APPLICATION OF SKEW-TRANSVERSELY-REPETITIVE ANALYSIS FOR BUCKLING OF PLATE ARRAYS WITH CURVED OR STRAIGHT INTERNAL SUPPORTS.

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Abstract.

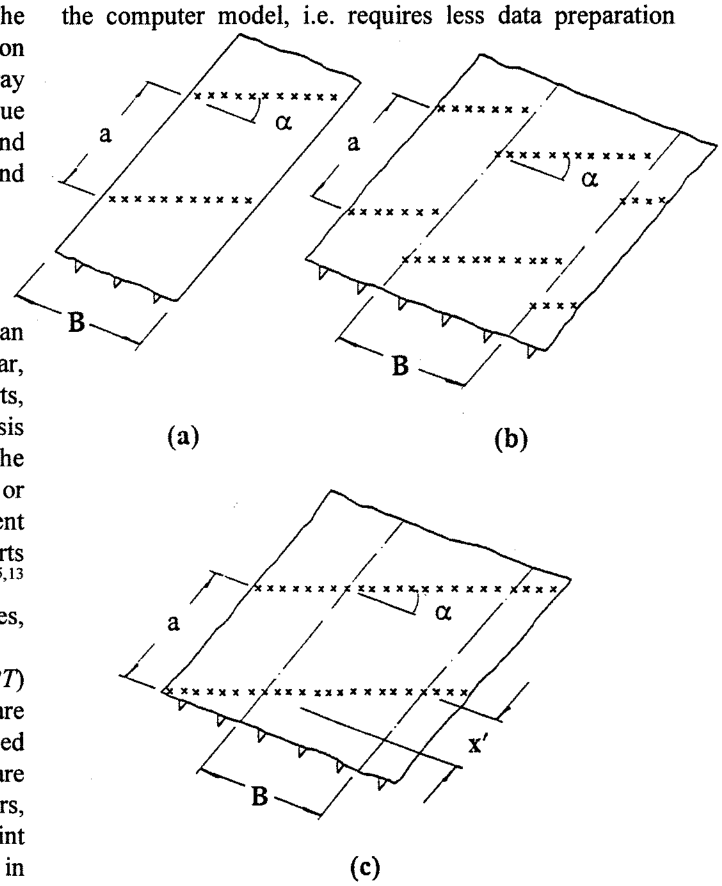
This paper presents an efficient buckling analysis method for a continuous array of plates, extending over regular internal supports, which can be either curved or straight. Practical examples include flight control surfaces subject to local buckling between the internal rectangular supports over which the plate is continuous. The investigation includes a preliminary set of comparison results for plates that are continuous over a regular array of adjacent rectangular, circular, hexagonal, oblique (skew) or triangular supports. The results may find application in the design of super-plastically-formed and diffusion-bonded panels.

Introduction.

This paper presents new buckling results for an array of plates, continuous over either rectangular, circular, hexagonal, oblique (skew) or triangular supports, and for which few published results[1] exist. The analysis procedure was developed[14] originally to investigate the potential limitations of modelling an isolated plate or plate assembly, as is often done in practice, to represent the real problem where there is continuity over supports along one or both in-plane directions. Previous results[15,13] dealt specifically with skew plates and plate assemblies, of which rectangular plates were the limiting case.

Analysis is based on classical plate theory (*CPT*) and while longitudinal plate boundaries (or edges) are modelled exactly, the transverse boundaries are enforced by a sufficient number of point constraints that are introduced by the method of Lagrangian multipliers, which is described in a later section. These point constraint were arranged to form skew supports in previous work, but this arrangement may be modified to give hexagonal, circular or triangular supports. The support pattern repeats at intervals of the panel length, see Fig. 1(a), since the analysis accounts for an infinitely long plate, thus forming a series of plates joined end to end, which typify the continuity found in aircraft wing or fuselage construction.

