CHATTERING ARC IN THREE-DIMENSIONAL FLIGHT

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Abstract

The chattering arcs of a winged space vehicle at its hypersonic reentry flight have been studied. With the use of dimensionless variables, we need only two parameters to specify the aerodynamic characteristics of the vehicle, one is the ballistic coefficient and the other is the maximum lift-to-drag ratio. Also, the Earth's atmosphere is specified by one parameter, the so-called Chapman's parameter. The two control variables are the normalized lift coefficient and the bank angle. The normalized lift coefficient is defined to be the ratio of the lift coefficient and the lift coefficient at the maximum lift-to-drag ratio. We use the chattering control on the normalized lift coefficient to obtain the maximum drag or, say, the maximum deceleration. When the maximum and minimum normalized lift coefficients have the same absolute value, the theoretical chattering arc only exists in the vertical plane flight. The bank angle control is invalid. The resulted reentry trajectory is a two-dimensional ballistic trajectory with maximum drag. When the maximum and minimum normalized lift coefficients have different absolute values, we can have either two-dimensional or three-dimensional reentry chattering flight. For numerical computation, we assume that the absolute value of the minimum normalized lift coefficient is smaller than the maximum normalized lift coefficient. In the case of two-dimensional reentry chattering flight, the trajectory is ballistic type again with the drag-effective normalized lift coefficient equals to the average of the absolute minimum and maximum normalized lift coefficients. The lift-effective normalized lift coefficient is the difference of the two values. In the case of three-dimensional reentry chattering flight, the lift-effective normalized lift coefficient is used as the lift control. It is then used with the bank angle control to obtain the lateral range of the winged reentry vehicle. A three-dimensional reentry chattering trajectory for maximum lateral range has been computed under the assumption of equilibrium glide condition. The three-dimensional reentry chattering trajectory with constant bank angle has also been investigated. In summary, the theoretical three-dimensional chattering arc will be existing only when the maximum lift coefficient is different from the absolute value of the minimum lift coefficient.

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Introduction

Theoretically, in chattering arc of the first kind, the control chatters between its maximum and minimum values at an infinite rate.[1] Its existence in optimal trajectories has been discussed extensively in two eminent books[2,3] and some published papers.[4-8] In Ref. 4, a study of chattering cruise flight was presented. In Ref. 5, it was found that the chattering arc is a part of the minimum time optimal trajectory at constant altitude flight. Then the approximate chattering arc was developed in which the control switching is at a finite rate.[6-7] This development is for the purpose of practical maneuver since the control switching at an infinite rate is not possible. In these discussions, the flight is at constant altitude.

The theoretical chattering arc for hypersonic flight in a vertical plane has been developed in Ref. 8. With the use of dimensionless variables, we need only two physical values of the space vehicle for numerical computation: the maximum lift-to-drag ratio and the ballistic coefficient.(9) The ballistic coefficient also represents the initial altitude through the atmospheric density. The atmosphere of the planet is specified by the so-called Chapman's parameter. The control variable is the normalized lift coefficient. From the singularity analysis, it is found that the theoretical chattering arc in the vertical plane is a kind of ballistic trajectory with maximum drag.

The purpose of this paper is to investigate the chattering arc for hypersonic flight in the three-dimensional space. The theoretical chattering arc will be studied and analyzed. This study will be useful for solving minimum time aerobraking trajectory in three-dimensional flight.

Model and Equations of Motion

Equations of Motion

For reentry flight in three-dimensional space, the motion of the center of mass of a winged space vehicle can be defined by six state variables r, V, γ, θ, ϕ, and ψ, where r is the distance measured from the earth center, V is the vehicle speed, γ is the flight path angle, θ is the longitude, ϕ is the latitude, and ψ is the heading angle. The geometry is shown in Fig. 1. Since there will be no thrust during the reentry flight, the governing differential equations for the six state variables are(3)

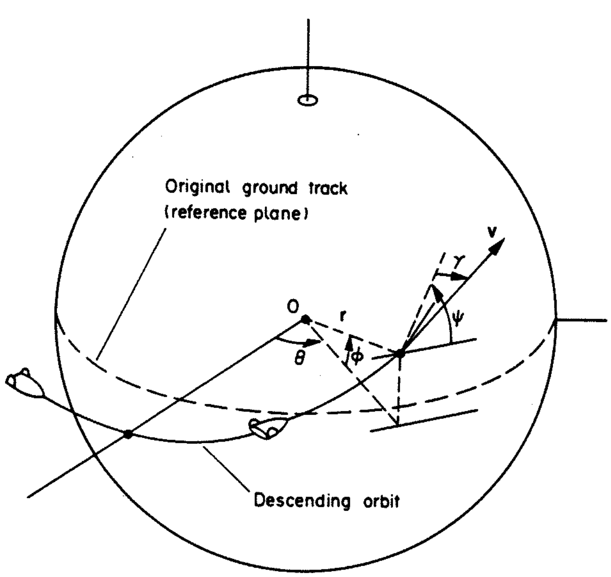


FIGURE 1 - Geometry of Reentry Trajectory.

$$\frac{dr}{dt} = V\sin\gamma$$

$$\frac{dV}{dt} = -\frac{\rho S C_D V^2}{2m} - g\sin\gamma$$

$$V\frac{d\gamma}{dt} = \frac{\rho S C_L V^2}{2m}\cos\sigma - (g - \frac{V^2}{r})\cos\gamma \tag{1}$$

$$\frac{d\theta}{dt} = \frac{V\cos\gamma\cos\psi}{r\cos\phi}$$

$$\frac{d\phi}{dt} = \frac{V\cos\gamma\sin\psi}{r}$$

$$V\frac{d\psi}{dt} = \frac{\rho S C_L V^2}{2m\cos\gamma}\sin\sigma - \frac{V^2}{r}\cos\gamma\cos\psi\tan\phi$$

where ρ is the atmospheric density, S is the reference area, C_D is the drag coefficient, C_L is the lift coefficient, σ is the bank angle, g is the gravitational acceleration, and t is the flight time. The two control variables are C_L and σ.

