SIMULATION OF UNSTEADY AERODYNAMIC EFFECTS OF ISOLATED VORTICES CLOSE TO THE AIRFOIL

Časlav Mitrović
Faculty of Mechanical Engineering, University of Belgrade
Aeronautical Engineering Department, 27.marta St. 80
11000 Belgrade, F.R. of Yugoslavia

Abstract

This abstract presents a model of airflow around an airfoil which includes variations of position of an isolated vortex, thus simulating unsteady lift characteristics of a helicopter rotor blade. The problem, based on airfoil flow mapping to a cylinder is spread by introduction if a vortex, which, when angle of attack changes, varies its position and satisfies the permanency of trailing edge stagnation point. This request is satisfied by generating a new vortex of a certain intensity fulfilling the given condition, and so on. The aim is to define the dependence of airfoil lift in unsteady conditions and the vortex intensities and it their distances from the airfoil that simulate the real airflow. In this investigation the aim is to simulate separated flow conditions by a series of vortices that change their positions in time. The separated flow velocity profile is approximated by superposition of displacement thicknesses of these vortices coupled with the potential solution model. After every time step the position of free vortices is changed, which requires generation of new vortices that would all together satisfy the airfoil contour boundary conditions. In this analysis models particle-in-cell, vortex and method of singularities are applied.

Nomenclature

V_∞	=	freestream velocity
Γ	=	vortex intensity
z	=	complex variables, $x + iy$
$\bar{z}$	=	$x - iy$
x,y,z	=	Cartesian coordinates
f	=	potential function
$\bar{f}$	=	complex conjugate potential function
ϕ	=	velocity potential
ψ	=	stream function
w	=	complex potential
V	=	complex velocity
Cp	=	pressure coefficient
n	=	number of vortices in wake
δ	=	wake thickness
β	=	stagnation point position
α	=	angle of attack
u_x', u_x''	=	velocities at upper and lower side of the shear layer
a	=	radius
$z_{1/2}$	=	stagnation point position
i_ω	=	unit vector in z direction
R_e	=	Reynolds number
ν	=	kinematic viscosity
μ	=	dynamic viscosity
ρ	=	density
θ	=	mapping angle
$d\zeta/dz$	=	mapping derivative
δ	=	Dirac δ function
π	=	3.14159...
L	=	characteristic length
t	=	time

Introduction

According to the experimental results, a remarkable difference exist between steady and unsteady lift. This paper presents an attempt to establish a model that will obtain unsteady lift curves using stationary lift and mapping.

This model is based on introduction of an isolated vortex close to the airfoil and defining of how much it contributes to the change of pressure coefficient and the airfoil lift. The finale results is obtaining of unsteady lift dependence of angle of attack.

Model established in such a way is characteristic for the helicopter rotor blade airflow and it is based on the influence of the previous lifting surface's wake influence on the next coming blade.

Problem setting

The main idea in this paper is the following moving vortex close to the cylinder. This vortex, by its moving, causes the necessity for releasing of a vortex of intensity Γv from the rear stagnation point, which by its influence induces a certain speed that keeps the position of stagnation point unchanged. The condition that stagnation point position should not be changed sets the request for introduction of a mapped vortex inside the airfoil contour.

The whole problem can be analyzed by mapping of an airfoil to the circle. The presence of concentrated vortices is compensated by application of Thomson's circular theorem. The intensity of the released vortices will be determined by application of Kelvin's theorem of the constancy of circulation in potential flow field. The variation of circulation about the airfoil is satisfied by unsteady Kutta-Joukowsky condition.

The transition of vorticity is modeled according to the Gauss distribution. The law of Biot-Savart is used for calculation of local velocities. Discretization of the vortex area is done according to the assumptions of Lagrange.

The generalization of the problem implies a clear physical understanding of based on the analysis of the flow field, a complex description of the vortex system, approximation of the wake, free vortices trace, position of the moving vortex, etc.

According to such analysis and a chosen model, a computer program is developed, used for the helicopter blade airflow analysis.

Foundations of the Irrotational 2-D Flow

The planar potential flow of incompressible fluid can be treated in Cartesian coordinates x and y. If physical plane is mapped to the complex plane by $z = x + iy$ where $i = \sqrt{-1}$. The symmetrical point with respect to the x-axis is $\bar{z} = x - iy$.

So that is:

$$z + \bar{z} = 2x \qquad z - \bar{z} = 2iy$$

$$x = \frac{1}{2}(z + \bar{z}) \qquad y = -\frac{i}{2}(z - \bar{z}) \tag{1}$$

Two dimensional potential incompressible flow is completely defined by the speed potential and stream function and is presented by Cauchy-Reimann equations:

$$\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} = 0 \quad \text{i} \quad \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = 0 \tag{2}$$

where u and v are velocities in x and y directions respectively.

The Laplace partial differential equations can also be introduced:

$$\nabla^2 \phi = 0 \quad \text{i} \quad \nabla^2 \psi = 0 \tag{3}$$

Where ∇^2 is the Laplace's operator.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{4}$$

The fulfillment of Cauchy-Reimann conditions enables combining of the velocity potential and stream function:

$$w(z) = \phi\,(x,y) + i\psi\,(x,y) \tag{5}$$

of complex variable $z = x + iy$.

This complex function entirely defines the planar potential flow of incompressible fluid as a function of a complex coordinate.

The complex analytical function $w(z)$, called the complex flow potential, always has a unique value for the first derivative. This derivative of the complex potential is equal to the complex velocity at that point, i.e.:

$$\frac{dw}{dz} = u - iv = \bar{V} \tag{6}$$

where u is its real part (velocity component in x-direction) and v is the imaginary part (velocity component in y-direction).