The results obtained for this investigation, which are the result of an enhancement[14] to existing theory[11,8] and the associated computer code[9], account for *skew* plate assemblies that are continuous over supports both at regular longitudinal and transverse intervals, compare Fig. 1(b) and 1(c). As a result of this enhancement, efficiency is achieved through a reduction in the size of the computer model, i.e. requires less data preparation



(a) (b)

(c)

Fig. 1. Plate assembly (a) of width B, continuous over skew supports, α, at longitudinal intervals *a* was later modified (b) to allow for continuity transversely. The enhancement (c) allows for skew-transverse-continuity, where $x' = B.\tan\alpha$.

and CPU time[16]. This fact is demonstrated in Fig. 2, which shows alternative ways of modelling the plate, and is of particular significance in the context of the current investigation, since many point supports are required to define the various new geometries.

Formulation.

The analysis method used[11,9,7] is based on the Kirchoff-Love hypothesis. The general form of the differential equation of equilibrium is given by:

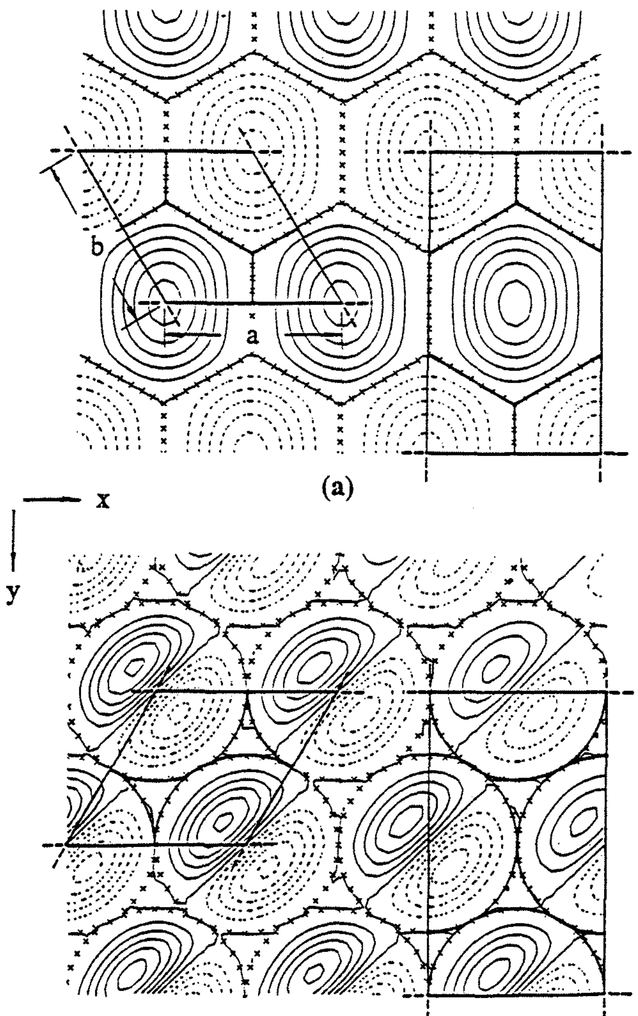


Fig. 2. Buckling modes for (a) compression loaded, hexagon and (b) shear loaded, circular supported plate array. Bold lines illustrate the alternative model configurations, i.e. the portion of plate assembly and supports required, for both transversely- and skew-transversely-repetitive analysis. Reference axes are illustrated.

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} + N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2} = 0 \quad (1)$$

The stress-strain relationship for each lamina is given by

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}\overline{Q}_{11} & & \text{Sym}\\ \overline{Q}_{12} & \overline{Q}_{22} & \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \quad (2)$$

where the $\overline{Q}_{ij}$ represent the transformed reduced stiffnesses, the relationships for which are derived and defined elsewhere[4].

The solution permits orthotropic in-plane material properties (so that $A_{16} = A_{26} = 0$) and uncoupled anisotropic out-of-plane (i.e. flexural) properties, so that $\mathbf{B} = \mathbf{0}$. Balanced and symmetric laminates eliminate shear- and bending-extension coupling respectively.

