^{(1)}\frac{\partial w^{(2)}}{\partial r}\right]$$

$$(r = x, y, z)$$

$$\frac{\partial N_{xx}^{(j)}}{\partial x} + \frac{\partial N_{xy}^{(j)}}{\partial y} - j\,N_{xz}^{(j-1)} + \frac{\partial D_{xx}^{(j)}}{\partial x} + \frac{\partial D_{xy}^{(j)}}{\partial y} - j\,D_{xz}^{(j-1)} + F_x^{(j)} = 0$$

$$\frac{\partial N_{yy}^{(j)}}{\partial y} + \frac{\partial N_{xy}^{(j)}}{\partial x} - j\,N_{yz}^{(j-1)} + \frac{\partial D_{yy}^{(j)}}{\partial y} + \frac{\partial D_{yx}^{(j)}}{\partial x} - j\,D_{yz}^{(j-1)} + F_y^{(j)} = 0$$

$$\frac{\partial N_{xz}^{(j)}}{\partial x} + \frac{\partial N_{yz}^{(j)}}{\partial y} - j\,N_{zz}^{(j-1)} + \frac{\partial D_{zx}^{(j)}}{\partial x} + \frac{\partial D_{zy}^{(j)}}{\partial y} - j\,D_{zz}^{(j-1)} + F_z^{(j)} = 0$$

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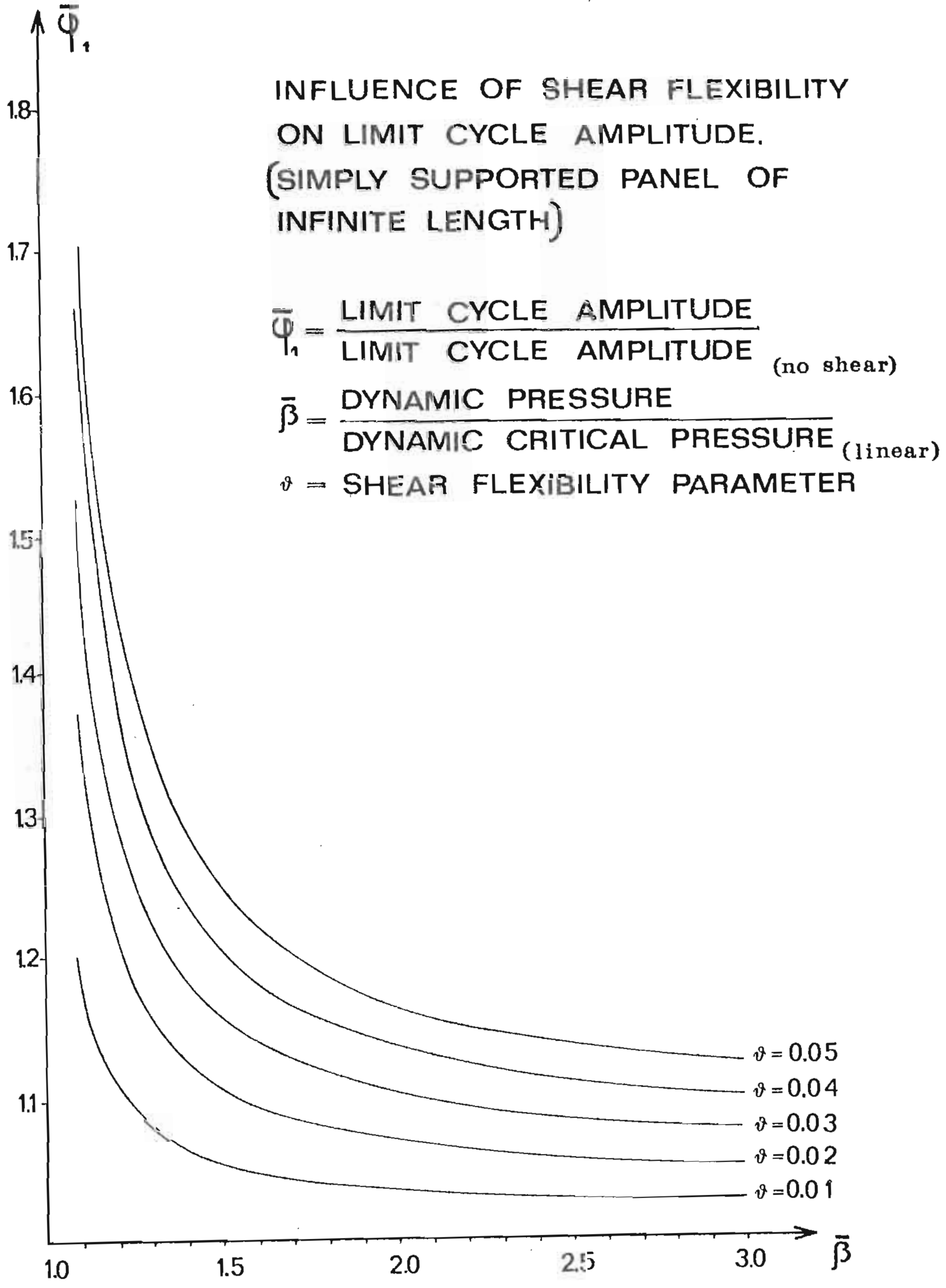