Circulation and flow are equal to zero for any closed curve in complex plane. Complex potential has no singularities except at the stagnation point.

Flow About Circular Cylinder

The flow of incompressible ideal fluid is described by the differential equation of Laplace (3a) with velocity potential as a variable. This equation was derived from the continuity equation for incompressible flow.

The airfoil boundary condition is:

$$\left(\nabla\phi + \vec{v}\right)\cdot\vec{n} = 0 \tag{7}$$

where v is the velocity of a certain point on the airfoil contour, n is the normal vector and $\nabla\phi$ is the flowfield velocity. The problem will first be treated on a cylinder and afterwards, by mapping, on an airfoil. We will settle the Cartesian coordinate system so that velocity is parallel to the x-axis.

The equation of the zero streamline is

$$y=0, \qquad x^2 + y^2 = a^2$$

where a is radius of the circle (Fig. 1). Streamlines are symmetrical, while fluid particles that move inside the circle do not leave that area. This becomes a symmetrical acyclic flow about a circular cylinder disturbed in any finite point of the flowfield.

Let the $f(z)$ be a complex potential of the flow without firm boundaries and let the singularities be at the distance greater than a. In that case the resultant flow is obtained by adding a steady flow around the doublet to the steady straight acyclic flow, i.e. by adding their complex potentials:

$$w(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \tag{8}$$

or for the flow around the cylinder whose center does not coincide with the origin of the coordinate system:

$$w(z) = f(z - z_0) + \bar{f}\left(\frac{a^2}{z - z_0}\right) \tag{9}$$

This formulation is the first Thomson's theorem for the circle where:

$$f(z), f(z - z_0)$$

is the complex potential of the uniform flow when center of cylinder does and does not coincide with the origin.

The conjugate complex functions are:

$$\bar{f}\left(\frac{a^2}{z}\right), \bar{f}\left(\frac{a^2}{z - z_0}\right)$$

Applying the Milne-Thomson theorem for airflow at a certain angle of attack α (with respect to the x-axis) the following form is:

$$f(z) = V_\infty e^{-i\alpha}(z - z_0) \tag{10}$$

And conjugate complex functions are:

$$\bar{f}\left(\frac{a^2}{z}\right) = V_\infty \frac{a^2}{z - z_0} e^{+i\alpha} \tag{11}$$

For a circle with center out of origin, the complex potential for flow at an angle of attack α is:

$$w(z) = V_\infty\left[(z - z_0)e^{-i\alpha} + \frac{a^2}{z - z_0} e^{+i\alpha}\right] \tag{12}$$

By simplifying equation (12) by assuming that z_0=0 is new origin, we get the equation for complex potential (without solid boundaries) at α:

$$w(z) = V_\infty z e^{-i\alpha} + V_\infty \frac{a^2}{z} e^{+i\alpha} \tag{13}$$

where V_∞ is velocity at the infinity.

Complex velocity is given by the equation:

$$\bar{V}(z) = V_\infty e^{-i\alpha} - V_\infty \frac{a^2}{z^2} e^{+i\alpha} \tag{14}$$

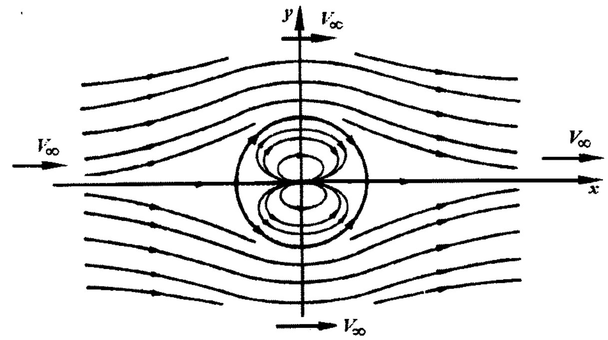

Figure 1.

Steady Cyclic Flow About Circular Cylinder. Introduction of a Vortex at the Center of Cylinder

Now if we add a steady flow of a vortex of magnitude -Γ to the previously mentioned case, which is also positioned at the point z_v which is greater than z_0, then a circular cylinder of the radius a is introduced in the airflow (Fig.2). By the application of the Thomson's theorem, and if complex potential of the newly introduced vortex has a form:

$$f_\Gamma(z) = \frac{i\Gamma}{2\pi} ln(z - z_0 - z_v) \tag{15}$$

$$w_\Gamma(z) = \frac{i\Gamma}{2\pi} ln(z - z_0 - z_v) - \frac{i\Gamma}{2\pi} ln\left(\frac{a^2}{z - z_0} - \bar{z}_v\right)$$

we obtain a complex potential equation for the flow about a cylinder of circulation Γ:

$$w(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\ln(z - z_0) \tag{16}$$

or, for the case with angle of attack α :

$$w(z) = V_\infty(z - z_0)e^{-i\alpha} + V_\infty \frac{a^2}{z - z_0} e^{+i\alpha} + \frac{i\Gamma}{2\pi}\ln(z - z_0) \tag{17}$$

We will define a condition that circulation Γ. Has to satisfy so that point z on the circle would become a stagnation point. Complex velocity is obtained dy differentiation by z:

$$\bar{V} = \frac{dw}{dz} \tag{18}$$

$$\bar{V} = f'(z - z_0) - \frac{a^2}{(z - z_0)^2} \bar{f}'\left(\frac{a^2}{z - z_0}\right) + \frac{i\Gamma}{2\pi(z - z_0)}$$

Circulation is obtained from the condition that complex velocity must be equal to zero at the stagnation point:

$$\Gamma = 4\pi \cdot a \cdot V_\infty sin(\alpha - \beta) \tag{19}$$

where: α -is angle of attack and
β - angle of rear stagnation point position.