Figure 3(a) shows a component plate of width b, together with the basic longitudinally invariant in-plane forces which it carries. These are forces of N_L, N_T and N_S per unit length, corresponding to uniform longitudinal and transverse compressive forces and shear flow, respectively. The deflections of the plate assembly are assumed to vary sinusoidally in the longitudinal direction with half wavelength λ. The nodal lines of the deflection pattern, shown dashed on Fig. 3(b), are perpendicular to the longitudinal direction when all the plates of a plate assembly are isotropic or orthotropic and are subject only to N_L and/or N_T. The nodal lines are then consistent with transverse simple supports at the ends of each plate of the assembly, and so exact results are obtained for such end conditions if λ is taken as $\lambda_j = a/j$, where the integer j = 1,2,3... and a is the length of the assembly. Skewed nodal lines result when some of the component plates are anisotropic or carry in-plane shear loads N_S. They are inconsistent with transverse simple supports and so form only approximate solutions for such supports.

Displacements at nodes, i.e. at junctions between the longitudinal plates, are given by the real part of $\mathbf{D}_j' exp(i\pi x/\lambda_j)$, where $i = \sqrt{-1}$, x is the longitudinal co-ordinate and $\mathbf{D}_j'$ contains the four complex displacement amplitudes for each node which correspond, in order, to the ψ, w, v and u of Fig. 3(a). All possible types of mode are included by permitting the junctions between individual plates to flex[11]. The displacement amplitude ψ_y (ψ_z) for rotation about the y axis (z-axis), which can be set to zero for clamped conditions, is obtained by differentiating the displacement function in the z (y) direction, e.g. $-i\pi w/\lambda$ replaces the displacement amplitude for rotation about the y axis since

$$-\frac{\partial}{\partial x}(w.e^{i\pi x/\lambda}) = -(i\pi/\lambda)\, w.e^{i\pi x/\lambda}$$

(3)

Critical loads are the eigenvalues corresponding to $\mathbf{K}_j\mathbf{D}_j = \mathbf{0}$, where $\mathbf{D}_j$ is obtained by multiplying every fourth element of $\mathbf{D}_j'$, associated with longitudinal displacement, by i. This i takes account of a 90° spatial phase difference between these displacements and others which occur for plate assemblies consisting of orthotropic

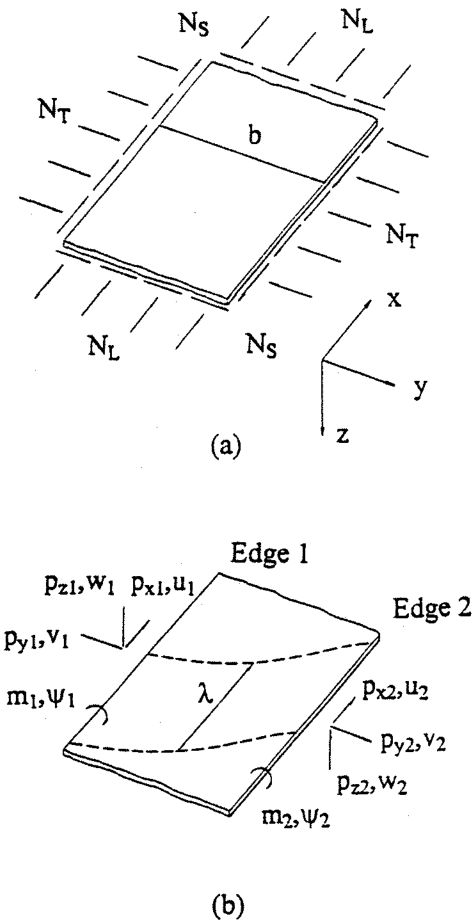


Fig. 3. (a) Loading and reference axis system for a component plate of width b and; (b) skew mode with half-wavelength λ and the perturbation force (denoted by p and m) and displacement amplitudes at the longitudinal edges of the plate, which are multiplied by $exp(i\pi x/\lambda)$.

plates with no shear loading, i.e. $N_S = 0$.

