MOMENTUM AND HEAT TRANSFER IN LAMINAR BOUNDARY LAYER FLOWS OF NON-NEWTONIAN POWER-LAW FLUIDS PAST EXTERNAL SURFACES

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Abstract

The basic equation of energy for non-Newtonian fluid is developed in a non-orthogonal accelerated curvilinear coordinate system. Resulting equation is applied to incompressible plane flows of power-law fluids together with the equations of motion to investigate the effects of longitudinal surface curvature on thermal boundary layer flow. Possible similarity solutions including curvature effects are given.

1. Introduction

Non-Newtonian fluids have a wide range of application in the various manufacturings and processing industry such as those dealing with plastics, polymers, foods and journal bearing lubrication, etc.[1,2]. Some of the flow problems appearing in these applications can be solved by using the well-known boundary layer approximation. Conventional boundary layer theory (i.e., first order boundary layer theory) for laminar flow of non-Newtonian fluids on any surface is based on the approximation that the pressure change normal to the surface is small enough to be neglected for thin boundary layers. The second order boundary layer theory requires to account effects of the surface curvature, the displacement thickness and the vorticity. In order to correct the conventional theory for the effects of longitudinal surface curvature, one has to consider the pressure gradient normal to the surface. Such consideration requires that some additional terms have to be included into the equations of the motion and the outer boundary condition must be modified to account for curvature effects.

Modification for curvature effects of the conventional velocity boundary layer theory for power-law fluids were studied recently by the present authors [3,4]. Their numerical results indicated that convex surfaces have lower skin friction parameters and greater boundary layer thickness than concave surfaces.

When the energy transport phenomena are considered, the equation of energy must be investigated together with the equations of motion. Among the investigators of the conventional thermal boundary layer equation, Acrivos et al.[5]; Lee and Ames[6], Thompson[7] and Chen and Radulovic[8] may be mentioned. However, there appears no solution available in the literature for the thermal boundary layer equation of non-Newtonian fluid flows including curvature effects.

In the present paper, the basic equation of energy is developed for incompressible, non-Newtonian fluid flows in a curvilinear, non-orthogonal coordinate system. Resulting equation is applied to incompressible plane flows of power-law fluids together with the equations of motion to investigate the effects of longitudinal surface curvature on thermal boundary layer flow.

2. Basic equations

Consider an inertial reference system $\overline{S}$ with the origin $\overline{O}$ and a non-inertial Cartesian reference system S' attached to a body moving in an arbitrary manner relative to the inertial reference $\overline{S}$ as shown in Fig.1. The position of the origin O' of the primed frame relative to

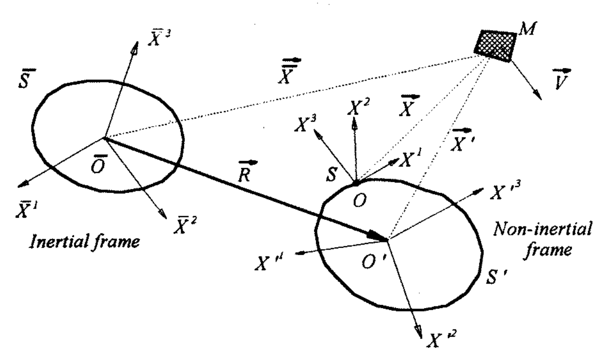


Figure 1: Inertial and non-inertial references

the barred frame is given by the position vector $\vec{R}$. Let $\vec{\omega}'(t)$ be the angular velocity of the frame S' relative to $\bar{S}$ and $\bar{O}$ be its center of rotation [9]. We may introduce also another reference system S rigidly attached to the non-inertial frame S', but non-cartesian in general. The transformation from S' to S frame is purely geometrical whereas the transformation from inertial frame $\bar{S}$ to the accelerated frame S' is a time dependent process [9].

The geometrical relation between the frame S and S' is given by

$$\vec{X} = \vec{X}(\vec{X}') \qquad \vec{X}' = \vec{X}'(\vec{X}) \tag{1a}$$

or

$$X^i = X^i(X^{j'}) \qquad X^{k'} = X^{k'}(X^l) \tag{1b}$$

where $\vec{X}$ and $\vec{X}'$ are the position vectors, X^i and X'^j are the position coordinates relative to the frames S and S', respectively.

Following relations can be obtained from Eq. (1)

$$\beta^i_{j'} = \frac{\partial X^i}{\partial X^{j'}}, \qquad \beta^{k'}_l = \frac{\partial X^{k'}}{\partial X^l}, \qquad i,j',k',l = 1,2,3 \tag{2}$$

Using the relations (2), the covariant coordinates of the metric tensor can be obtained as

$$g_{jk} = \beta^{i'}_j \cdot \beta^{i'}_k = g_{kj} = \sum_{i'=1}^{3} \beta^{i'}_j \cdot \beta^{i'}_k \tag{3}$$

and contravariant metric tensor is given by orthogonality relation as

$$g_{jk} \cdot g^{kl} = \delta^l_j \tag{4}$$

where δ^l_j represents the Kronocker tensor. Christoffel symbols can be determined from the following relations [10]

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \frac{1}{2} g^{im} \left[\frac{\partial g_{jm}}{\partial X^k} + \frac{\partial g_{km}}{\partial X^j} - \frac{\partial g_{kj}}{\partial X^m} \right], \quad i,j,k = 1,2,3 \tag{5}$$