Equation (17) can be written in the following way:

$$w(z) = V_\infty (z - z_0) e^{-i\alpha} + V_\infty \frac{a^2}{z - z_0} e^{+i\alpha} + i2a \cdot V_\infty \cdot sin(\alpha - \beta) \ln(z - z_0) \tag{20}$$

Then complex velocity is equal to:

$$\bar{V}(z) = V_\infty e^{-i\alpha} - V_\infty \frac{a^2}{z^2} e^{+i\alpha} + \frac{i\Gamma}{2\pi z} \tag{21}$$

From this equation it can be seen that complex velocity will be equal to zero in the points in the complex plane whose complex number satisfies the obtained quadratic equation. Analysis of the quadratic equation implies that positions of stagnation points are defined by its discriminate. Stagnation points will be moved down while velocities will be increased on upper and decreased on lower side. Displacement of stagnation points will depend on the circular velocity of the cylinder and the velocity V_∞ of the undisturbed flow.

In case when α=0, there could be three cases:

a) case when discriminate is greater than zero (Fig. 2), i.e.:

$$0 < \frac{\Gamma}{V_\infty a} < 4\pi$$

Then stagnation points are at the lower half of the cylinder , symmetrically distributed are at positions:

$$z_{1/2} = \pm \sqrt{a^2 - \left(\frac{\Gamma}{4\pi V_\infty}\right)^2} - i \frac{\Gamma}{4\pi V_\infty} \tag{22}$$

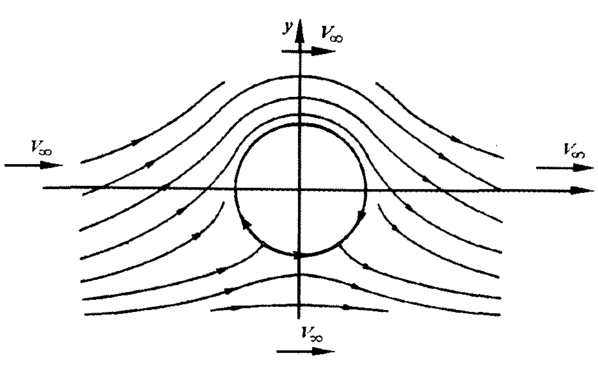

Figure 2.

b) case when discriminate is equal to zero (Fig. 3), i.e.:

$$\frac{\Gamma}{V_\infty a} = 4\pi$$

Then quadratic equation has only one solution:

$$z_{1/2} = -i \frac{\Gamma}{4\pi V_\infty} = -ia \tag{23}$$

so stagnation point is at the negative part of the y-axis (Fig. 3).

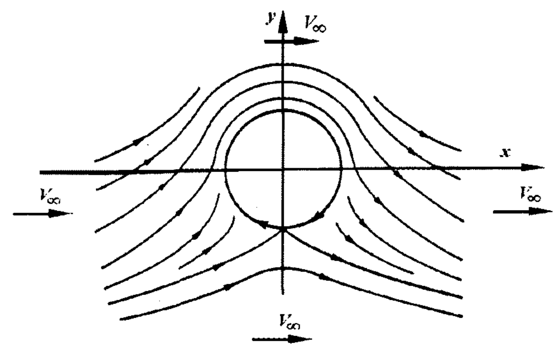

Figure 3.

c) case when discriminate is less than zero (Fig. 4), i.e.:

$$\frac{\Gamma}{V_\infty a} > 4\pi$$

Then complex velocity is equal to zero in points that are at the negative part of the y-axis:

$$z_{1/2} = \pm i \sqrt{-a^2 + \left(\frac{\Gamma}{4\pi V_\infty}\right)^2} - i \frac{\Gamma}{4\pi V_\infty} \tag{24}$$

while one of them is inside and the other outside the circle. It can easily be proved that circulation and flow are equal to zero for any curve that does not enclose the origin, and equal to the vortex intensity for any curve that encloses it.

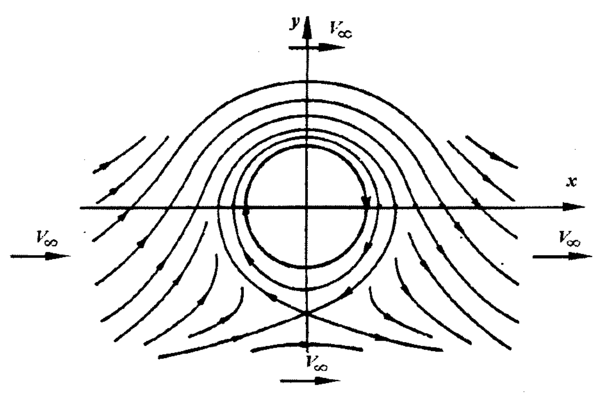

Figure 4.