Note that $\mathbf{K}_j$ is a transcendental function of λ and load factor, which changes from being complex and Hermitian to being real and symmetric when all component plates are isotropic or orthotropic and $N_S = 0$. Due to this transcendental nature, usual linear eigenvalue methods are inapplicable. However for such exact stiffness matrix analysis the Wittrick-Williams algorithm[10] removes the possibility of eigenvalues ever being missed despite the transcendental nature of the problem. Therefore this algorithm was used to ensure that for any value of λ_j, the lowest critical buckling load is not confused with higher ones.

For skew plate assemblies, in which the prismatic nature of the plate assembly is maintained, point supports are used to produce the (skew) transverse boundaries. They are enforced by the method of Lagrangian multipliers, which was already present in the theory because it was needed to overcome the problem associated with shear loaded rectangular plates[3].

To include such point supports the equations become:

$$a\mathbf{K}_m\mathbf{D}_m + \mathbf{e}_m^H \gamma_n = \mathbf{0} \qquad (m = n + qM,\ q = 0, \pm 1, \pm 2, \dots) \tag{4}$$

$$\sum \mathbf{e}_m\mathbf{D}_m = \mathbf{0} \tag{5}$$

where H denotes Hermitian transpose and it is sufficient here to note that γ and **e** are the Lagrangian multiplier vectors and constraint matrices defined later in eqns (11) and (12) respectively, while $\mathbf{K}_m$ and $\mathbf{D}_m$ are defined beneath eqn. (8). The equations apply to any infinitely long plate assembly which repeats at longitudinal intervals, to form identical bays of length a. The mode is assumed to repeat over M bays, i.e. over a length L = Ma. All modes can be obtained by simultaneously satisfying these equations in turn for each of the integers n given by

$$-M'' \le n \le M' \tag{6}$$

where M" and M' are, respectively, the integer parts of (M-1)/2 and M/2. A complete solution is obtained by repeating the computations which follow at sufficient values of M. For the values of M chosen, the analysis assumes that the nodal displacements and forces of the plate assembly can be expressed, respectively, as the Fourier series:

$$\mathbf{D}_A = \sum_{m=-\infty}^{\infty} \mathbf{D}_m exp\left(\frac{2i\pi mx}{L}\right) \tag{7}$$

$$\mathbf{P}_A = \sum_{m=-\infty}^{\infty} \mathbf{K}_m\mathbf{D}_m exp\left(\frac{2i\pi mx}{L}\right) \tag{8}$$

where $\mathbf{D}_m$ and $\mathbf{K}_m$ are the $\mathbf{D}_j$ and $\mathbf{K}_j$ defined above, for $\lambda = \lambda_m$, where $\lambda_m = L/2m$ and m = 1, 2, 3,.... The total energy of a length L of the panel is expressed in terms of the stiffness matrices $\mathbf{K}_m$. The governing equations are now obtained by the method of Lagrangian Multipliers, by which the total energy is minimised subject to the constraints needed to represent the point attachments of the plate assembly to the rigid point supports. Equation (9) follows (It is similar in form to eqns (4) - (5) written as a single equation).

$$\begin{bmatrix} LK_0 & & & & & E_0^T \\ & LK_1 & & & & E_1^H \\ & & LK_{-1} & & & E_{-1}^H \\ & & & LK_2 & & E_2^H \\ & & & & \ddots & \vdots \\ E_0 & E_1 & E_{-1} & E_2 & \cdots & 0 \end{bmatrix} \begin{Bmatrix} D_0 \\ D_1 \\ D_{-1} \\ D_2 \\ \vdots \\ P_L \end{Bmatrix} = \mathbf{0} \qquad (9)$$

where negative signs indicate complex conjugates. This is valid for any prismatic plate assembly with responses which repeat over length Ma. The Lagrangian multipliers will repeat over this length such that

$$\mathbf{P}_L^T = [\mathbf{P}_{L0}^T, \mathbf{P}_{L1}^T, \mathbf{P}_{L2}^T, \ldots] \qquad (10)$$

with $\mathbf{P}_{Lk} = \mathbf{P}_{L,k+M}$ representing the Lagrangian multipliers in the interval $ka \le x < (k + 1)a$.