For an incompressible, non-Newtonian power-law fluid flow, the equation of continuity, Cauchy's first law of motion and the equation of energy are written in a curvilinear, non-orthogonal coordinate system, as follows [4,9]

$$\nabla_i V^i = 0 \tag{6}$$

$$\frac{\partial V^i}{\partial t} + V^j \nabla_j V^i + 2\varepsilon^{ijk} \beta^{k'}_j g_{kl} \omega_{k'} V^l + \varepsilon^{ijk} \beta^{k'}_j \beta^{l'}_k \omega_{k'} U_{l'}$$

$$+ \beta^i_{k'} W^{k'} + \beta^i_{k'} R^{k'} = -\frac{1}{\rho} g^{ij} \frac{\partial p}{\partial X^j} + \beta^i_{k'} f^{k'} + \frac{1}{\rho} \nabla_j \sigma^{ij} \tag{7}$$

$$\rho C_p \frac{DT}{Dt} = g^{ij} \nabla_i \left[\aleph \frac{\partial T}{\partial X_j} \right] + \phi \tag{8a}$$

or

$$\rho C_p \left[\frac{\partial T}{\partial t} + V^i \frac{\partial T}{\partial X^i} \right] = g^{ij} \nabla_i \left[\aleph \frac{\partial T}{\partial X_j} \right] + \phi \tag{8b}$$

where ∇_i represents covariant derivative, p is the pressure, ρ the density of the fluid, f the resultant of the given orthonormeous force per unite mass, ε^{ijk} a permutation tensor, D/Dt the material derivative, T the temperature, $\aleph$ the thermal conductivity which is a scalar function of the thermodynamic state only (for example $\aleph = \aleph(T)$) and ϕ the dissipation function for the non-Newtonian fluid, σ^{ij} the extra stress tensor, C_p spesific heat per unit mass and

$$\begin{aligned} \vec{\omega}' \wedge \vec{X}' &= \vec{U}' \quad or \quad U_{k'} \\ \dot{\omega}' \wedge \vec{X}' &= W' \quad or \quad W^{k'} \end{aligned} \tag{9}$$

Extra stress tensor in equation (7) is given by

$$\sigma^{ij} = \mu(s) D^{ij} = \mu \frac{1}{2} \left[g^{im} \nabla_m V^j + g^{lj} \nabla_l V^i \right] \tag{10}$$

where μ represents the viscosity coefficient of non-Newtonian power-law fluid and can be expressed as follows [4,11,12].

$$\mu(s) = \mu_0 \, s^{(n-1)/2} \tag{11}$$

where n and μ_0 are constant parameters called power-law index and consistency, respectively [11,12], and

$$s = 2\, tr(\tilde{\bar{D}}^2) = 2\,(D^i_j D^j_i) \tag{12}$$

For an incompressible fluid, the dissipation function ϕ is defined by

$$\phi = 2\mu\,(D^j_i D^i_j) \tag{13a}$$

or

$$\phi = \mu_0 \, s^{(n-1)/2} \, s \tag{13b}$$

3. Example applications

3.1. Similarity solutions with the method of separation of variables

For two dimensional fluid flow, surface oriented coordinate system illustrated in Fig. 2 is conveniently used in order to investigate the flow problems such as the general elliptical cylinders, the ditch or the mound, etc. [3,4]. In Fig. 2 $(O'X'^1X'^2X'^3)$ denote the Cartesian coordinates, while $(OX^1X^2X^3)$ denote surface oriented orthogonal coordinate system.

Substituting the following non-dimensional quantities into the equations (6-8) and (10-13)

$$\bar x = \frac{x}{L},\quad \bar y = \frac{y}{L} Re^{1/(1+n)},\quad \bar k = k\,L\,Re^{-1/(1+n)}$$

$$\bar u = \frac{u}{U_\infty},\quad \bar v = \frac{v}{U_\infty} Re^{1/(1+n)},\quad \bar t = \frac{U_\infty}{L} t \tag{14}$$

$$\bar T = \frac{T-T_\infty}{T_w - T_\infty},\quad \bar p = \frac{p}{\rho U_\infty^2},\quad \bar \mu = \frac{\mu}{\mu_0},\quad \bar \aleph = \frac{\aleph}{\aleph_0}$$

$$Re = \frac{\rho}{\mu_0} U_\infty^{2-n} L^n,\ Pr = \frac{C_p U_\infty \rho L}{\aleph_0} Re^{2/(1+n)},\ E = \frac{U_\infty^2}{C_p (T_w - T_\infty)}$$

and applying the boundary layer approximation for the cases of $\omega' = 0$ and $\dot\omega' = 0$, we can obtain the following equations for two dimensional non-Newtonian fluid flow in the O, X^1, X^2 plane of the surface oriented coordinate system

$$\frac{\partial \bar u}{\partial \bar x} + \frac{\partial}{\partial \bar y}\left[\left(1+\bar k \bar y\right)\bar v\right] = 0 \tag{15}$$

$$\frac{\partial \bar u}{\partial \bar t} + \frac{\bar u}{1+\bar k\bar y}\frac{\partial \bar u}{\partial \bar x} + \frac{\bar v}{1+\bar k \bar y}\frac{\partial}{\partial \bar y}\left[\left(1+\bar k\bar y\right)\bar u\right]$$

$$= -\frac{1}{1+\bar k\bar y}\frac{\partial \bar p}{\partial \bar x} + n\left[\frac{\partial \bar u}{\partial \bar y} - \frac{\bar k}{1+\bar k\bar y}\right]^{n-1}\left[\frac{\partial^2 \bar u}{\partial \bar y^2}\right.$$

$$\left. - \frac{n-2}{n}\frac{\bar k}{1+\bar k\bar y}\left(\frac{\partial \bar u}{\partial \bar y} - \frac{\bar k}{1+\bar k\bar y}\bar u\right)\right] \tag{16a}$$

$$\frac{\bar k}{1+\bar k\bar y}\bar u^2 = \frac{\partial \bar p}{\partial \bar y} \tag{16b}$$

$$\frac{\partial \bar T}{\partial \bar t} + \frac{\bar u}{1+\bar k\bar y}\frac{\partial \bar T}{\partial \bar x} + \bar v\frac{\partial \bar T}{\partial \bar y}$$