Equations with center of the circle at the origin

If we assume that center of the circle coincides with the origin, the equations …take the following forms :

$$f(z)=V_\infty e^{-i\alpha}z \quad ; \quad f\left(\frac{a^2}{z}\right)=V_\infty \frac{a^2}{z}e^{+i\alpha}$$

$$w_U(z)=f(z)+\bar{f}\left(\frac{a^2}{z}\right) \qquad (25)$$

$$w_\Gamma(z)=\frac{i\Gamma}{2\pi}lnz=i2a\cdot V_\infty sin(\alpha-\beta)$$

$$w(z)=w_U(z)+w_\Gamma(z) \qquad (26)$$

$$w(z)=V_\infty ze^{-i\alpha}+V_\infty\frac{a^2}{z}e^{+i\alpha} +i2aV_\infty sin(\alpha-\beta)\ln z \qquad (27)$$

Simulation of the moving vortex

If we assume that the moving vortex of intensity Γ_0 is at the distance z_0, than the perturbance potential of such a flow is:

$$w(z)=V_\infty ze^{-i\alpha}+V_\infty\frac{a^2}{z}e^{+i\alpha}+i2a\cdot V_\infty\cdot sin(\alpha-\beta)\ln z +\frac{i\Gamma_0}{2\pi}\ln(z-z_0)-\frac{i\Gamma_0}{2\pi}\ln\left(\frac{a^2}{z}-\bar{z}_0\right) \qquad (28)$$

and its complex velocity:

$$\bar{V}=V_\infty e^{-i\alpha}-V_\infty\frac{a^2}{z^2}e^{+i\alpha}+\frac{i}{z}2a\cdot V_\infty\cdot sin(\alpha-\beta) +\frac{i\Gamma_0}{2\pi}\frac{1}{(z-z_0)}+\frac{i\Gamma_0(a^2/z^2)}{2\pi\left(\frac{a^2}{z}-\bar{z}_0\right)} \qquad (29)$$

In this case displacement of the stagnation point appears, which is now at the distance:

$$z=a\cdot e^{i\beta'} \qquad (30)$$

According to the request that complex velocity at the stagnation point must be zero:

$$\frac{dw}{dz}=\bar{V}=0 \qquad (31)$$

we can determine position z and according to that β' i.e.:

$$\bar{V}(ae^{i\beta})=0 \qquad (32)$$

The angle β' defines new position of stagnation point, while the difference of angles is:

$$\Delta\beta=\beta-\beta' \qquad (33)$$

In order to keep the stagnation point at the steady position, a vortex of intensity Γ_1 must be released.

Intensity of the vortices that are released can be determined according to the Kelvin's theorem:

$$d\Gamma/dt=0$$

So every change in circulation must be compensated by vortex inside of cylinder/airfoil of the opposite sign, whose intensity is a function of the variation of circulation around the cylinder:

$$\Gamma_i=-(d\Gamma/dt)\Delta t \qquad (34)$$

and it is equal do the difference of the intensities before and after the free moving vortex is introduced:

$$\Gamma_1=\Delta\Gamma_1=4\pi a\cdot V_\infty\left[sin(\alpha-\beta_1')-sin(\alpha-\beta)\right] \qquad (35)$$

Now the total complex potential has the value:

$$w(z)=w_0(z)+w_{\Gamma_0}(z)+w_{\Gamma_1}(z) \qquad (36)$$

where:

$$w_0(z)=V_\infty ze^{-i\alpha}+V_\infty\frac{a^2}{z}e^{+i\alpha}+i2aV_\infty sin(\alpha-\beta)\ln z$$

$$w_{\Gamma_0}(z)=\frac{i\Gamma_0}{2\pi}\ln(z-z_0)-\frac{i\Gamma_0}{2\pi}\ln\left(\frac{a^2}{z}-\bar{z}_0\right)$$

$$w_{\Gamma_1}(z)=\frac{i\Gamma_1}{2\pi}\ln(z-z_1)-\frac{i\Gamma_1}{2\pi}\ln\left(\frac{a^2}{z}-\bar{z}_1\right)$$

or:

$$w(z)=V_\infty ze^{-i\alpha}+V_\infty\frac{a^2}{z}e^{+i\alpha}+i2a\cdot V_\infty sin(\alpha-\beta)\ln z +\frac{i\Gamma_0}{2\pi}\ln(z-z_0)-\frac{i\Gamma_0}{2\pi}\ln\left(\frac{a^2}{z}-\bar{z}_0\right) +\frac{i\Gamma_1}{2\pi}\ln(z-z_1)-\frac{i\Gamma_1}{2\pi}\ln\left(\frac{a^2}{z}-\bar{z}_1\right) \qquad (37)$$

Complex velocity is:

$$\bar{V} = V_\infty e^{-i\alpha} - V_\infty \frac{a^2}{z^2} e^{+i\alpha} + \frac{i}{z} 2a \cdot V_\infty \cdot sin(\alpha - \beta)$$
$$+ \frac{i\Gamma_0}{2\pi} \frac{1}{(z - z_0)} + \frac{i\Gamma_0 (a^2/z^2)}{2\pi\left(\frac{a^2}{z} - \bar{z}_0\right)}$$
$$+ \frac{i\Gamma_1}{2\pi} \frac{1}{(z - z_1)} + \frac{i\Gamma_1 (a^2/z^2)}{2\pi\left(\frac{a^2}{z} - \bar{z}_1\right)} \tag{38}$$

After a period of time Δt induced velocity at the trailing edge is a consequence of the disposition of both moving and released vortex.

Setting again the condition that a point at the circle is stagnation point and complex velocity at that point equal to zero, we determine a new position of the stagnation point by new angle β'.