Models of Aerodynamics, Atmosphere and Gravitation

For aerodynamic model of the winged space vehicle, we shall use a parabolic drag polar of the form

$$C_D = C_{D0} + KC_L^2 \tag{2}$$

where C_{D0} is the zero-lift drag coefficient, and K is the induced drag factor. By defining the normalized lift coefficient λ such that

$$\lambda = \frac{C_L}{C_L^*} \tag{3}$$

where C_L^* is the lift coefficient corresponding to maximum lift-to-drag ratio, we have

$$C_L^* = \sqrt{\frac{C_{D0}}{K}}, \; C_D^* = 2C_{D0}, \; E^* = \frac{C_L^*}{C_D^*} = \frac{1}{2\sqrt{C_{D0}K}} \tag{4}$$

In this paper, we shall assume that both C_{D0} and K have constant values at hypersonic speed during reentry flight.

The atmospheric density is assumed to be locally exponential, that is

$$\rho = \rho_0 e^{\beta(r)(r_0 - r)} \tag{5}$$

where r_0 is the specified reference radial distance, ρ_0 is the atmospheric density at r_0, and $\beta(r)$ is the inverse of the scale height. By defining the dimensionless altitude h such that

$$h = \frac{r - r_0}{r_0} \tag{6}$$

and using the so-called Chapman's parameter $\beta(r) r_0 = 1/\varepsilon(r)$, we have

$$\rho = \rho_0 \varepsilon^{-h/\varepsilon(r)} \tag{7}$$

Finally, the gravitational field is modelled to be Newtonian:

$$\frac{g}{g_0} = \frac{r_0^2}{r^2} \tag{8}$$

where g_0 is the gravitational acceleration at r_o.

Dimensionless Equations of Motion

With the definition of the dimensionless kinetic energy u, the dimensionless arc length s, and the ballistic coefficient B as given below,

$$u = \frac{V^2}{g_0 r_0}, \; s = \int_0^t \frac{V}{r} \cos\gamma dt, \; B = \frac{\rho_0 S C_L^* r_0}{2m} \tag{9}$$

we have the following dimensionless equations of motion:

$$\frac{dh}{ds} = (1+h)\tan\gamma$$

$$\frac{du}{ds} = -\frac{B(1+h)e^{-h/\varepsilon(r)}u}{E^* \cos\gamma}(1+\lambda^2) - \frac{2\tan\gamma}{(1+h)}$$

$$\frac{d\gamma}{ds} = \frac{B(1+h)e^{-h/\varepsilon(r)}}{\cos\gamma}\lambda\cos\sigma - \frac{1}{u(1+h)} + 1 \tag{10}$$

$$\frac{d\theta}{ds} = \frac{\cos\psi}{\cos\phi}$$

$$\frac{d\phi}{ds} = \sin\psi$$

$$\frac{d\psi}{ds} = \frac{B(1+h)e^{-h/\varepsilon(r)}}{\cos^2\gamma}\lambda\sin\sigma - \cos\psi\tan\phi$$

The two control variables are subject to the constraints

$$\lambda_{\min} \le \lambda \le \lambda_{\max}, \quad \sigma_{\min} \le \sigma \le \sigma_{\max} \tag{11}$$

Variational Formulation

Hamiltonian

Using the maximum principle, we introduce the adjoint vector $\vec{p}$ to form the Hamiltonian(10)

$$H = p_h(1+h)\tan\gamma$$
$$- \frac{B(1+h)e^{-h/\varepsilon}up_u}{E^*\cos\gamma}(1+\lambda^2) - \frac{2p_u\tan\gamma}{(1+h)}$$
$$+ \left[\frac{B(1+h)e^{-h/\varepsilon}}{\cos\gamma}\lambda\cos\sigma - \frac{1}{u(1+h)} + 1\right]p_\gamma$$

$$+\frac{\cos\psi}{\cos\phi}p_\theta + p_\phi \sin\psi$$

$$+\left[\frac{B(1+h)e^{-h/\varepsilon}}{\cos^2\gamma}\lambda\sin\sigma - \cos\psi\tan\phi\right]p_\psi \qquad (12)$$

Domain of Maneuverability

The controls C_L and σ must be selected such that at each instant the Hamiltonian is an absolute maximum. Regarding the optimal lift control, it suffices to consider the part of the Hamiltonian containing λ:

$$\overline{H} = \frac{B(1+h)e^{-h/\varepsilon}}{\cos\gamma}\left(-\frac{1}{E^*}up_u\lambda^2 + p_\gamma\lambda\cos\sigma + \frac{p_\psi\lambda\sin\sigma}{\cos\gamma}\right) \qquad (13)$$