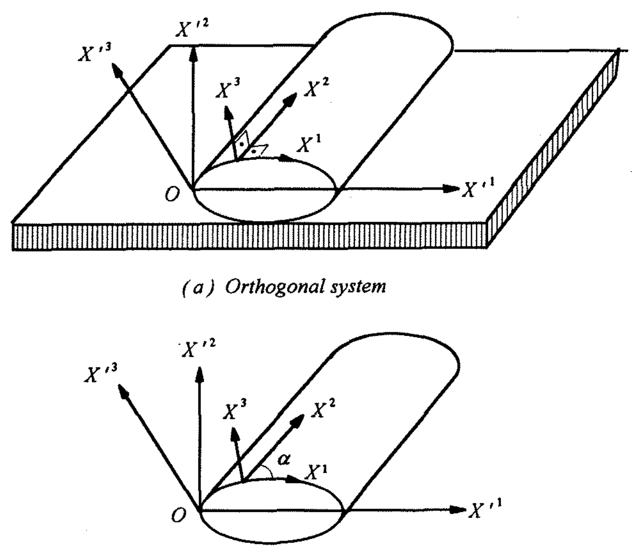
$$= -\frac{1}{1+\bar k\bar y}\cdot\frac{\partial}{\partial \bar y}\left[\left(1+\bar k\bar y\right)\bar\aleph\frac{\partial \bar T}{\partial \bar y}\right]\frac{1}{Pr} +$$

$$\frac{E}{Pr}\left|\frac{\partial \bar u}{\partial \bar y} - \frac{\bar k}{1+\bar k\bar y}\bar u\right|^{n-1}\left[\frac{\partial \bar u}{\partial \bar y} - \frac{\bar k}{1+\bar k\bar y}\bar u\right]^2 \tag{17}$$

with the following boundary conditions,

$$\begin{aligned} &\bar y = 0 \qquad \bar u = 0,\ \bar v = 0,\ \bar T = 1 \\ &\bar y = \infty \qquad \bar u = \bar U_e,\ \bar T = 0 \end{aligned} \tag{18}$$

where, L, U_∞, T_∞ are the reference length, the velocity and the temperature, respectively. T_w, Re, Pr and E represent the wall temperature and the Reynolds, Prandtl and Eckert numbers, respectively. The non-dimensional vorticity $\bar\zeta$ is given by



(a) Orthogonal system

(b) Non-orthogonal system

Figure 2: Surface oriented coordinates systems

$$\bar{\zeta} = -\frac{1}{(1+\bar{k}\,\bar{y})}\frac{\partial}{\partial \bar{y}}\left[\left(1+\bar{k}\,\bar{y}\right)\bar{u}\right] \tag{19}$$

We obtain the free stream velocity from the irrotationality condition as

$$\bar{U}_e = \frac{\bar{U}_0}{1+\bar{k}\bar{y}} \tag{20}$$

then the pressure at the edge of the velocity boundary layer may be expressed as,

$$\bar{p} = -\frac{1}{2}\frac{\bar{U}_0^2(\bar{x})}{1+\bar{k}\bar{y}} = -\frac{\bar{U}_e^2}{2} \tag{21}$$

where $\bar{U}_0$ and $\bar{U}_e$ are the potential flow velocity at the wall and at the edge of the boundary layer, respectively. $\bar{U}_0$ depends on only the variable $\bar{x}$. This function is expressed for the wedge flows by

$$\bar{U}_0 = |\bar{x}|^m \tag{22a}$$

and for the Goldstein type flows by

$$\bar{U}_0 = e^{c\bar{x}} \tag{22b}$$

where c is any real constant except zero characterizing the Goldstein flow.

Eqs. (15) and (16) were obtained by Erim [3] and Yükselen and Erim [4]. They introduced the following non-dimensional quantities into Eqs. (16)

$$\begin{aligned}
\bar{\psi} &= \left[\bar{u}(\bar{x})/\alpha_1\bar{x}^{\alpha_2}\right]f(\eta)\\
\eta &= \alpha_1\bar{y}\,\bar{x}^{\alpha_2}\\
x &= \bar{x}\\
\bar{k}\,\bar{y} &= A\,\eta\\
A\,\eta^* &= \ln(1+A\,\eta)\\
F &= F(\eta^*)\\
\bar{U}_e &= \bar{x}^m
\end{aligned} \tag{23}$$

to obtain the following generalized Falkner-Skan similarity equation, as an application of the method of separation of variables.