This new angle β' defines a new difference $\Delta\beta = \beta - \beta'$. So the new position of the stagnation point is defined by distance:

$$z = a \cdot e^{i\beta'} \tag{39}$$

This again causes generation of a new vortex Γ_2 which is equal do the difference of vortex intensity before and after displacing of the free moving vortex from time t_1 to time t_2.

$$\Gamma_2 = \Delta\Gamma = 4\pi a V_\infty sin(\alpha - \beta') - 4\pi a V_\infty sin(\alpha - \beta) \tag{40}$$

Now the total complex potential has the value:

$$w(z) = w_0(z) + w_{\Gamma_0}(z) + w_{\Gamma_1}(z) + w_{\Gamma_2}(z) \tag{41}$$

or:

$$w(z) = V_\infty z e^{-i\alpha} + V_\infty \frac{a^2}{z} e^{+i\alpha} + i2aV_\infty sin(\alpha - \beta) \ln z$$
$$+ \frac{i\Gamma_0}{2\pi} \ln(z - z_0) - \frac{i\Gamma_0}{2\pi} \ln\left(\frac{a^2}{z} - \bar{z}_0\right)$$
$$+ \frac{i\Gamma_1}{2\pi} \ln(z - z_1) - \frac{i\Gamma_1}{2\pi} \ln\left(\frac{a^2}{z} - \bar{z}_1\right)$$
$$+ \frac{i\Gamma_2}{2\pi} \ln(z - z_2) - \frac{i\Gamma_2}{2\pi} \ln\left(\frac{a^2}{z} - \bar{z}_2\right)$$

Complex velocity is:

$$\bar{V} = V_\infty e^{-i\alpha} - V_\infty \frac{a^2}{z^2} e^{+i\alpha} + \frac{i}{z} 2a \cdot V_\infty sin(\alpha - \beta)$$
$$+ \frac{i\Gamma_0}{2\pi} \frac{1}{(z - z_0)} + \frac{i\Gamma_0 (a^2/z^2)}{2\pi\left(\frac{a^2}{z} - \bar{z}_0\right)}$$
$$+ \frac{i\Gamma_1}{2\pi} \frac{1}{(z - z_1)} + \frac{i\Gamma_1 (a^2/z^2)}{2\pi\left(\frac{a^2}{z} - \bar{z}_1\right)} \tag{42}$$
$$+ \frac{i\Gamma_2}{2\pi} \frac{1}{(z - z_2)} + \frac{i\Gamma_2 (a^2/z^2)}{2\pi\left(\frac{a^2}{z} - \bar{z}_2\right)}$$

Now we can give general equations of:

- the stagnation point position:

$$z = a \cdot e^{i\beta'} \tag{43}$$

- intensity of the released vortex:

$$\Gamma_m = 4\pi a \cdot V_\infty \left[sin(\alpha - \beta'_m) - sin(\alpha - \beta) \right] \tag{44}$$

- total circulation:

$$\Gamma_\Sigma = \Gamma + \Gamma_0 + \sum_1^n \Gamma_m \tag{45}$$

- circulation inside the cylinder/airfoil:

$$\Gamma_n = \Gamma_s \pm \Gamma_m \tag{46}$$

- complex potential:

$$w(z) = w_0(z) + \left[w_{\Gamma_0}(z) + \sum\nolimits_1^n w_{\Gamma_m}(z) \right] \tag{47}$$

or:

$$w(z) = V_\infty z e^{-i\alpha} + V_\infty \frac{a^2}{z} e^{+i\alpha} + \frac{i(\Gamma - \Gamma_m)}{2\pi} \ln z +$$
$$+ \frac{i\Gamma_0}{2\pi} \ln(z - z_0) - \frac{i\Gamma_0}{2\pi} \ln\left(\frac{a^2}{z} - \bar{z}_0\right)$$
$$+ \sum_1^n \left[\frac{i\Gamma_m}{2\pi} \ln(z - z_m) - \frac{i\Gamma_m}{2\pi} \ln\left(\frac{a^2}{z} - \bar{z}_m\right) \right]$$

complex velocity:

$$\frac{d\bar{z}_m}{dt} = \bar{V}_m = V_\infty e^{-i\alpha} - V_\infty \frac{a^2}{z^2} e^{+i\alpha} + \frac{i(\Gamma - \Gamma_m)}{2\pi z} + \frac{i\Gamma_0}{2\pi} \frac{1}{(z - z_0)} + \frac{i\Gamma_0}{2\pi\left(z - \frac{z^2}{a^2}\bar{z}_0\right)} \quad (48)$$

$$+ \sum_1^n \left[\frac{i\Gamma_m}{2\pi} \frac{1}{(z - z_m)} + \frac{i\Gamma_m}{2\pi\left(z - \frac{z^2}{a^2}\bar{z}_m\right)} \right]$$

Vortices travel down the flowfield by its velocity so that their position is determined by solving the system of equations:

$$\frac{dx_i}{dt} = \left.\frac{\partial\phi}{\partial x}\right|_i, \quad \frac{dy_i}{dt} = \left.\frac{\partial\phi}{\partial y}\right|_i. \quad (49)$$

Position of the vortex at a new moment can be determined by:

$$d\bar{z}_m = \bar{V}_m dt \quad (50)$$

and its elementary displacement:

$$\Delta\bar{z}_m = \bar{z}_m^n - \bar{z}_m^s = \Delta t \cdot \bar{V} \quad (51)$$