The optimal bank control can be derived from $\partial H / \partial\sigma = 0$. It gives

$$\tan\sigma = \frac{p_\psi}{p_\gamma\cos\gamma} \qquad (14)$$

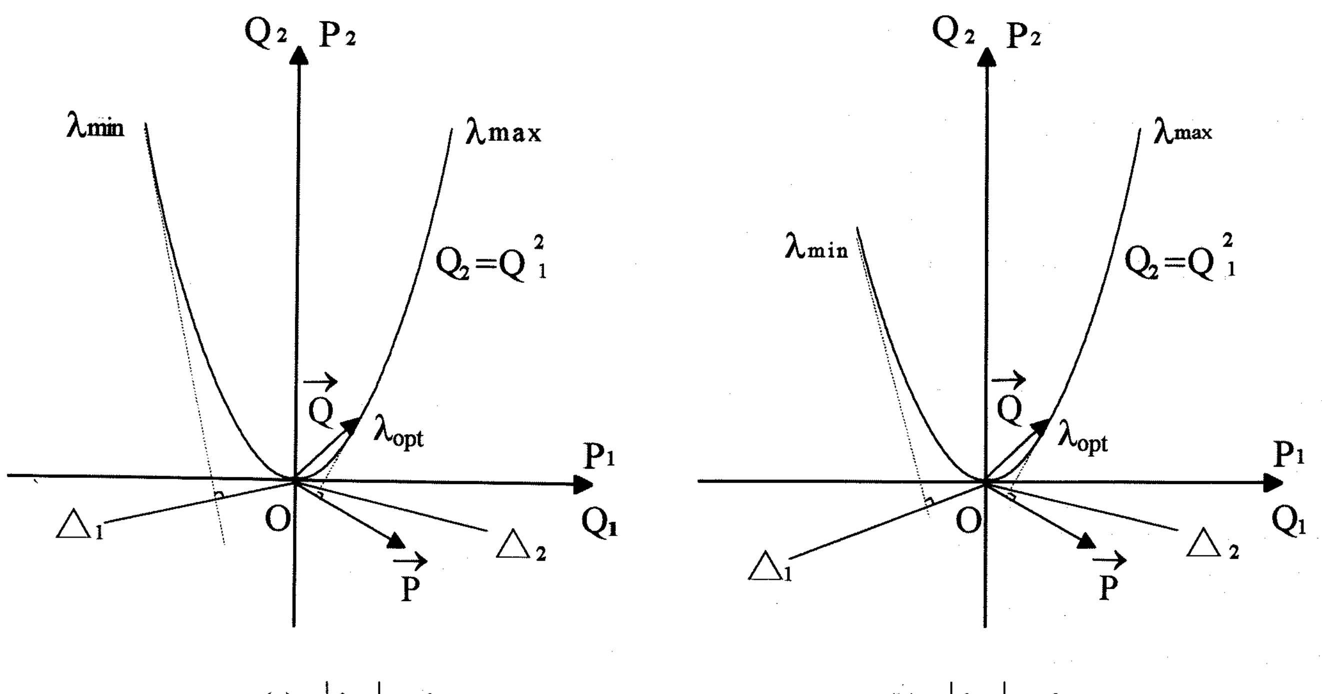
Now, we use Eq. (14) to eliminate p_ψ in Eq. (13). It becomes that

$$\overline{H} = \frac{B(1+h)e^{-h/\varepsilon}}{\cos\gamma}\left(-\frac{1}{E^*}up_u\lambda^2 + \frac{1}{\cos\sigma}p_\gamma\lambda\right) \qquad (15)$$

For the reentry flight we shall consider in this paper, the absolute value of the bank angle is constrained to be less than 90^0. That is, $|\sigma_{max}| < 90^0$ and we always have $\cos\sigma > 0$. Then the reduced Hamiltonian $\overline{H}$ can be expressed as the dot product of the two vectors (P_1, P_2) and (Q_1, Q_2) such that[3, 8]

$$P_1 = \frac{p_\gamma}{\cos\sigma}, \qquad P_2 = -\frac{1}{E^*}up_u,$$
$$Q_1 = \lambda, \qquad Q_2 = \lambda^2, \qquad (16)$$

and



(a). $|\lambda_{min}| = \lambda_{max}$ (b). $|\lambda_{min}| < \lambda_{max}$

FIGURE 2 - Domain of Maneuverability for Normalized Lift Coefficient.

$$\overline{H} = \frac{B(1+h)e^{-h/\varepsilon}}{\cos\gamma}(P_1Q_1 + P_2Q_2) \tag{17}$$

When λ varies, the vector (Q_1, Q_2) describes the domain of maneuverability which is the parabola as shown in Fig. 2,

$$Q_2 = Q_1^2 \tag{18}$$

To maximize $\overline{H}$, if the vector (P_1, P_2) is inside $\Delta_1 O \Delta_2$, the optimal lift used is an interior value such that the tangent to the parabola is perpendicular to (P_1, P_2) as shown in Fig. 2. This is expressed by

$$\frac{\partial Q_2}{\partial Q_1} = 2Q_1 = -\frac{P_1}{P_2},$$

or

$$\lambda = \frac{E^* p_\gamma}{2up_u \cos\sigma} = \frac{E^*}{2up_u \cos\gamma}\left[(p_\gamma \cos\gamma)^2 + p_\psi^{\ 2}\right]^{1/2} \tag{19}$$

Therefore, it is necessary that $P_2 < 0$, or in other words, $p_u > 0$. When the vector (P_1, P_2) is outside the angle $\Delta_1 O \Delta_2$, the optimal control to be used must be either $\lambda = \lambda_{min}$ when $p_\gamma < 0$ or $\lambda = \lambda_{max}$ when $p_\gamma > 0$. In the case when $p_\gamma = 0$ and $p_u < 0$ for a finite time interval, there may exist a singular arc in which the lift control switches rapidly between λ_{min} and λ_{max} at an infinite rate. This is the so-called chattering control and the resulted flight path is called the chattering arc. It is obvious that when the chattering control is applied, the drag is at its maximum value. Accordingly, the vehicle can use chattering arc to obtain maximum deceleration and to reduce its speed in the shortest time.