INFLUENCE OF SHEAR FLEXIBILITY ON LIMIT CYCLE AMPLITUDE.
(SIMPLY SUPPORTED PANEL OF INFINITE LENGTH)
$\bar{\varphi}_1 = \frac{\text{LIMIT CYCLE AMPLITUDE}}{\text{LIMIT CYCLE AMPLITUDE}_{\text{(no shear)}}}$
$\bar{\beta} = \frac{\text{DYNAMIC PRESSURE}}{\text{DYNAMIC CRITICAL PRESSURE}_{\text{(linear)}}}$
ϑ = SHEAR FLEXIBILITY PARAMETER
$\vartheta = 0.05$
$\vartheta = 0.04$
$\vartheta = 0.03$
$\vartheta = 0.02$
$\vartheta = 0.01$
$\bar{\varphi}_1$
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
1.5
2.0
2.5
3.0
$\bar{\beta}$

FIG. 1

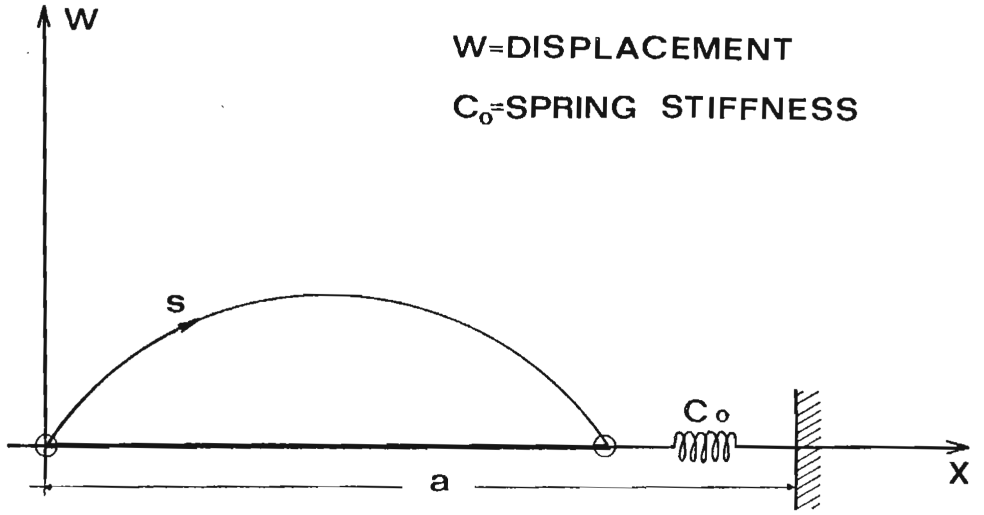

W
W=DISPLACEMENT
C_o=SPRING STIFFNESS
s
C_o
a
X

FIG. 1c

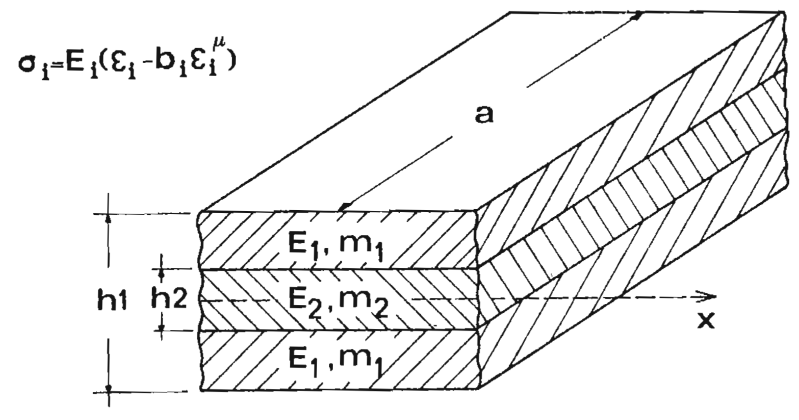
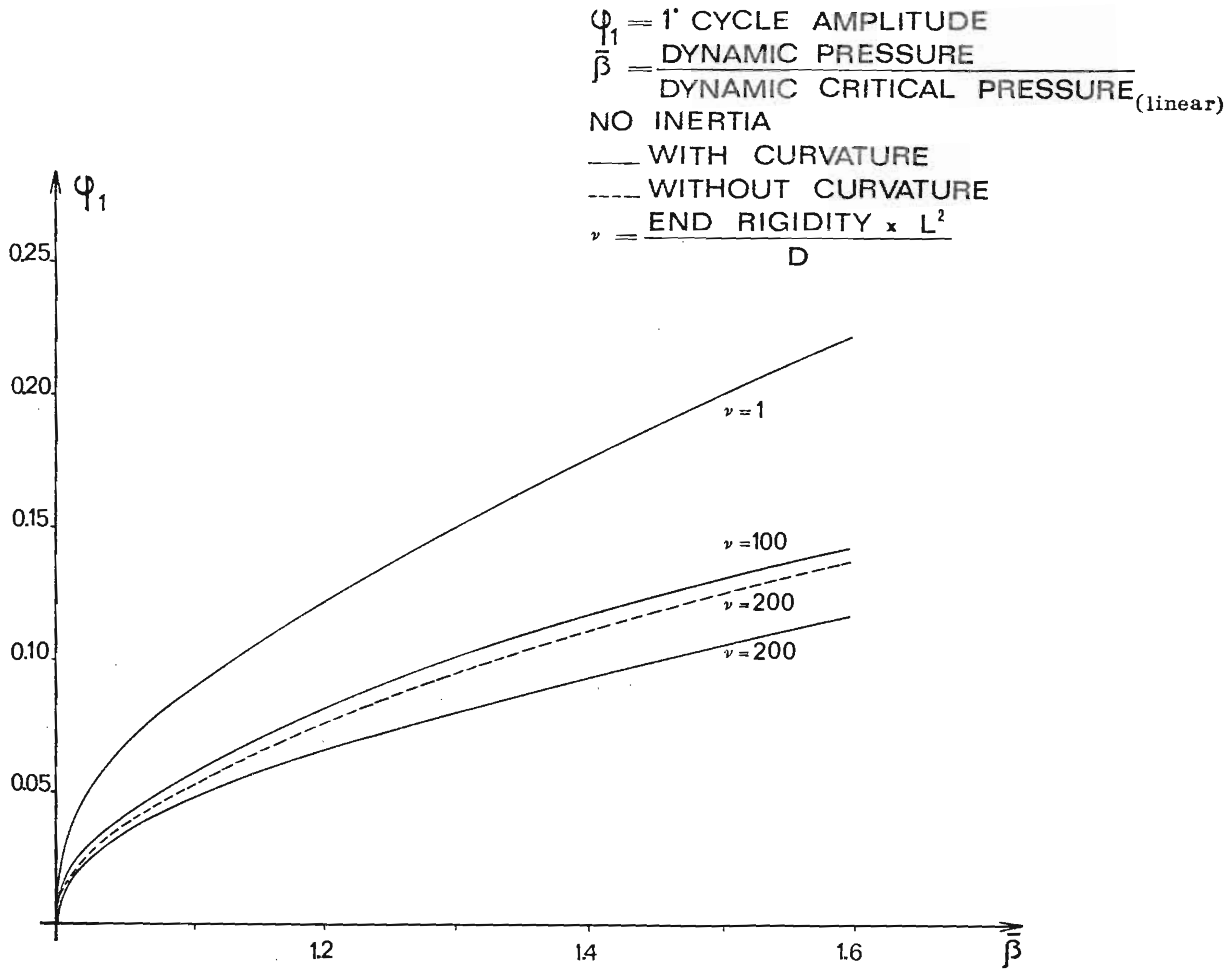
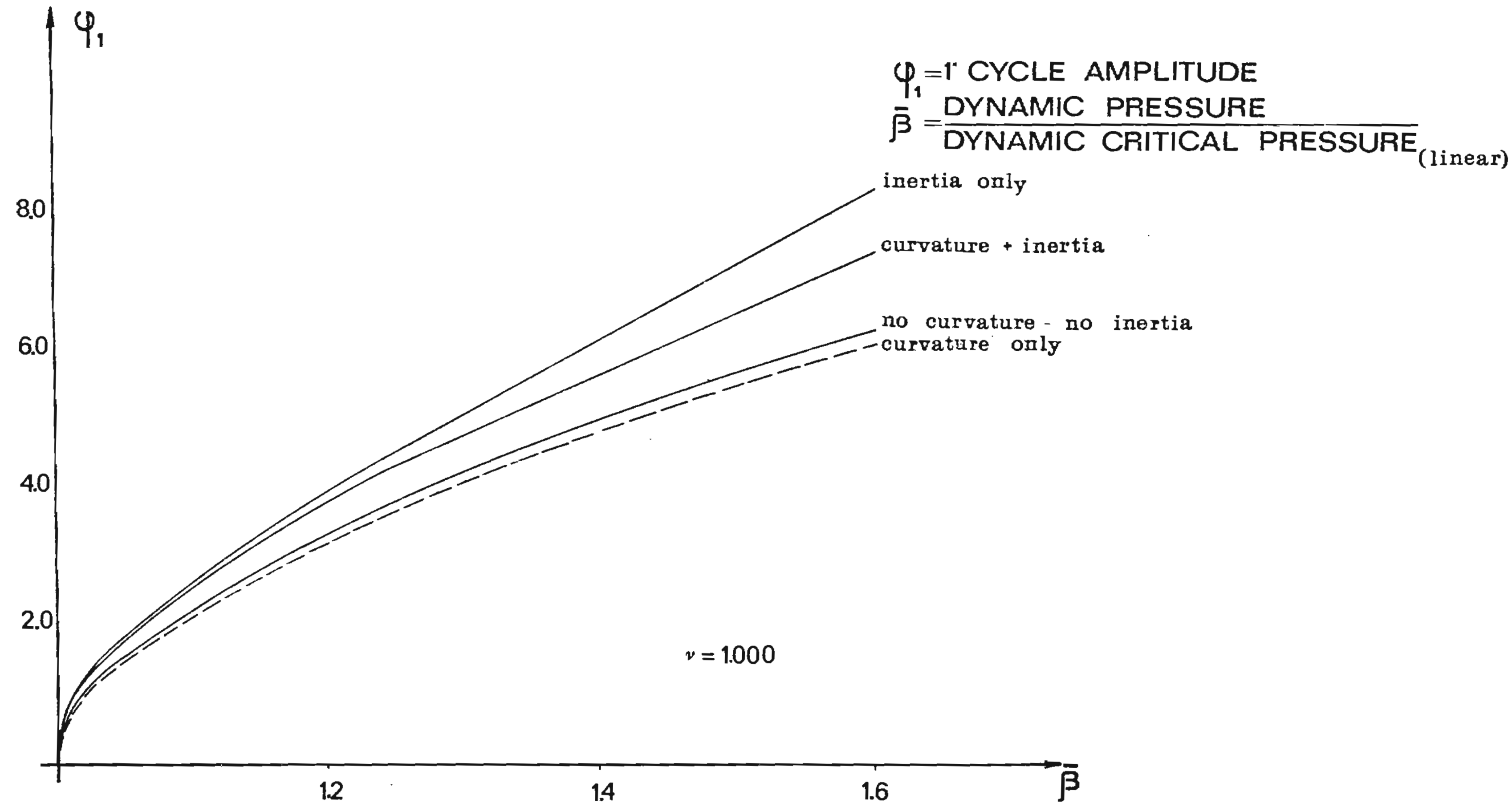