Now a new distance of each vortex as well as the trajectory of the vortex or any fluid particle is given by:

$$\bar{z}_m^n = \bar{z}_m^s + \Delta t \cdot \bar{V} \quad (52)$$

$$\bar{z}_m^n = \bar{z}_m^s + \Delta t \cdot \left\{ V_\infty e^{-i\alpha} - V_\infty \frac{a^2}{z^2} e^{+i\alpha} + \frac{i(\Gamma - \Gamma_m)}{2\pi z} + \frac{i\Gamma_0}{2\pi} \frac{1}{(z - z_0)} + \frac{i\Gamma_0}{2\pi\left(z - \frac{z^2}{a^2}\bar{z}_0\right)} + \sum_1^n \left[\frac{i\Gamma_m}{2\pi} \frac{1}{(z - z_m)} + \frac{i\Gamma_m}{2\pi\left(z - \frac{z^2}{a^2}\bar{z}_m\right)} \right] \right\}$$

According to the Bernoulli's equation pressure distribution can be calculated and then the pressure coefficient:

$$C_p' = 1 - \left(\frac{\bar{V}}{V_\infty}\right) \cdot \overline{\left(\frac{V}{V_\infty}\right)} \quad (53)$$

Vorticity in incompressible viscous flow

In many cases viscosity of the fluid can be neglected. If it has to be taken into account, some approximations and simplifications can be done, without affecting remarkably final results.

Incompressible fluid flow is governed by Navier-Stokes equation:

$$\Delta \mathrm{u} = 0 \quad (54)$$

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + (\mathrm{u} \cdot \nabla)\,\mathrm{u} = -\nabla p + \nu \nabla^2 \mathrm{u} \quad (55)$$

The vorticity of the flow is defined as:

$$\omega = \nabla \times \mathrm{u} \quad (56)$$

By taking the curl of equation (55) we obtain:

$$\frac{\partial \omega}{\partial \mathrm{t}} + (\mathrm{u} \cdot \nabla)\omega = \omega \cdot \nabla \mathrm{u} + \nu \nabla^2 \omega \quad (57)$$

For two-dimensional flows $\omega \cdot \nabla \mathrm{u} = 0$ and equation (56) reduces to:

$$\frac{\partial \omega}{\partial \mathrm{t}} + (\mathrm{u} \cdot \nabla)\omega = \nu \nabla^2 \omega \quad (58)$$

We can say that if ω is known than *u* can be computed using Biot-Savart law . Thus:

$$\mathrm{u}(\mathrm{x}) = \frac{1}{2\pi} i_\omega \times \int \frac{\mathrm{x} - \mathrm{x}'}{|\mathrm{x} - \mathrm{x}'|^2} \omega(\mathrm{x}) d\mathrm{x}' \quad (59)$$

The essence of the inviscid-vortex method is to replace .. by:

$$\omega(\mathrm{x}) = \sum \Gamma_i \delta(\mathrm{x} - \mathrm{x}_i) \quad (60)$$

where δ are functions approximating the Dirac δ - function and Γ_i is circulation of the *i*-the vortex. It has to satisfy the kinematic condition:

$$\frac{d\Gamma}{dt} = -\frac{(u_x' - u_x'')}{2} \quad (61)$$

Where:

1. $d\Gamma/dt$ - is the change of circulation in the shear layer moving over the stagnation point
2. u_x', u_x'' - are velocities on the upper and lower side of the shear layer

In order to satisfy the equation of motion of inviscid flow:

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) \cdot V + \nu \quad \nabla^2 \omega = 0 \quad (62)$$

the velocity of every vortex must be determined by the value of flow velocity at its instant position:

$$\frac{dx_i}{dt} = u(x_i,t) \tag{63}$$

Let us assume that rather small number of vortices per unit length represent the free shear layer. The known assumption of no-slip condition at the wall (airfoil) leads to the model of wake simulation. If the wake is properly simulated, it will leave airfoil at the trailing edge. In order to obtain proper simulation of viscous effects, it is necessary to assume that the size of the vortex is proportional to the thickens of the wake $\sigma \approx f\delta$. If the wake length is L then number of vortices can be defined as:

$$n \approx \frac{\delta \cdot l}{\sigma^2} \approx \frac{l}{\delta \cdot f^2} \approx \frac{\sqrt{R_e}}{f^2} \tag{64}$$

Where R_e is Reynolds number

$$R_e = \frac{V_\infty l}{\nu}$$

And ν - kinematic viscosity ($\nu = \mu / \rho$).

Mapping of a circular cylinder to airfoil

Mapping of the circular cylinder to airfoil is done by Joukowsky transformation:

$$\zeta = z + \frac{a^2}{z} \tag{65}$$

Where:

$$z = \varepsilon e^{i\delta} + \rho e^{i(\theta-\sigma)} \tag{66}$$

is the parametric equation of the function of mapping of θ and mapping derivative:

$$\frac{d\zeta}{dz} = 1 - \frac{a^2}{z^2} \tag{67}$$

Calculation

According to the whole mentioned analysis a numerical model was established. This model is based on the following:

- vortices move along the flowfield by the flowfield velocity
- trajectory of every vortex is defined by equation (52)
- for every point in the flowfield it is necessary to determine value of complex potential and complex velocity
- at a certain moment in a defined initial point a free moving vortex is simulated
- after every time interval Δt moving vortex changes the stagnation point position which must be compensated by introduction of new vortex which brings stagnation point back to its proper place
- new positions of all vortices are calculated by multiplying local velocities with ….. and adding these values to previous
- viscous effects should be simulated by generating vortices on the airfoil to satisfy no-slip condition; so introduced vortices must be moved by velocity defines by inviscid part of the equation of motion
- boundary condition of impermeability of the airfoil must be fulfilled
- vortex diffusion is simulated by variation of the vortex size and arbitrary step
- computer program is made so that flow parameters can be calculated for different angles of attack.