$$\left[F''' - \frac{2(2n-1)}{n}A\,F'' + \frac{4(n-1)}{n}A^2F'\right]\left[F''-2\,A\,F'\right]^{n-1}$$

$$+\left[F\,F'' + \beta\,(1-F'^2)\right]\cdot\left[1 + A\,\eta\right]^{2(n-1)} = 0 \tag{24}$$

with the following boundary conditions:

$$\begin{aligned}
\eta^* &= 0, & F(0) &= F'(0) = 0\\
\eta^* &= \infty, & F'(\infty) &= 1
\end{aligned} \tag{25}$$

where

$$\beta = \frac{m(n+1)}{m(2n-1)+1}$$

$$\alpha_1 = \left[\frac{\rho}{\mu_1}U_\infty^{2-n}\,L^n\,\frac{m(2n-1)+1}{n(n+1)}\right]^{1/(1+n)} \tag{26}$$

$$\alpha_2 = \frac{m(n-2)-1}{1+n}$$

Yükselen and Erim [4] solved Eq. (24) with the boundary conditions (25) for different values of the parameters n, A and β, in order to investigate the curvature effects on the velocity boundary layer. They concluded that convex surfaces have lower skin friction parameters $f''(0)$ and greater boundary layer thickness than concave surfaces. However, they did not studied the curvature effects on the thermal boundary layer.

For the thermal boundary layers, if we substitute the relations (23) and additionally $\bar{T} = \theta$ into the energy Eq. (17), disregarding the dissipation therm, the following equation can be obtained for an isothermal right angle wedge flow (β=0.5)

$$f\,\theta' + \frac{\alpha_3}{Pr}\left[(1+A\eta)\,\theta'' + A\,\theta'\right] = 0 \tag{27}$$

with the boundary conditions

$$\begin{aligned}
&\eta\to 0 \qquad f'=0,\ \ \theta=1 \quad or \quad \frac{\partial\theta}{\partial\eta}=Cons.\\
&\eta\to\infty \qquad f'=1,\ \ \theta=0
\end{aligned} \tag{28}$$

where

$$\alpha_3 = n^{-2/(1+n)}\,(2/3)^{(1-n)/(1+n)} \tag{29}$$

3.2. Group method for similarity solutions

The non-linearity in the equations of motion of non-Newtonian fluid systems limit the applicability of its similarity variables to the energy equation. Thus the classical method of separation of variables for determining similarity variables does not appear as a general procedure to indicate the possible cases giving the similarity solutions for complex flow problems. In addition, Ames [13] indicated that the method of one parameter group transformation is a simple and straightforward method to obtain similarity solutions of partial differential equations. This method was applied also successfully to the conventional boundary layer equations of non-Newtonian power-law fluids [6]. In this method, the problem of searching for similarity variables is reduced to that of solving for the invariant conditions of a system of differential equations under a certain group of continuous transformations. The conditions of absolute invariants of the subgroup consisting of transformations of independent variables enable us to give the similarity variables [6,13]. As an application of the method for the velocity and the thermal boundary layer including curvature effect of the non-Newtonian steady power-law fluid flows, the possibilities of similarity solutions will be treated below by using the one-parameter group method because of its convenience and simplicity.

The thermal conductivity $\aleph$ of a fluid can, in general, be regarded as a given function of the temperature and the pressure [2,9]. Although it has been postulated by Lee and Ames [6] that the heat conductivity varies as a function of the temperature only. They proposed the following relation

$$\aleph = \overline{T}^{r-1} \tag{30}$$

where r was a constant to be determined. For actual fluids the thermal conductivity changes slightly with the temperature and its dependence on the temperature can often be disregarded for practical purposes [2,6]. On the other hand, temperature dependence of the viscosity can be approximated by a decreasing function of the temperature [2]. At the same time the viscosity also varies slightly with the temperature. This variation can be disregarded for practical applications.

Introducing now into the Eqs. (16) and (17), the stream function given by

$$\overline{u} = \frac{\partial \overline{\psi}}{\partial \overline{y}}, \qquad \overline{v} = -\frac{1}{1+\overline{k}\overline{y}}\frac{\partial \overline{\psi}}{\partial \overline{y}} \tag{31}$$

and the one-parameter linear group defined as

$$\Gamma_1 \; : \; \begin{aligned} x^* &= a^{\alpha_1}\overline{x} \\ y^* &= a^{\alpha_2}\overline{y} \\ \psi^* &= a^{\alpha_3}\overline{\psi} \\ U_e^* &= a^{\alpha_4}\overline{U}_e \\ k^* &= a^{-\alpha_2}\overline{k} \\ T^* &= a^{s}\overline{T} \end{aligned} \tag{32}$$

and applying the constant conformaly invariant condition for them leads to the following algebraic equations

$$\alpha_1 + 2\alpha_2 - 2\alpha_3 = \alpha_1 - 2\alpha_4 = (2n+1)\alpha_2 - n\alpha_3 \tag{33a}$$

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_5 = 2\alpha_2 - r\,\alpha_5 = (2\alpha_2 - \alpha_3)(1+n) \tag{33b}$$

Because of the shear stress, the mechanical energy in the interior of the fluid is dissipated as heat which flows out to the surroundings. Consequently, a temperature gradient is created in the fluid. At moderate shear rates, as in the case of low speed flow, the temperature differences due to dissipation are of course so low that one can regard the fluid as an isothermal continuum to a close approximation.