From Fig. 2, it is found that we have different domains of maneuverability when $\lambda_{min} = -\lambda_{max}$ and $\lambda_{min} \neq -\lambda_{max}$. The domain of maneuverability for $\lambda_{min} = -\lambda_{max}$ is shown in Fig. 2(a). In this case, the effect of the λ^2 term on the chattering arc will be existing while the effect of the $\lambda \sin\sigma$ and $\lambda \cos\sigma$ terms will be cancelled. For the case when $|\lambda_{min}| < \lambda_{max}$, as shown in Fig. 2(b), we shall assume that on the chattering arc, the terms λ^2, $\lambda \sin\sigma$, and $\lambda \cos\sigma$ can be replaced by $(\frac{\lambda_{max} + |\lambda_{min}|}{2})^2$, $\lambda_{mm} \sin\sigma$, and $\lambda_{mm} \cos\sigma$, respectively. The definition of λ_{mm} is

$$\lambda_{mm} = \lambda_{max} - |\lambda_{min}| \tag{20}$$

In three-dimensional flight, it is well known that there are four integrals.(3) At first, H is not an explicit function of s, we have

$$H = C_0 \tag{21}$$

where C_0 is the first integral. Secondly, from

$$\frac{dp_\theta}{ds} = -\frac{\partial H}{\partial \theta} = 0,$$

we have

$$p_\theta = C_1 \tag{22}$$

where C_1 is the second integral. Then for p_ϕ and p_ψ, we have

$$\frac{dp_\phi}{ds} = -\frac{\partial H}{\partial \phi} = \frac{\cos\psi}{\cos^2\phi}\left(-C_1 \sin\phi + p_\psi\right),$$

$$\frac{dp_\psi}{ds} = -\frac{\partial H}{\partial \psi} = \frac{\sin\psi}{\cos\phi}\left(C_1 - p_\psi \sin\phi\right) - p_\phi \cos\psi. \tag{23}$$

By using the fourth equation of Eqs. (10) in Eqs. (22), they become

$$\frac{dp_\phi}{d\theta} = \tan\phi + \frac{p_\psi}{\cos\phi}, \tag{24}$$

$$\frac{dp_\psi}{ds} = -\left(1 + p_\psi \sin\phi\right)\tan\psi - p_\phi \cos\phi .$$

Hence,

$$\frac{d^2 p_\phi}{d\theta^2} = -p_\phi ,$$

and p_ϕ and p_ψ can be solved and expressed as

$$p_\phi = C_2 \sin\theta - C_3 \cos\theta ,$$
$$p_\psi = C_1 \sin\phi + \left(C_2 \cos\theta + C_3 \sin\theta\right)\cos\phi , \qquad (25)$$

where C_2 and C_3 are the third and fourth integrals, respectively. Finally, the differential equations for p_h, p_u, and p_γ are

$$\frac{dp_h}{ds} = -p_h \tan\gamma + \frac{B(1 - \frac{h}{\varepsilon} - \frac{h^2}{\varepsilon})e^{-h/\varepsilon}}{\cos\gamma} \times \left[\frac{up_u}{E^*}(1+\lambda^2) - \frac{\lambda p_\psi}{\cos\gamma \sin\sigma}\right] - \frac{2p_u \tan\gamma}{(1+h)^2} - \frac{p_\psi}{u(1+h)^2 \cos\gamma \tan\sigma},$$

$$\frac{dp_u}{ds} = \frac{B(1+h)e^{-h/\varepsilon} p_u}{E^* \cos\gamma}(1+\lambda^2) - \frac{p_\psi}{u^2(1+h)\cos\gamma \tan\sigma}$$

$$\frac{dp_\gamma}{ds} = \frac{1}{\cos^2\gamma}\{-p_h(1+h) + B(1+h)e^{-h/\varepsilon}\sin\gamma \times \left[\frac{up_u(1+\lambda^2)}{E^*} - \frac{\lambda p_\psi}{\cos\gamma}\left(\sin\sigma + \frac{1}{\sin\sigma}\right)\right] + \frac{2p_u}{(1+h)}\}. \qquad (26)$$

Numerical Computation and Results

For numerical computation, we assume the following values for the physical parameters of the winged space vehicle:

$$E^* = 1, \qquad B = 0.15,$$
$$\lambda_{max} = 2.5, \quad \lambda_{min} = -2.5 \text{ or } -2.0 . \qquad (27)$$

The property of the Earth's atmosphere is represented by $\varepsilon(r) = 1/\beta(r) r_0$=1/900, a constant.[9] The initial altitude is 120 km, the initial flight path angle is zero, and the initial vehicle speed is the circular speed at the initial altitude.