Analysis of the Calculation

Program vas run first for characteristic cases of flow around the cylinder (Figs 8, 9 & 10) described in the theoretical consideration by Figs. 1,2, & 3.

For the comparison, experimental wind tunnel investigation is shown (Figs. 5, 6 & 7) and computer simulation (Figs. 11-21).

Due to computer limitations, a rather small number of streamlines is shown. Model is calculated for different angles of attack (Figs. 11,12,14 & 19), different flowfield velocities (Figs. 15 & 16), different intensities of the moving vortex and its starting position (Figs. 15 & 16 ; 19,20 & 21). Also, wake shape with and without the influence of vorticity due to viscosity.

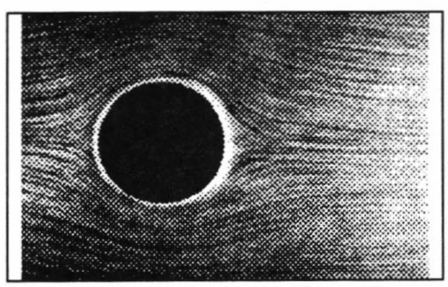

Figure 5.

Streamlines of steady flow (from left to right) past a circular cylinder of radius *a*

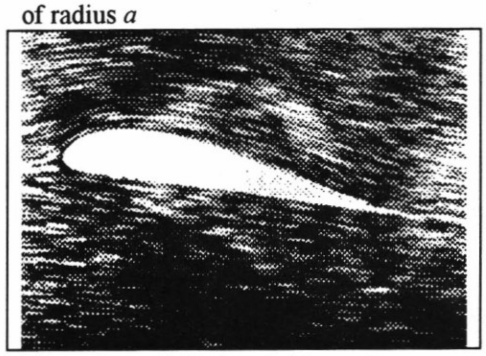

Figure6

Flow from left to right past an airfoil aligned roughly with stream


Figure 7

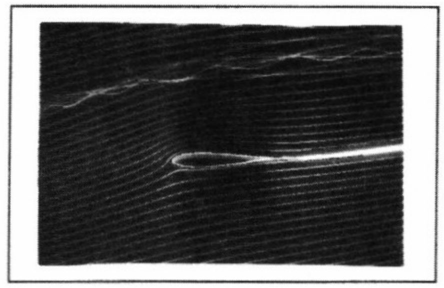
Figure 12

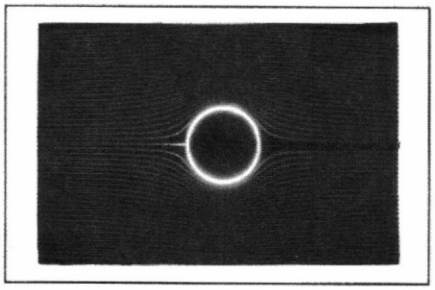
Figure 8

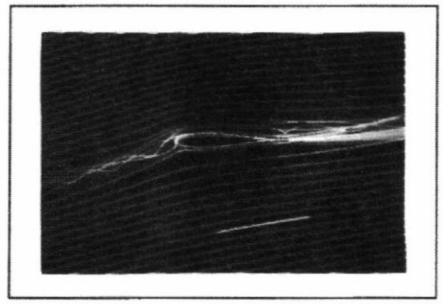
Figure 13

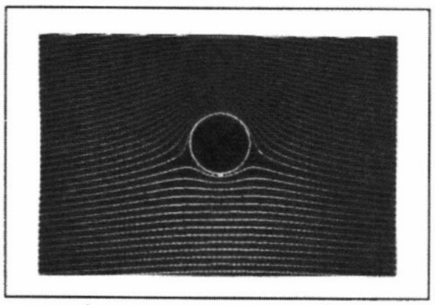
Figure 9

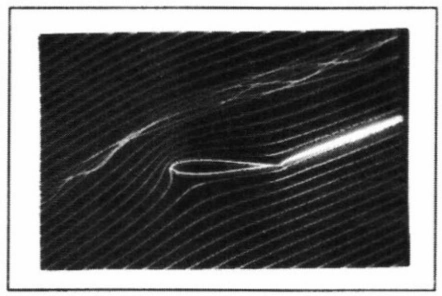
Figure 14

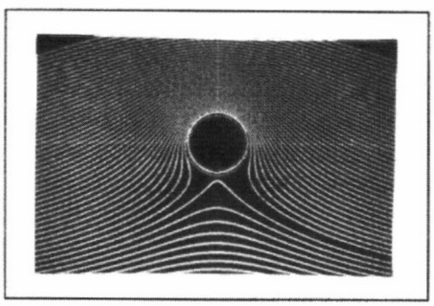
Figure 10

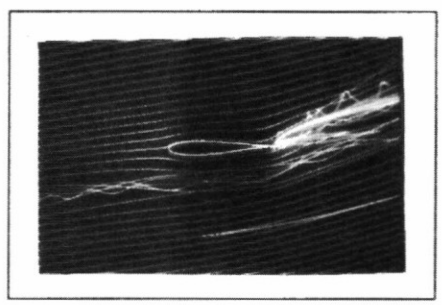
Figure 15

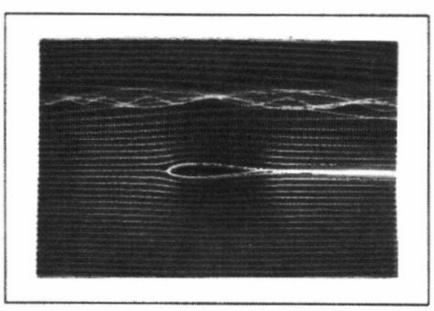
Figure 11