The above equation is satisfied by the complex Fourier series

$$\mathbf{P}_{LK} = \sum_{j=-M''}^{M'} \gamma_j exp\left(\frac{2i\pi jk}{M}\right) \qquad (11)$$

The constraint matrix $\mathbf{E}_m$ can be expressed as

$$\mathbf{E}_m^T = [\mathbf{e}_m^T, \mathbf{e}_{m1}^T, \mathbf{e}_{m2}^T, \ldots] \qquad (12)$$

$\mathbf{e}_{mk}$ is the constraint matrix for bay $ka \le x < (k + 1)a$.

The solution given by the above includes all modes with wavelength L, L/2, L/3, etc. However, by decoupling the equations and selecting m numbers that produce repetition over Ma and not also over some fraction of Ma, greater efficiency is achieved by avoiding computation involving values of m not contributing to the solution. Hence, because $\lambda_m = L/2m$ and $L = Ma$, the values of m previously defined in eqn. (4) give:

$$\lambda_m = a/\{(2n/M) + 2q\} \qquad q = 0,\pm1,\pm2,.... \qquad (13)$$

From eqn. (13), the λ_m's are functions of M/n and not of M and n independently. Therefore computational savings are made by considering only those combinations of M and n which share the same value of M/n. It is convenient here to express the resulting relationships in terms of the single parameter $\xi = 2n/M$, so that eqn. (13) can be rewritten as

$$\lambda_m = \frac{a}{(\xi + 2q)} \qquad q = 0,\pm1,\pm2,.... \qquad (14)$$

Higher accuracy is achieved, at the expense of increased solution time, by increasing both q_{max}, the maximum value of q used in eqn. (14), and also the number of ξ in the range $0 \le \xi \le 1$.

Transverse repetition.

Many plate assemblies exhibit repetitive cross-sections which can be analysed by assuming infinite-width and writing suitable recurrence equations. A brief summary of a recent publication[14] dealing with an extension of this theory for skew plate analysis follows.

For skew plate assemblies, constraints must be included in these recurrence equations such that the continuity of the line of supports is maintained in adjacent bays. This is achieved by introducing a constant longitudinal shift (x') to support locations at the start of each successive transversely adjacent portion. The fundamental equations for the repeating portion become:

$$\left.\begin{aligned} & a\underset{\sim}{K}_{m0}\underset{\sim}{D}_{m0} + \underset{\sim}{e}^H_{m0}\underset{\sim}{\gamma}_{n0} = \underset{\sim}{0} \quad (m=n+qM, \quad q=0,\pm1,\pm2,\ldots) \\ & \sum \underset{\sim}{e}_{m0}\underset{\sim}{D}_{m0} = \underset{\sim}{0} \end{aligned}\right\} \qquad (15)$$

where

$$\underset{\sim}{K}_{m0} = \underset{\sim}{K}_{m11} + \underset{\sim}{K}^H_{m12}\, exp\{-i(\phi - 2\pi m x'/Ma)\} + \underset{\sim}{K}_{m12}\, exp\{i(\phi - 2\pi m x'/Ma)\} \qquad (16)$$

Equations (15) must be solved for the same combinations of M and n, or values of ξ, as for plate assemblies that are not transversely repetitive. However, now suitable values of ϕ must be used for each combination.

When $\alpha = 0°$, eqn. (16) reduces to the previously defined form (Williams and Anderson, 1985)

$$\underset{\sim}{K}_{m0} = \underset{\sim}{K}_{m11} + \underset{\sim}{K}^H_{m12}\, exp(-i\phi) + \underset{\sim}{K}_{m12}\, exp(i\phi) \qquad (17)$$

and the values of ϕ can reasonably be restricted to those which give modes which repeat across twice the width of the assembly, so that, if P is the number of repeating portions of width b within the assembly,

$$\phi = \pi g/P \qquad g = -(P - 1),\ldots,-1,0,1,\ldots,P. \qquad (18)$$

and the transverse half-wavelength λ_T is

$$\lambda_T = Pb/g = \pi b/\phi \qquad (19)$$

Because $\alpha \neq 0°$ is now the general case, $x' \neq 0$ in eqn. (16) and so the mode repeats over twice the width Pb of the assembly except that it is now moved along the assembly by 2x', such that it is skewed by the angle α, where $x' = b.\tan\alpha$, see Fig. 1(c). Hence λ_T is the component, perpendicular to the longitudinal axis, of a half-wavelength that is skewed by the angle α.