Chattering Arc in the Vertical Plane

For flight in the vertical plane, we have $\sigma = \phi = \psi = 0$ and $s = \theta$. The dimensionless equations of motion are

$$\frac{dh}{d\theta} = (1+h)\tan\gamma ,$$

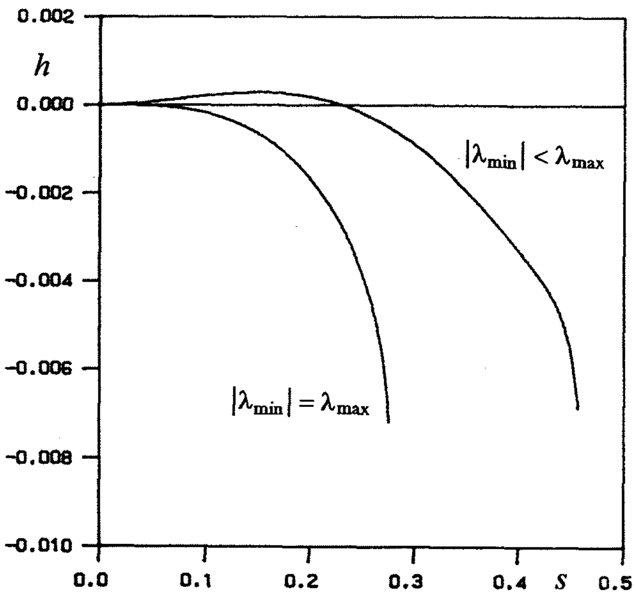
$$\frac{du}{d\theta} = -\frac{B(1+h)e^{-h/\varepsilon}u}{E^* \cos\gamma}\left[1 + \left(\frac{\lambda_{max} + |\lambda_{min}|}{2}\right)^2\right] - \frac{2\tan\gamma}{(1+h)},$$

$$\frac{d\gamma}{ds} = \frac{B(1+h)e^{-h/\varepsilon}}{\cos\gamma}\lambda_{mm} - \frac{1}{u(1+h)} + 1 \qquad (28)$$

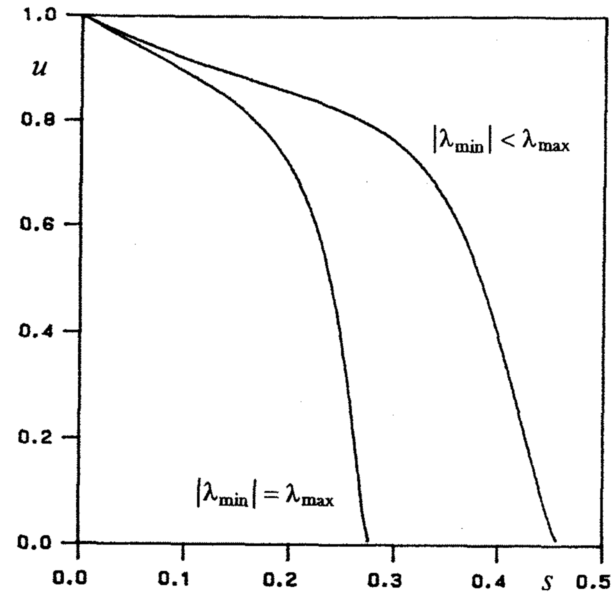
There is no control variable in Eqs. (28). Consequently, the chattering arc of a winged space vehicle in the vertical plane is equivalent to the flight trajectory of a ballistic type projectile with maximum drag. To obtain the chattering arc, we simply integrate Eqs. (28) from the initial state

$$\theta_0 = 0, \quad (h_0, u_0, \gamma_0) = (0, 1, 0^0) \qquad (29)$$

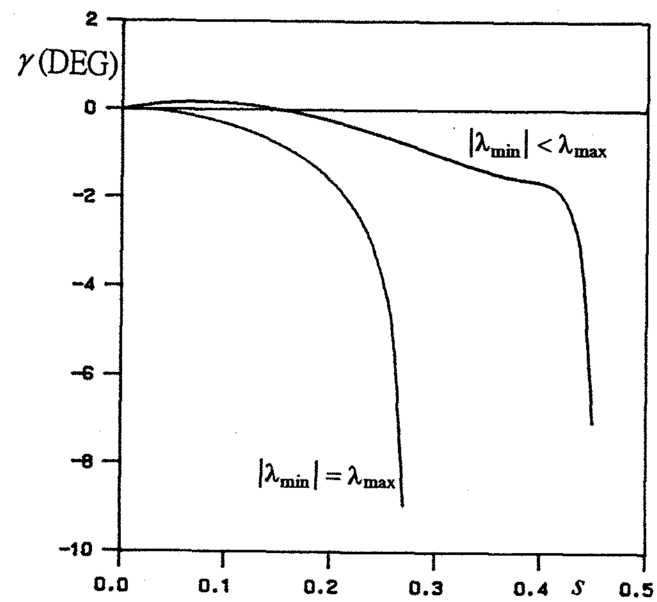
The integration is completed when $u_f = 0.01$ is reached. The chattering arcs for $\lambda_{max} = |\lambda_{min}| = 2.5$, and $\lambda_{max} = 2.5$ and $\lambda_{min} = -2.0$ are shown in Fig. 3. It is clear that $\lambda_{mm} = 0$ when $\lambda_{max} = |\lambda_{min}| = 2.5$, and $\lambda_{mm} = 0.5$ when $\lambda_{max} = 2.5$ and $\lambda_{min} = -2.0$.



(a). Variation of Altitude



(b). Variation of Speed



(c) . Variation of Flight Path Angle.

FIGURE 3-Chattering Arcs in the Vertical Plane.