$\sigma_i=E_i(\varepsilon_i-b_i\varepsilon_i^{\mu})$
a
E_1, m_1
h1
h2
E_2, m_2
x
E_1, m_1

FIG. 1D



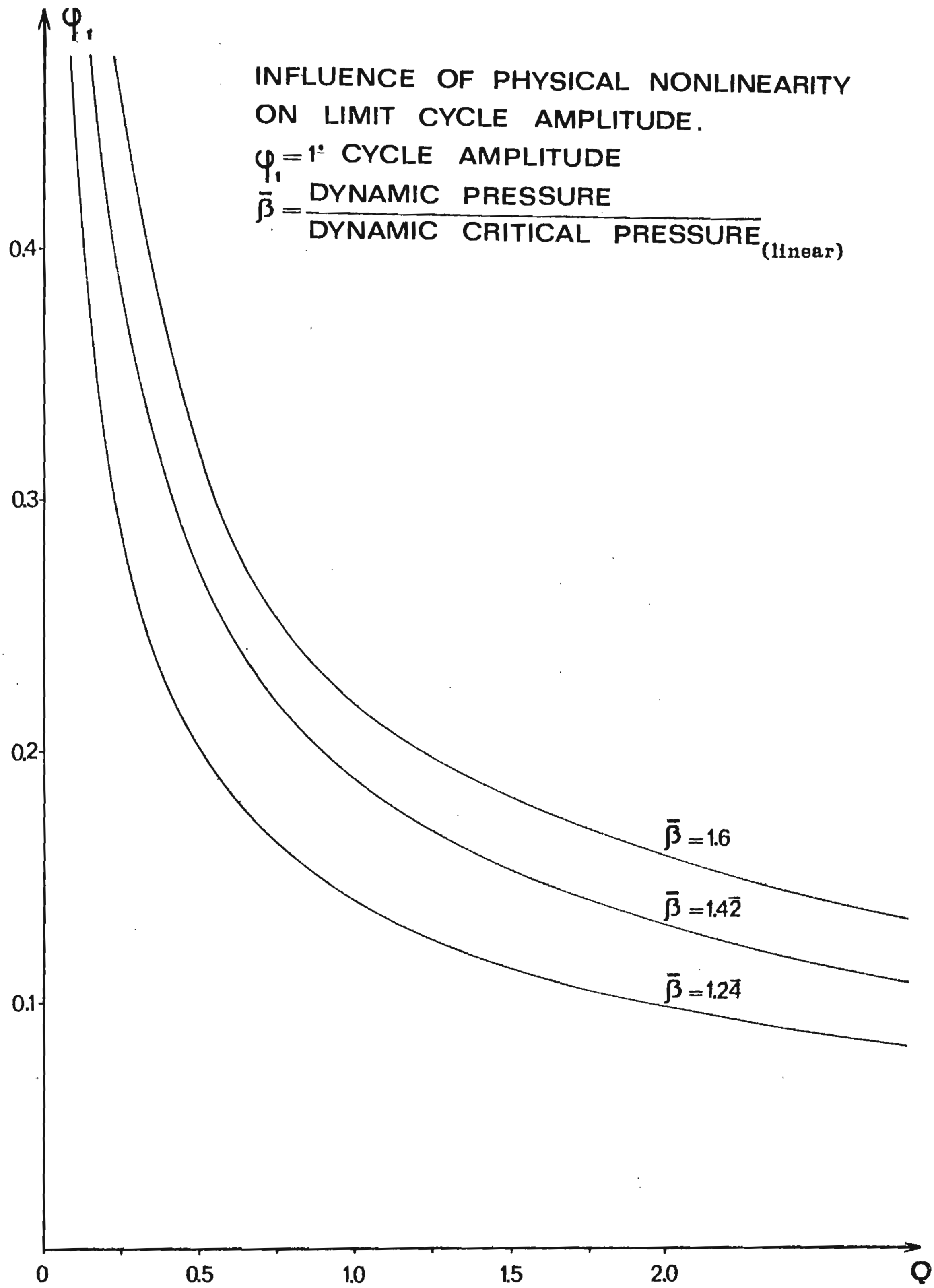
φ1 = 1° CYCLE AMPLITUDE
β̄ = DYNAMIC PRESSURE / DYNAMIC CRITICAL PRESSURE(linear)
NO INERTIA
WITH CURVATURE
WITHOUT CURVATURE
ν = END RIGIDITY x L² / D
φ1
0.25
0.20
0.15
0.10
0.05
ν = 1
ν = 100
ν = 200
ν = 200
1.2
1.4
1.6
β̄

FIG. 2



φ1
φ1 =1' CYCLE AMPLITUDE
β̄ = DYNAMIC PRESSURE / DYNAMIC CRITICAL PRESSURE(linear)
inertia only
curvature + inertia