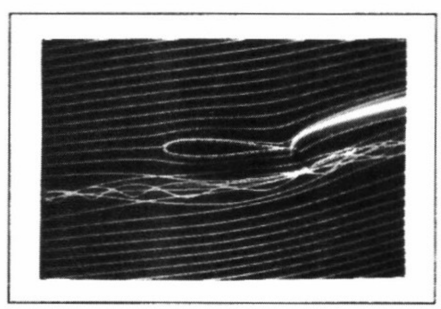
Figure 16

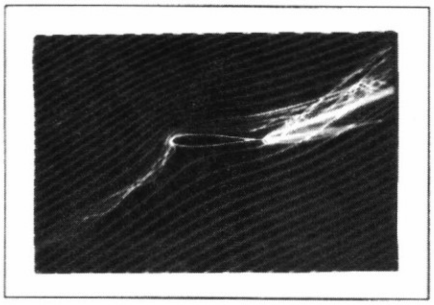

Figure 17

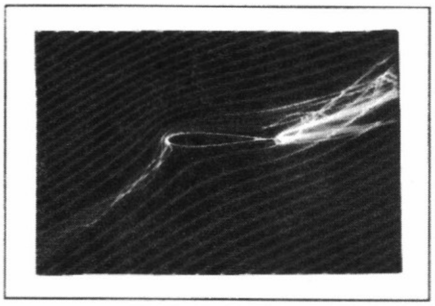

Figure 18

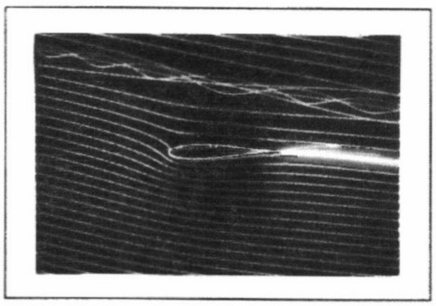

Figure 19

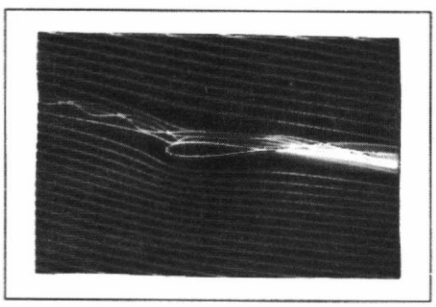

Figure 20

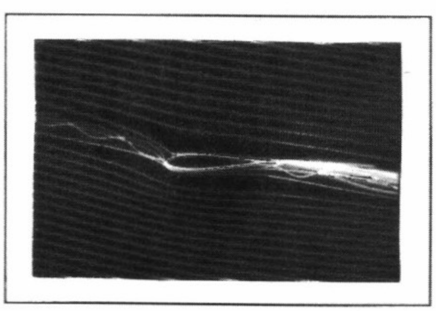

Figure 21

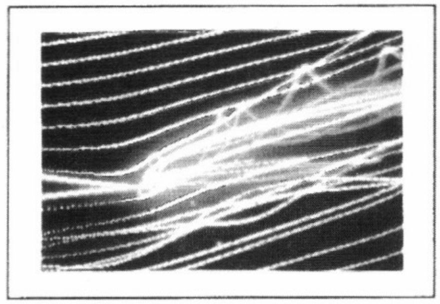

Figure 22

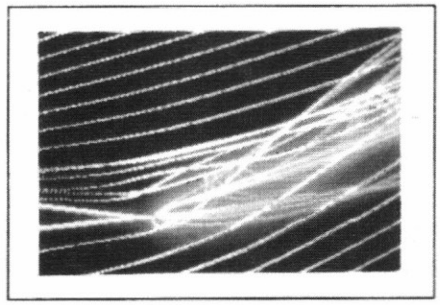

Figure 23

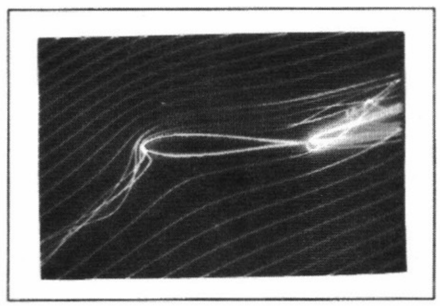

Figure 24

References

- Betchelor, G.K.,"An introduction to Fluid Dynamics", *Cambrige University Press*, 1983
- White, F.M, "Viscous Fluid Flow", 1974
- Chia-Shun Y., "Fluid Mechanics", *The University of Mich.*,1979
- Rand O.,"Extension of the Circle Theorems by Source Distribution", JFE Vol 11 1989.
- Saffman P.G.,and Sheffield, J.S., "Flow over a Wing with an Attached Free Vortex", *Studies in Applied Mathematics*, Vol.57, No 2, 1977, pp.107-117
- Higuci, H., and Park. O.W., " Computations of the Flow Past Solid and Slotted Two-Dimensional Bluff Bodies with Vortex Tracing Method", AIAA Paper 89-0929.
- *NASA* TM 100068, Vortex Methods for Separated Flows & *NASA* TM 84328, Numerical Simulation of Separated Flows
- Stepniewski W.Z.," Rotary-Wing Aerodynamics" New York, 1984
- Fishelov,D., " A New Vortex Schemefor Viscous Flows", *Journal of Computational Physics*, Vol.27, 1990, pp.211-224 .