3-D Chattering Arc with Constant Bank

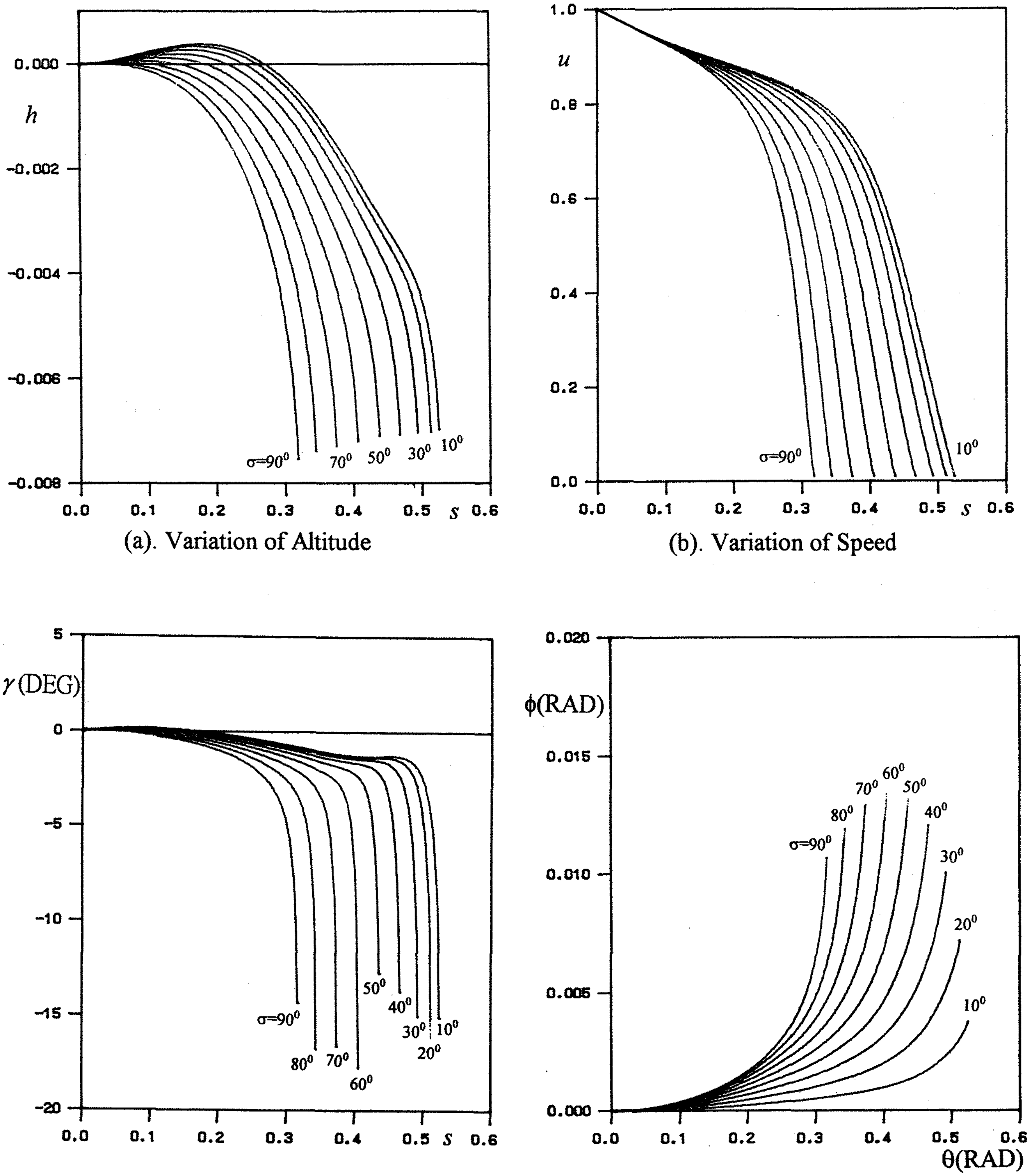
When the bank angle is kept at a certain constant value, we can obtain the three-dimensional chattering arc by integrating the Eqs. (10) from the initial condition

$$(h_0, u_0, \gamma_0, \theta_0, \phi_0, \psi_0) = (0,1,0,0,0,0) \qquad (30)$$

It is obvious that the three-dimensional chattering arc will be existing only when $|\lambda_{min}| < \lambda_{max}$. Therefore, in the second equation of Eqs. (10), the λ^2 will be replaced by $(\frac{\lambda_{max} + |\lambda_{min}|}{2})^2$ and has the value $(2.25)^2$. Also, the λ in the third and sixth equations of Eqs. (10) will be replaced by λ_{mm} and has the value 0.5. The integration is stopped at the final speed $u_f = 0.01$. The chattering arcs for $\sigma = 10^0, 20^0, \ldots,$ and 90^0 are presented in Fig. 4.

3-D Chattering Arc with Optimal Bank Control

It is intended to use the optimal bank control to maximize the final latitudinal range for the three-dimensional chattering flight. In other words, we try to reduce the vehicle speed as soon as possible by using the maximum drag, and at the same time try to obtain the maximum lateral range by using



(a). Variation of Altitude

(b). Variation of Speed

(c). Variation of Flight Path Angle.

(d). Trajectory Mapping on θ–φ Plane.