no curvature - no inertia
curvature only
8.0
6.0
4.0
2.0
ν = 1.000
1.2
1.4
1.6
β̄

FIG. 3



φ1
INFLUENCE OF PHYSICAL NONLINEARITY ON LIMIT CYCLE AMPLITUDE.
φ1 = 1st CYCLE AMPLITUDE
β̄ = DYNAMIC PRESSURE / DYNAMIC CRITICAL PRESSURE(linear)
0.4
0.3
0.2
0.1
β̄ = 1.6
β̄ = 1.42̄
β̄ = 1.24̄
0
0.5
1.0
1.5
2.0
Q

FIG. 4

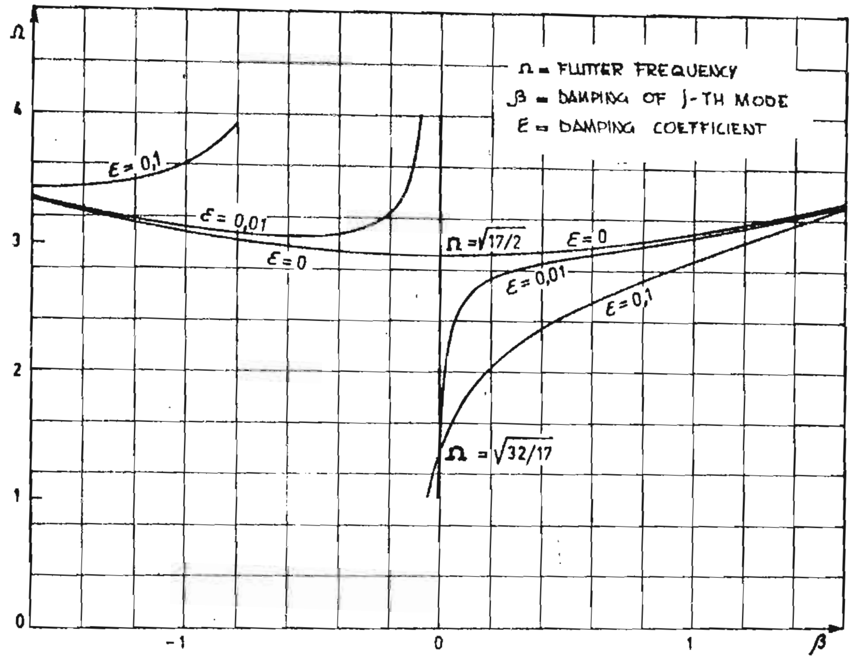

Ω = FLUTTER FREQUENCY
β = DAMPING OF j-TH MODE
ε = DAMPING COEFFICIENT
ε = 0,1
ε = 0,01
ε = 0
Ω = √17/2
Ω = √32/17
Ω
β
-1
0
1
2
3
4