The theory presented above is now incorporated in an existing 36,000 line, FORTRAN 77 computer program[9]. The program also has an optional numerical procedure[2] allowing for transverse shear deformation, and has been applied to some of the results that follow.

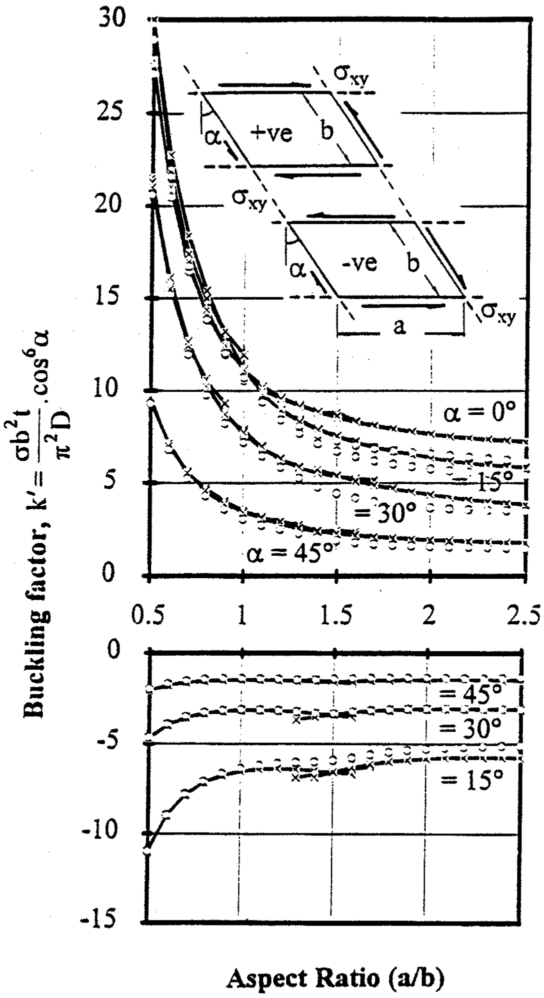


Fig. 5. Skew-transversely-repetitive buckling factor curves ($k' = k.\cos^6\alpha$) for simply supported shear loaded plates. Discrete results (O) represent finite width plates.

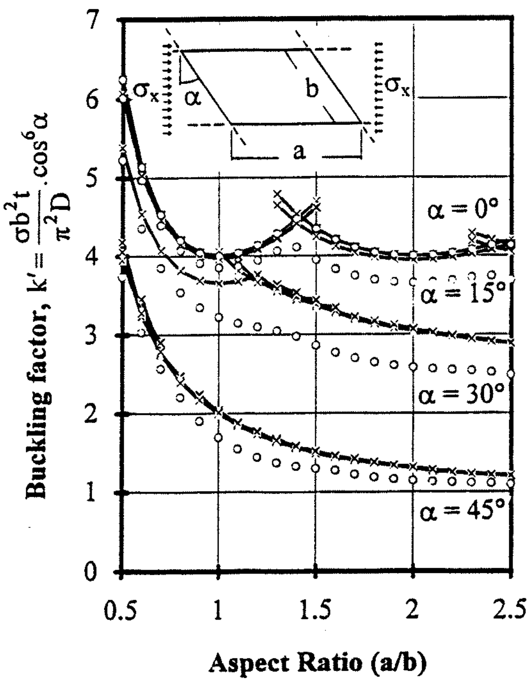


Fig. 4. Skew-transversely-repetitive buckling factor curves ($k' = k.\cos^6\alpha$) for simply supported compression loaded plates. Discrete results (O) represent finite width plates.

Results

Buckling factors ($k = \sigma b^2 t/\pi^2 D$) were obtained using Poisson's ratio $\nu = 0.3$ with parameters $\xi = 0, 0.1, 0.2, ..., 0.9, 1$ and $q_{max.} = 10$ in the computer model, unless stated otherwise.