FIGURE 4 - Three-Dimensional Chattering Arcs with Constant Bank Angle.

the optimal control on the bank angle. Thus the performance index is

$$J = \max \phi_f \tag{31}$$

In Eq. (21), since the dimensionless final time is free, we have

$$H = C_0 = 0 \tag{32}$$

Also, from Eq. (22) we have

$$p_\theta = C_1 = 0, \tag{33}$$

since θ_f is free. For the final condition of the other state variables, we assume that $h_f = free$, $u_f = 0.01$, $\gamma_f = free$, and $\psi_f = free$. Therefore, from Eqs. (25) we have

$$\begin{aligned} p_{\phi_f} &= 1 = C_2 \sin\theta_f - C_3 \cos\theta_f \\ p_{\psi_f} &= 0 = (C_2 \cos\theta_f + C_3 \sin\theta_f)\cos\phi_f \end{aligned} \tag{34}$$

Solving for the integrals C_2 and C_3 we can obtain the solutions for p_ϕ and p_ψ,

$$\begin{aligned} p_\phi &= \cos(\theta_f - \theta) \\ p_\psi &= \sin(\theta_f - \theta)\cos\phi \end{aligned} \tag{35}$$

To obtain the optimal trajectory, we integrate Eqs. (10) from the initial condition given in Eq. (30), with λ^2 replaced by $(2.25)^2$ and λ replaced by 0.5, and using the control law (14) and the Eqs. (35) for p_ϕ and p_ψ. There are four parameters, namely, θ_f and the three initial values p_{h_0}, p_{u_0}, and p_{γ_0}. Since we can use the Hamiltonian integral *H=0* to eliminate one of the parameters, this is a three-parameter problem.

A simplification has been made in Ref. 3 by using the so-called equilibrium glide condition. It is a condition with the assumption that the glide angle is small and varies very slowly, so that we can have

$$\gamma \cong 0, \qquad \frac{d\gamma}{ds} \cong 0 \tag{36}$$

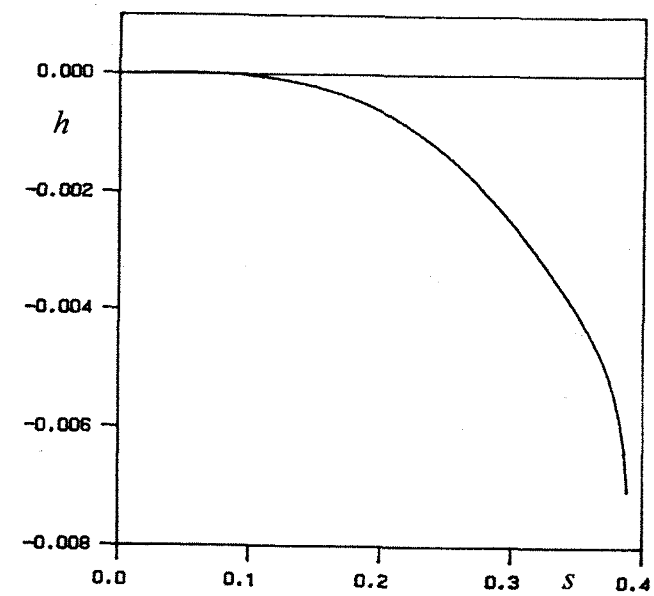
With this condition, an explicit law for the bank angle control can be derived[3]

$$\tan\sigma = \left[\frac{1}{u(1+h)} - 1\right] \frac{\sin(\theta_f - \theta)\cos\phi}{\cos(\theta_f - \theta)\sin\psi - \sin(\theta_f - \theta)\sin\phi\cos\psi} \tag{37}$$

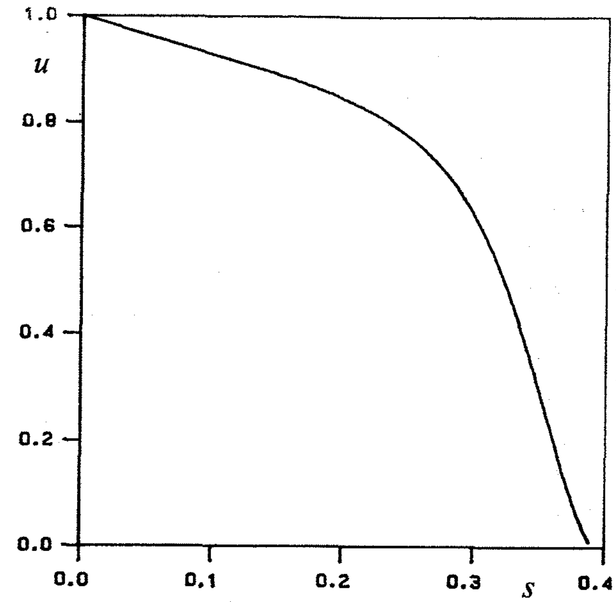
For numerical computation, we use the control law (37) to integrate the full set of exact state equations (10), with a guessed value for θ_f. This value is to be adjusted such that at the final time when $\theta = \theta_f$, the prescribed final value $u_f = 0.01$ is satisfied. The sub-optimal trajectory obtained is shown in Fig. 5. The maximum lateral range obtained is $\phi_f = 0.013592$. This value is greater than the maximum value 0.013141 obtained in the constant bank angle case.