Each of (33a) and (33b) contains two equations. However, if the dissipation is neglected (33b) is reduced to one equation. Thus we obtain the following three equations with four unknowns

$$\alpha_1 + (1-2n)\alpha_2 + (n-3)\alpha_3 = 0 \tag{34a}$$

$$\alpha_2 - \alpha_3 + \alpha_4 = 0 \tag{34b}$$

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_5 = 2\alpha_2 - r\,\alpha_5 \tag{34c}$$

This system of equations has no unique solution. This difficulty may be eliminated by choosing a particular outside flow field. For example, if wedge flow is considered, the velocity field is obtained from Equations (20) and (22a) as

$$\overline{U}_e = \frac{\overline{x}^m}{1+\overline{k}\overline{y}} \tag{35}$$

and it may be seen that

$$\alpha_4 = m\,\alpha_1 \tag{36}$$

For the remainder of the unknowns there exist two cases possible depending on the value of α_1.

i) Case $\alpha_1 \neq 0$

For $r \neq 1$ equations (34) and (36) yield the following relations

$$\frac{\alpha_2}{\alpha_1} = \frac{(n-2)m+1}{n+1} \quad (37a)$$

$$\frac{\alpha_3}{\alpha_1} = \frac{(2n-1)m+1}{n+1} \quad (37b)$$

$$\frac{\alpha_5}{\alpha_1} = \frac{(n-1)(1-3m)}{(n+1)(r-1)} \quad (37c)$$

If we suppose that x^* is the independent variable to be eliminated, absolute invariants of Γ_1 group are

$$\begin{aligned} \eta &= \bar{y}/\bar{x}^{\alpha_2/\alpha_1} \\ \bar{\psi} &= \bar{x}^{\alpha_3/\alpha_1} f(\eta) \\ \bar{T} &= \bar{x}^{\alpha_5/\alpha_1} \theta(\eta) \end{aligned} \quad (38)$$

For $r=1$, the relation (37c) is no longer valid. In this case, absolute invariants of Γ_1 group can only be calculated for $m=1/3$ as,

$$\begin{aligned} \eta &= \bar{y}/\bar{x}^{1/3} \\ \bar{\psi} &= \bar{x}^{2/3} f(\eta) \\ \bar{T} &= \bar{x}^{t} \theta(\eta) \end{aligned} \quad (39)$$

where t is an arbitrary real number [6]. This means that the boundary condition for $\bar{T}$ on the wall is any power function of $\bar{x}$, namely

$$\bar{T}_w = \bar{x}^{t} \quad (40)$$

This include a constant boundary temperature for $t=0$.

Moreover, as seen from (33) and (34), for the values

$$\begin{aligned} \alpha_2/\alpha_1 &= 1/3 \\ \alpha_3/\alpha_1 &= 2/3 \\ \alpha_5/\alpha_1 &= 2/3 \end{aligned} \quad (41)$$

there exist also a similarity solution for $m=1/3$ satisfying automatically the energy dissipation term. In this similarity solution the temperature field is expressed as

$$\bar{T} = \bar{x}^{2/3} \theta(\eta) \quad (42)$$

and the boundary condition on the wall becomes

$$\bar{T}_w = \bar{x}^{2/3} \quad (43)$$

Note that for the other values of m, there appears no similarity solution.

Substituting the relations (37) and (38) together with (33) into the equation (17) without dissipation term yields

$$\frac{\alpha_5}{\alpha_1}\theta\frac{\partial f}{\partial \eta} - \frac{\alpha_3}{\alpha_1} f\frac{\partial \theta}{\partial \eta} = \frac{1}{Pr}\frac{1}{1+A\eta}\frac{\partial}{\partial \eta}\left[(1+A\eta)\theta^{r-1}\frac{\partial \theta}{\partial \eta}\right] \quad (44)$$

with the following boundary conditions

$$\begin{aligned} &\eta \to 0 \qquad f'=0,\ \theta=1 \quad or \quad \frac{\partial \theta}{\partial \eta}=0 \\ &\eta \to \infty \qquad f'=1,\ \theta=0 \end{aligned} \quad (45)$$

If $r=1$, this equation is no longer valid. Therefore, for the right angle wedge flow $m=1/3$ we must recalculate similarity solution. Indeed, substitutions of the equations (39) and (42) into Eqs. (17) yield

$$\frac{2}{3}\frac{1}{1+A\eta}(\theta f' - f\theta) - \frac{1}{Pr}\frac{1}{1+A\eta}\frac{\partial}{\partial \eta}\left[(1+A\eta)\frac{\partial \theta}{\partial \eta}\right] - \frac{E}{Re}|f''-Af'|^{n-1}(f''-Af')^2 = 0 \quad (46)$$

with the boundary conditions (45). For the isothermal right angle flow, this equation can be written as

$$\frac{2}{3}f\theta' + \frac{1}{Pr}\left[A\theta' + (1+A\eta)\theta''\right] + \frac{E}{Re}|f''-Af'|^{n-1}(f''-Af')^2(1+A\eta) = 0 \quad (47)$$

with the boundary conditions (45). For $A=0$, without dissipation term, this equation is the same as that of Lee and Ames [6].

ii) Case $\alpha_1 = 0$

In this case another group Γ_2 named the spiral group may be defined as in [6,13]

$$\Gamma_2: \quad \begin{aligned} x^* &= \bar{x} + \ln a \\ y^* &= a^{\alpha_2}\,\bar{y} \\ \psi^* &= a^{\alpha_3}\,\bar{\psi} \\ A^* &= \bar{k} / a^{\alpha_2} \\ T^* &= a^{t}\,\bar{T} \end{aligned} \tag{48}$$

For Γ_2 group, the edge velocity may be written by the aid of the equations (20) and (22b) as

$$\bar{U}_e = \frac{e^{cx}}{1+\bar{k}\bar{y}} \tag{49}$$

where c can take any real value except zero.