Previous results[13] for compression and shear loaded arrays of continuous skew plates are shown in Figs 4 and 5 respectively. They also illustrate the comparison with continuity along a single axis, i.e. plates of finite width. Cusps denote a change in mode.

New results for hexagonal, circular and triangular compression and shear loading geometries follow.

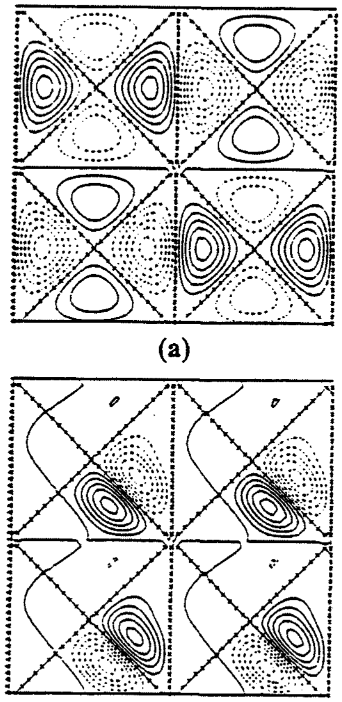
Note: It is important to realise that the aspect ratio (a/b) is not constant for the various geometries. The circular and hexagonal geometries are further complicated by the fact that their boundaries overlap, see Fig. 2. It can however be seen that when centre-lines are drawn, see Fig. 2(a), an oblique shape is produced, equivalent to that of a skew plate with its centres at angle $\alpha = 30°$. Hence the aspect ratio is defined as the longitudinal centre-line length a divided by the skew-transverse centre-line length b.

The triangular array is a special case and is defined by a square planform into which the internal supports forming the triangular arrangement are positioned, see the buckling modes in Fig. 6.

The circular supported plate array requires more point constraints to satisfactorily define the geometry. The number of terms $q_{max.}$ required for good convergence are shown in Table 1.

Table 1. Buckling factor ($k = \sigma b^2 t/\pi^2 D$) results for Circular, Hexagonal, Triangular, Skew and Square geometries. Convergence results are included. All plates have aspect ratio a/b = 1, see preceding note.

		$q_{max.}$				
Geometry	Load	5	10	20	25	30
Circular	Comp.	-	-	9.956	9.940	9.933
	Shear	-	-	16.84	16.79	16.76
Hexagonal	Comp.	6.63	6.60	-	-	-
	Shear	18.23	18.12	-	-	-
Triangular	Comp.		17.99	-	-	-
	Shear		38.29	-	-	-
Skew	Comp.	-	8.654	-	-	-
(α= 30°)	Shear		18.04 / -7.48		-	-
Square	Comp.	-	4.00	-	-	-
	Shear	-	11.07	-	-	-



(a)

(b)

Fig. 6. Buckling modes for (a) compression and (b) shear loaded plate array with triangular internal supports.

To allow comparison with the few results in the literature, an option within the computer code[2] was used to obtain the following results for an array of continuous plates with triangular internal supports using transverse shear deformation theory, see Table 2.

Table 2. Transverse shear buckling results and Classical Plate Theory results for compression loaded square plates (a/b = 1) with triangular internal supports.

	t/b	k
Wang *et al.*[6]	0.05	16.6074
Liew and Wang[5]	*CPT*	18.055
Author	0.047	17.1280
Author	*CPT*	17.99

Concluding remarks

An efficient buckling analysis method for a continuous array of plates, extending over regular internal supports has been presented. The results include both curved (circular) or straight (hexagonal, skew, triangular and square) supports.

Despite the limited number of results in this preliminary investigation, the merits of using internal supports other than rectangular would seem, in certain cases, to be of significance, especially in applications where Super-Plastic-Forming processes are available for fabrication.

Futher work

Clamped results have been shown to give good agreement[13] with other results in the literature for skew plates. This agreement should also apply to the other geometries investigated in this paper.

Acknowledgements.

The results presented herein were obtained while under EPSRC support, grant No. GR/J79430.

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