Conclusions

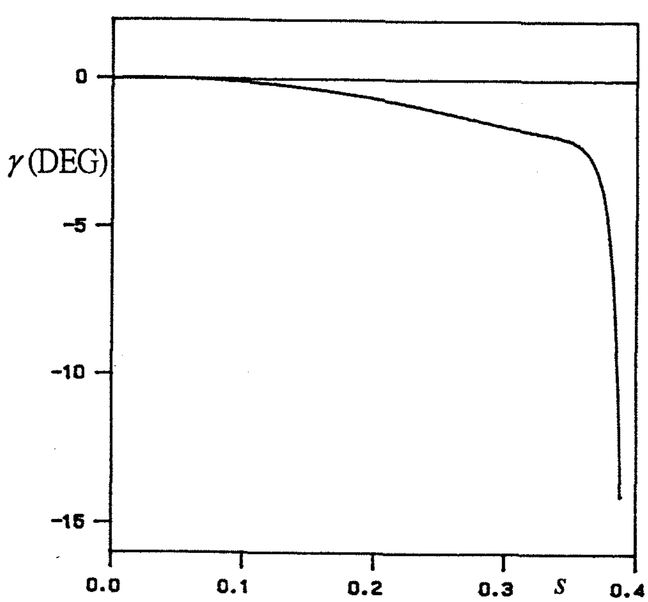
The theoretical two- and three-dimensional chattering trajectories of a winged space vehicle at its hypersonic reentry flight have been investigated. The aerodynamic characteristics of the vehicle are specified by the ballistic coefficient and the maximum lift-to-drag ratio, and the Earth's atmosphere is specified by the so-called Chapman's parameter. We use the normalized lift coefficient and the bank angle as the control variables, and the chattering on the lift control is studied. The resulted chattering arc, either two- or three-dimensional, is a flight trajectory with maximum drag and maximum deceleration. The net effective lift coefficient for the drag is equal to the average of the maximum lift coefficient and the absolute value of the minimum lift coefficient, while the net effective lift coefficient for the lift is equal to their difference. Therefore, we can have theoretical three-dimensional chattering reentry flight only when the two extreme lift coefficients have different absolute values. The two-dimensional chattering reentry trajectory is a maximum deceleration ballistic trajectory. The three-dimensional chattering reentry is



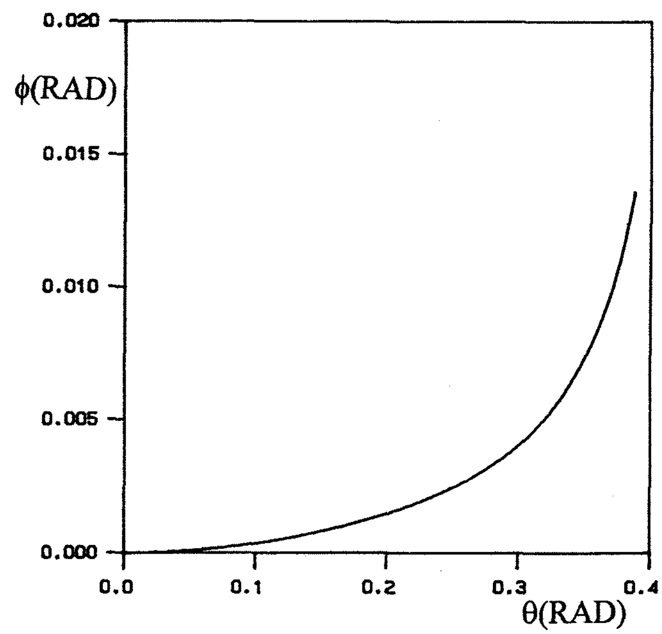
(a). Variation of Altitude



(b). Variation of Speed



(c). Variation of Flight Path Angle



(d). Trajectory Mapping on θ–ϕ Plane

FIGURE 5 - Sub-Optimal Trajectory for Maximum Lateral Range in Three-Dimensional Chattering Reentry Flight.

mainly at maximum deceleration, with a certain part of the lift for lateral maneuver.

References

1. Marchal, C., "Chattering Arcs and Chattering Controls," Journal of Optimization Theory and Applications, Vol. 11, No. 5, 1973, pp. 441-468.

2. Marec, J. P., *Optimal Space Trajectories*, Elsevier Scientific Publishing Co., Amsterdam, 1979.

3. Vinh, N. X., *Optimal Trajectories in Atmospheric Flight*, Elsevier Scientific Publishing Co., Amsterdam, 1981.

4. Houlihan, S. C., Cliff, E. M., and Kelly, H. J., "Study of Chattering Cruise," Journal of Aircraft, Vol. 19, No. 2, 1982, pp. 119-124.
5. Chern, J. S., Yang, C. Y., and Lai, C. C., "Minimum-Time Aerobraking Maneuver at Constant Altitude," Acta Astronautica, Vol. 26, No. 11, 1992, pp. 763-771.
6. Chern, J. S., and Lai, C. C., "Approximate Chattering Arc for Aerobraking Maneuver," AAS-93-175, Pasadena, California, Feb. 22-24, 1993.
7. Chern, J. S., Hong, Z. C., and Chen, Y. T., "Approximate Chattering Arc for Minimum-Time Flight," IAF-93-A.2.14, Graz, Austria, Oct. 16-22, 1993 ; also Acta Astronautica, Vol. 35, No. 9-11, 1995, pp. 623-631.
8. Chern, J. S., "Chattering Arc in the Vertical Plane," presented at the PICAST'1, Tainan, Taiwan, Requblic of China, Dec. 6-10, 1993.
9. Vinh, N. X., and Han, D., "Optimal Reentry Trajectories by Asymptotic Matching," IAF-93-A.2.12, Graz, Austria , Oct. 16-22, 1993.
10. Bryson , A. E., and Ho, Y. C., *Applied Optimal Control*, Ginn , Waltham, Mass., 1969.