Setting the relations (48) and (49) into Eqs. (16) and (17), its invariant conditions require the following equations

$$2\alpha_2 - 2\alpha_3 = -2c = (2n+1)\alpha_2 - n\alpha_3 \tag{50a}$$

$$\alpha_2 - \alpha_3 - \alpha_5 = 2\alpha_2 - r\alpha_5 \tag{50b}$$

The relation (50a) gives

$$\alpha_2 = \frac{n-2}{n+1}c, \qquad \alpha_3 = \frac{2n-1}{n+1}c \tag{51}$$

and

$$\begin{aligned} \eta &= \bar{y} / e^{\frac{n-2}{n+1}c\bar{x}} \\ \bar{\psi} &= e^{\frac{2n-1}{n+1}c\bar{x}} f(\eta) \\ \bar{k} &= A\, e^{\frac{n-2}{n+1}c\bar{x}} \end{aligned} \tag{52}$$

Setting the relation (52) into eqn. (16) for velocity boundary layer, one can obtain

$$\frac{n}{c}\left|f'' - \frac{A}{1+A\eta}f'\right|^{n-1}\left[f''' - \frac{n-2}{n}\frac{A}{1+A\eta}\left(f'' - \frac{1}{1+A\eta}f'\right)\right]$$

$$+\frac{1}{1+A\eta}\left[\frac{2n-1}{n+1}\left(f f'' + \frac{A}{1+A\eta} f f'\right) + \frac{1}{(1+A\eta)^2} - f'^2\right] = 0 \tag{53}$$

with the following boundary conditions

$$\begin{aligned} f' &= 0 \quad \text{for} \quad \eta = 0 \\ f' &= \frac{1}{1+A\eta} \quad \text{for} \quad \eta = \infty \end{aligned} \tag{54}$$

Substituting the following transformation to eqs. (53)

$$\begin{aligned} A\,\eta^* &= \ln(1+A\eta) \\ F(\eta^*) &= f(\eta) \end{aligned} \tag{55}$$

we obtain

$$\frac{n}{c}\left|F'' - 2AF'\right|^{n-1}\left[F''' - \frac{2(2n-1)}{n}AF'' + \frac{4(n-1)}{n}A^2F'\right]$$

$$+\left[\frac{2n-1}{n+1}F F'' + 1 - F'^2\right]e^{2(n-1)A\eta^*} = 0 \tag{56}$$

with the boundary conditions

$$\begin{aligned} F' &= 0 \quad \text{for} \quad \eta^* = 0 \\ F' &= 1 \quad \text{for} \quad \eta^* = \infty \end{aligned} \tag{57}$$

It is seen from (50b) that there is only a similarity solution of the energy equation for Newtonian fluid flow ($n=1$). This result was shown also by Lee and Ames [6] in the case of conventional boundary layer flow.

In connection with the similarity solutions, Thompson's [7] work will be the last example. His transformed momentum equation for flat plate is not changed, but the energy equation is limited by local similarity. We can generalize Thompson's solution so that curvature effects are included in the similarity solution. For the generalized case, the solution of the equation of motion may be obtained easily by substituting $\beta=0$ in the equation (24) with the boundary condition (25). For the solution of the energy equation (17), the absolute invariants are

$$\begin{aligned} \zeta &= \bar{y} / \bar{x}^{1/2} \\ T &= \theta_1 \\ \bar{\psi}_1 &= \bar{x}^{1/2} f_1(\zeta) \end{aligned} \tag{58}$$

and from here

$$\begin{aligned} \bar{u}_1 &= f_1'(\zeta) \\ \bar{v}_1 &= \frac{1}{2\sqrt{\bar{x}}}\frac{1}{1+A\zeta}\left[\zeta f_1' - f_1\right] \\ \zeta &= \bar{x}^{(1-n)/2(1+n)}\,\eta \\ \bar{k} &= A_1 / \bar{x}^{1/2} \end{aligned} \tag{59}$$

In the case of isothermal flow of the power-law fluid over a flat-plate, disregarding dissipation term and setting the relations (58) and (59) into (17) we can obtain the following equation

$$\frac{f_1}{1+A_1\zeta}\frac{\partial\theta_1}{\partial\zeta} + \frac{1}{Pr}\left[\frac{\partial}{\partial\zeta}\left(\theta_1^{r-1}\frac{\partial\theta_1}{\partial\zeta}\right) + \frac{A_1}{1+A_1\zeta}\theta_1^{r-1}\frac{\partial\theta_1}{\partial\zeta}\right] = 0 \tag{60}$$

with the boundary conditions

$$\theta_1(\zeta) = 0 \quad for \quad \zeta = 0 \tag{61}$$

$$\theta_1(\zeta) = 1 \quad or \quad \frac{\partial\theta_1}{\partial\zeta} = 0 \quad for \quad \zeta = \infty$$

The details of all these analysis has been given by Yükselen and Erim [14].

4. Numerical Results

Using a fourth order Runge-Kutta method modified by Gill [15], the Eqn. (24) with the boundary conditions (25) was integrated numerically by Yükselen and Erim [4]. The same method of integration was applied to solve of the Eqn (27) with the boundary conditions (28), disregarding the dissipation term.

Since the numerical method, its accuracy and precision etc. was explained widely by Yükselen and Erim [4,14] in comparison with the results of other publications, detailed calculation procedure is not given here. However, to see how the method works in order to solve the similarity equations of momentum and energy simultaneously, an example application was made on the similarity equations of momentum and energy given by Lee and Ames [6] as following

$$3n\,f''' + 2f\,(f'')^{2-n} + \left[1-(f')^2\right](f'')^{1-n} = 0 \tag{62}$$

$$\theta'' + (2/3)\,Pr\,f\,\theta' = 0 \tag{63}$$

The comparison of the momentum equation was previously carried out by Yükselen and Erim [4], and that was found to be quite satisfactory. Comparison of the energy equation is given in Fig. 3. Results of the present method shows a very good agreement with the results of Lee and Ames.