FIG.5

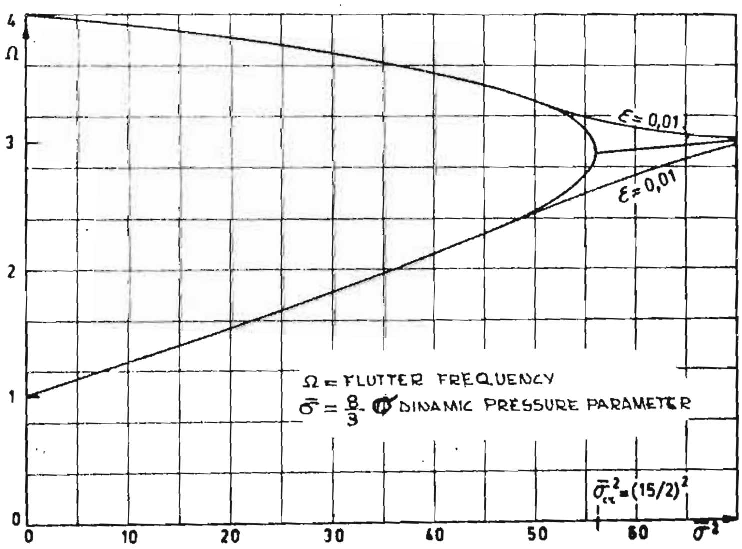

Ω = FLUTTER FREQUENCY
σ̄ = 8/3 Φ DINAMIC PRESSURE PARAMETER
ε = 0,01
σ̄cr² = (15/2)²
Ω
σ̄²
0
10
20
30
40
50
60
1
2
3
4

FIG.6

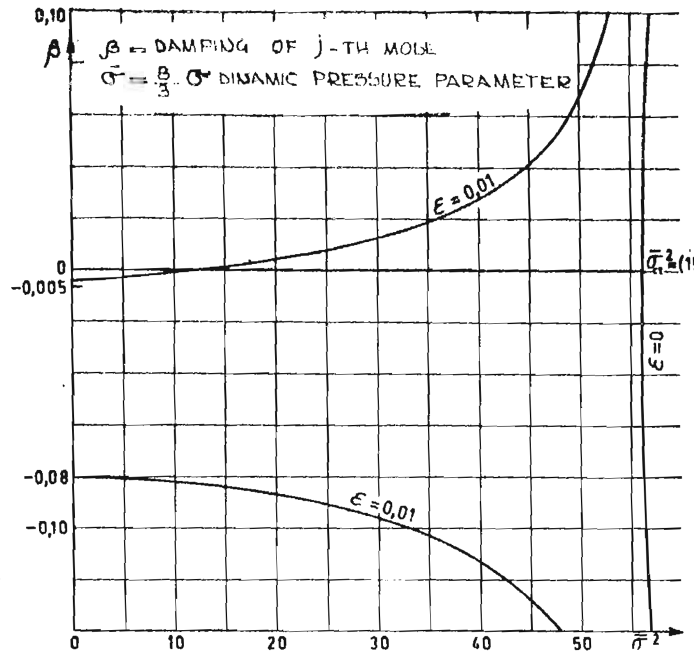

0,10
β