As a result of the numerical solutions of Eqn (27), the Figs. 4 to 7 are plotted for several values of the power-law index n, curvature parameter A and Prandtl number Pr.

Temperature profiles and corresponding velocity profiles are illustrated in Fig. 4 for various values of Prandtl number Pr. For the Prandtl numbers less than 5, the thermal boundary layer is thicker than the velocity boundary layer for the values of $n \geq 1$, while the thermal boundary layer for $n<1$ is influenced by the velocity boundary layer which is thicker. The slope of the temperature profile increase together with an increase in the Prandtl number.

Most important effect of the surface curvature is the modification of the temperature profile for large η. Its modification is imposed by the outer boundary condition. On the other hand, the thermal boundary layer thickness of the concave surfaces are slightly smaller than that of the convex surfaces.

If $\alpha(x)$ denotes the coefficient of heat transfer at any x point, Nusselt number can be defined as

$$Nu_x = \frac{\alpha(x)\,L}{\aleph} = -\left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{64}$$

and, from here we can write

$$Nu_x\left(\bar{x}^{1/3}\,Re_x^{1/(1+n)}\right) = \theta'(0) \tag{65}$$

This last equation indicates clearly that $\theta'(0)$ is a measure of the local, non-dimensional coefficient of heat transfer from the wall by the conduction. Consequently, it is useful and sufficient to know the effect of the curvature, power-law index and Prandtl number on $\theta'(0)$ instead on the coefficient of heat transfer known as Nusselt number Nu_X.

For this purpose, the curves $\theta'(0)$ versus n for various values of the Prandtl number and the curvature parameter, and the curves $\theta'(0)$ versus Pr for various

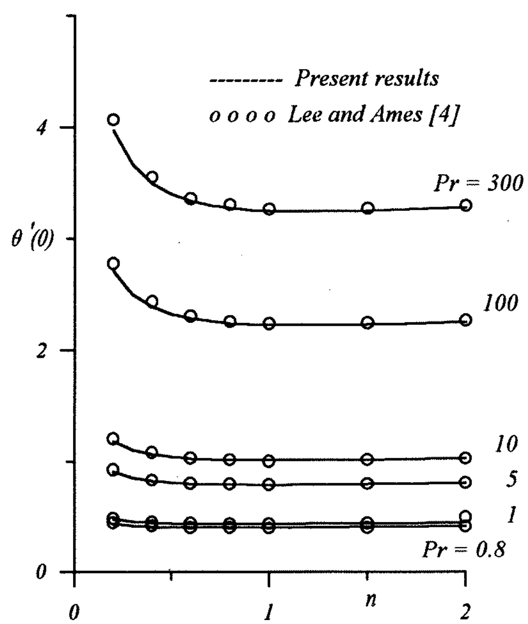
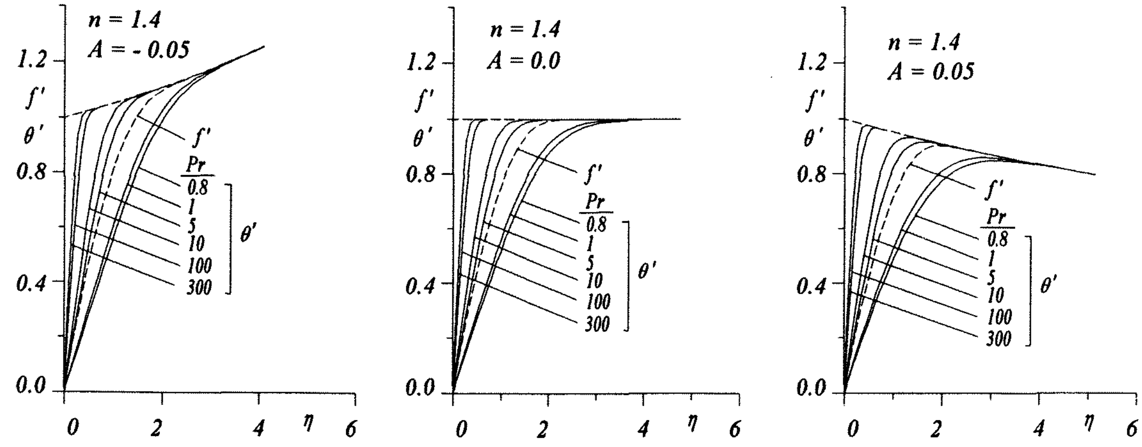
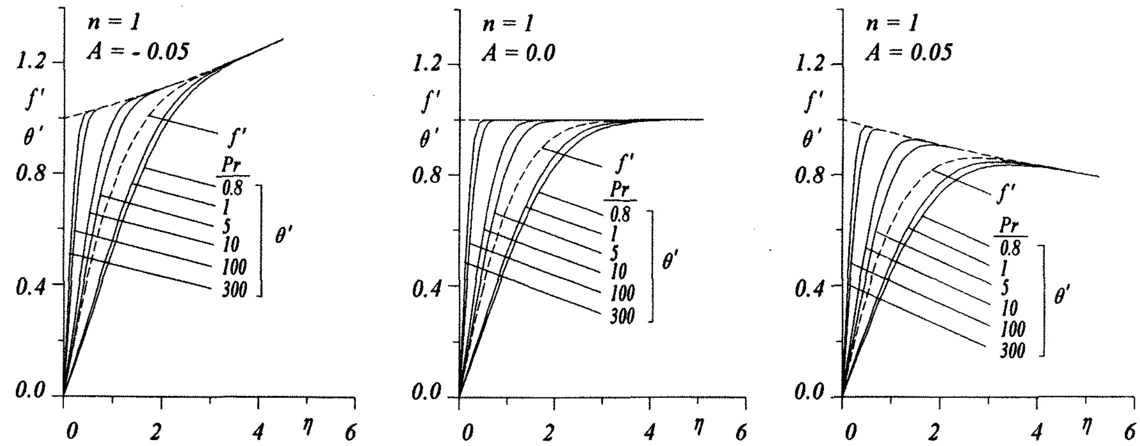