β = DAMPING OF j-TH MODE
DINAMIC PRESSURE PARAMETER
ε = 0,01
0
-0,005
-0,08
-0,10
ε = 0,01
ε = 0
0
10
20
30
40
50

FIG. 7

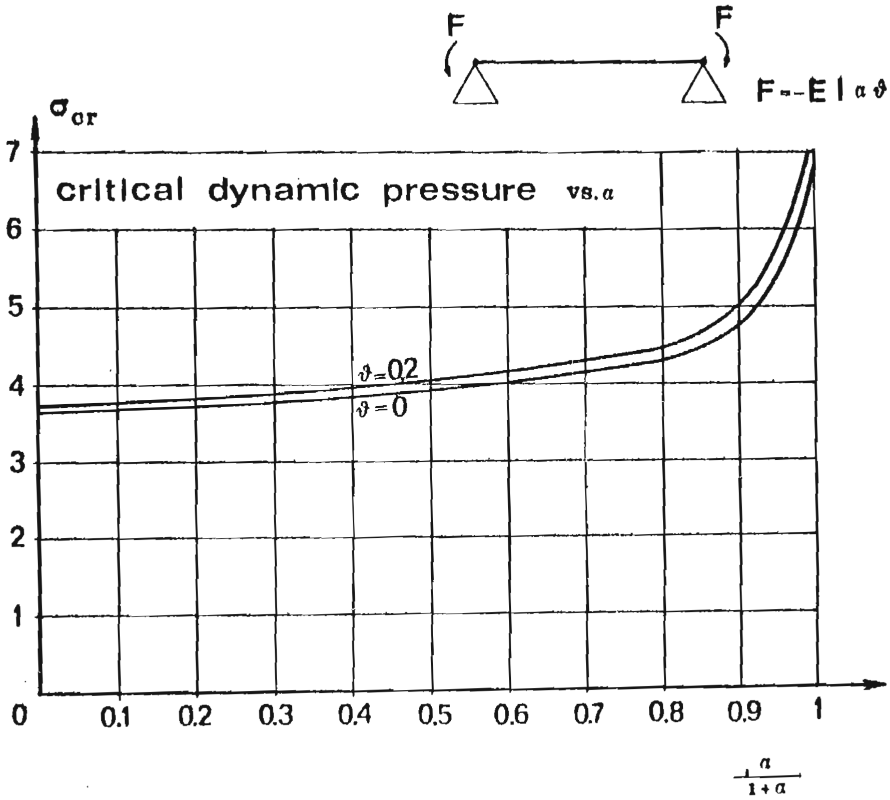
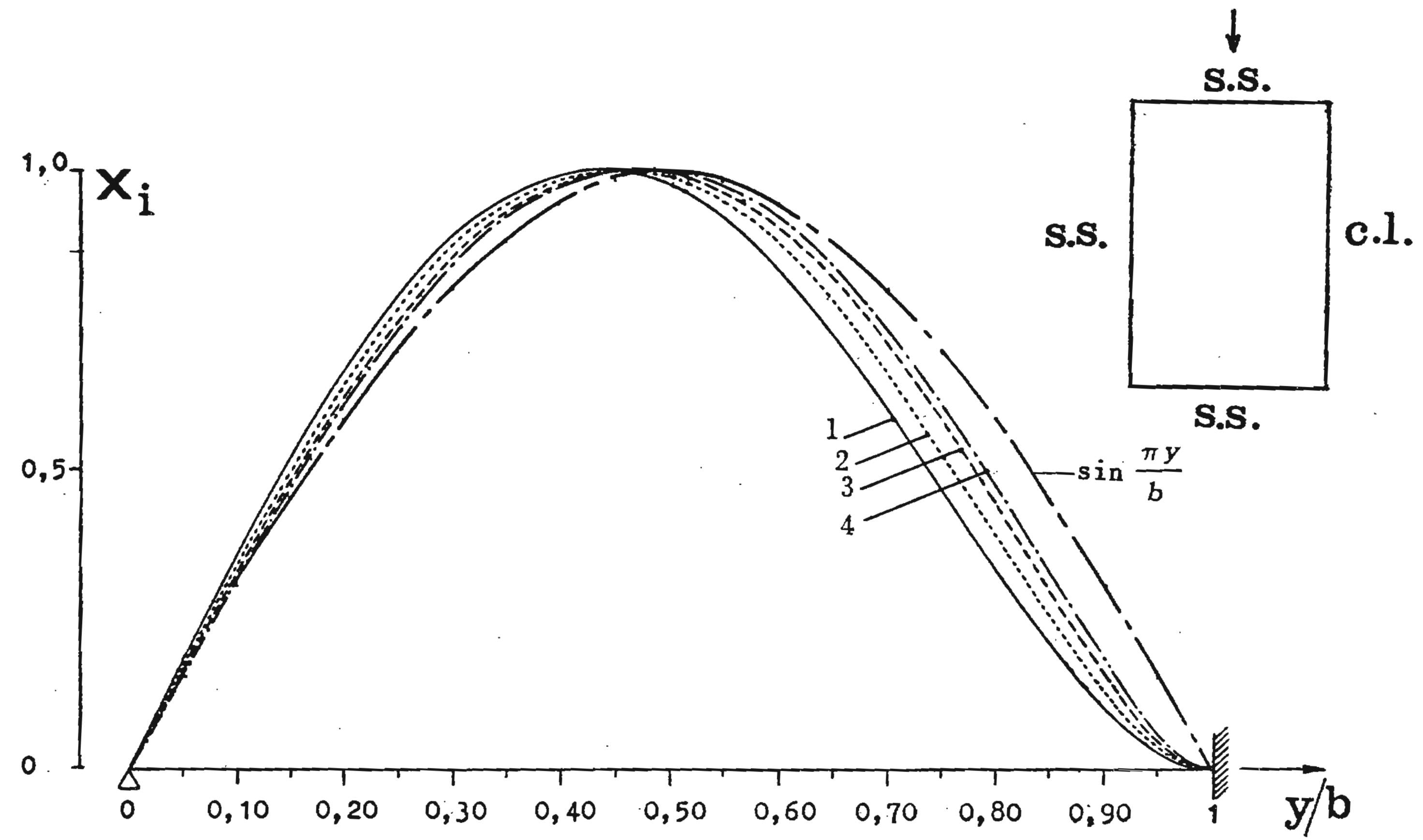
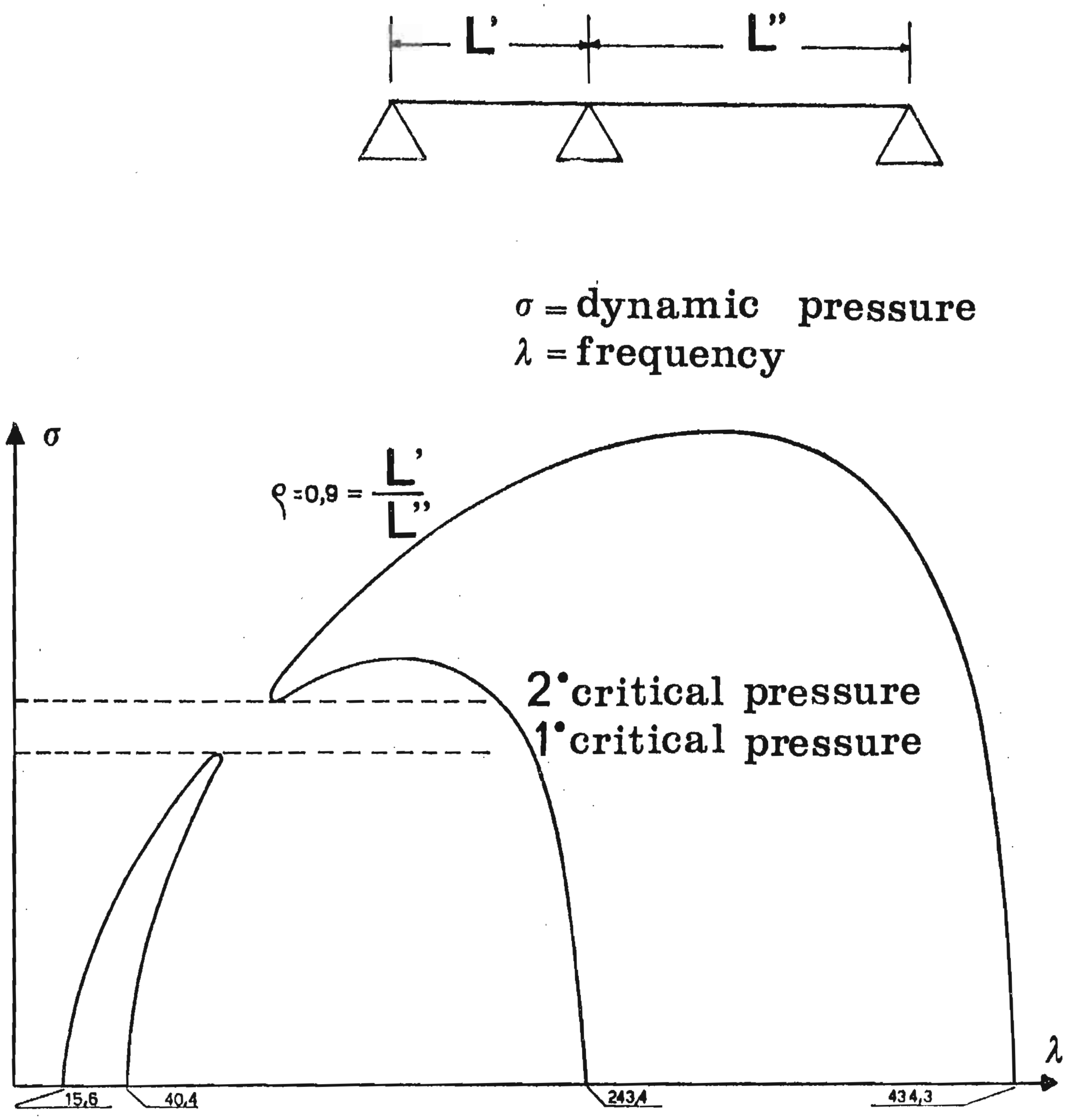

F
F
F=-EIαϑ
7
6
5
4
3
2
1
0
critical dynamic pressure vs.α
ϑ=0.2
ϑ=0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

FIG.8



Eigenfunction in y direction

FIG. 9



critical pressure for continuous panel

FIG. 10