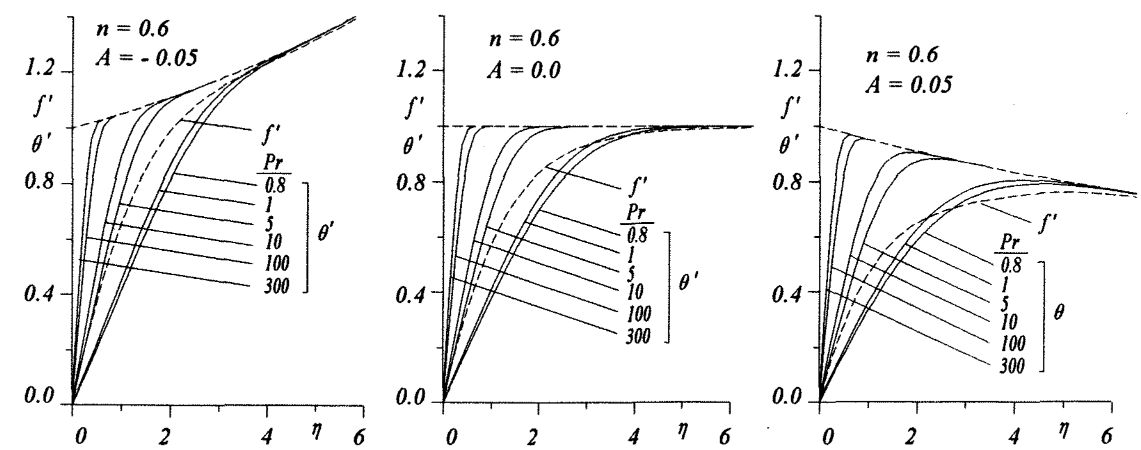
Figure 3: Comparisons for right angle wedge flow



a) Dilatant fluids (n>1)



b) Newtonian fluids (n=1)



c) Pseudoplastic fluids (n<1)

Figure 4: Curvature effects on temperature profiles

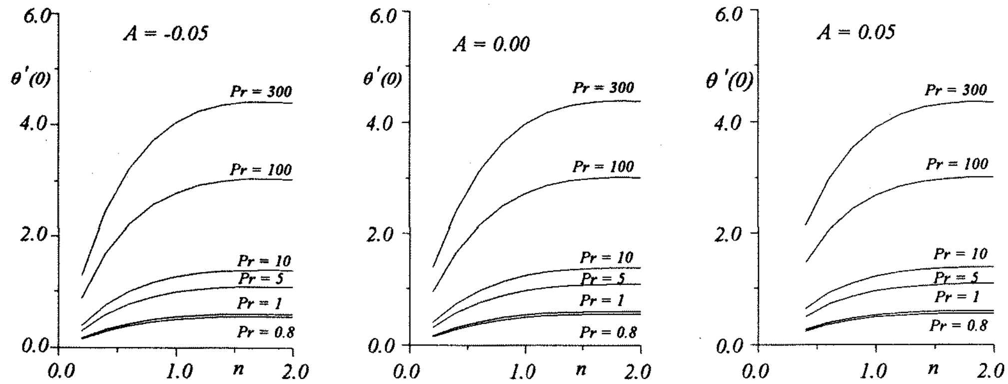


Figure 5 : Effect of power-law index on heat transfer for various values of Prandtl numbers and curvatures

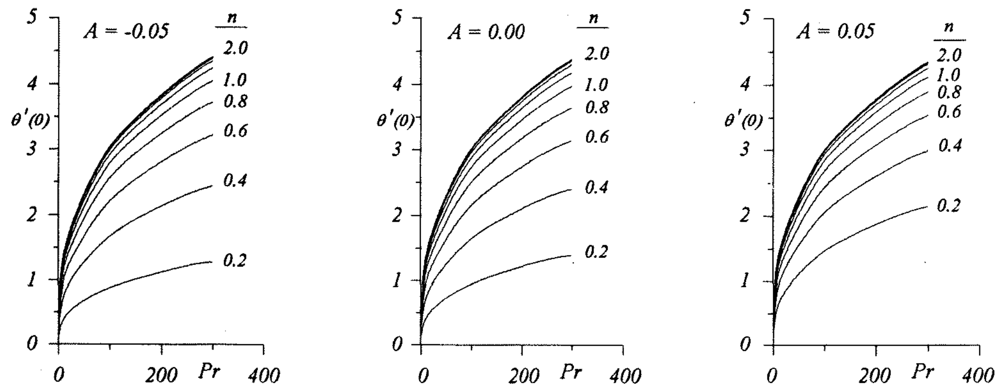


Figure 6 : Effect of Prandtl number on heat transfer for various values of power-law index and curvatures

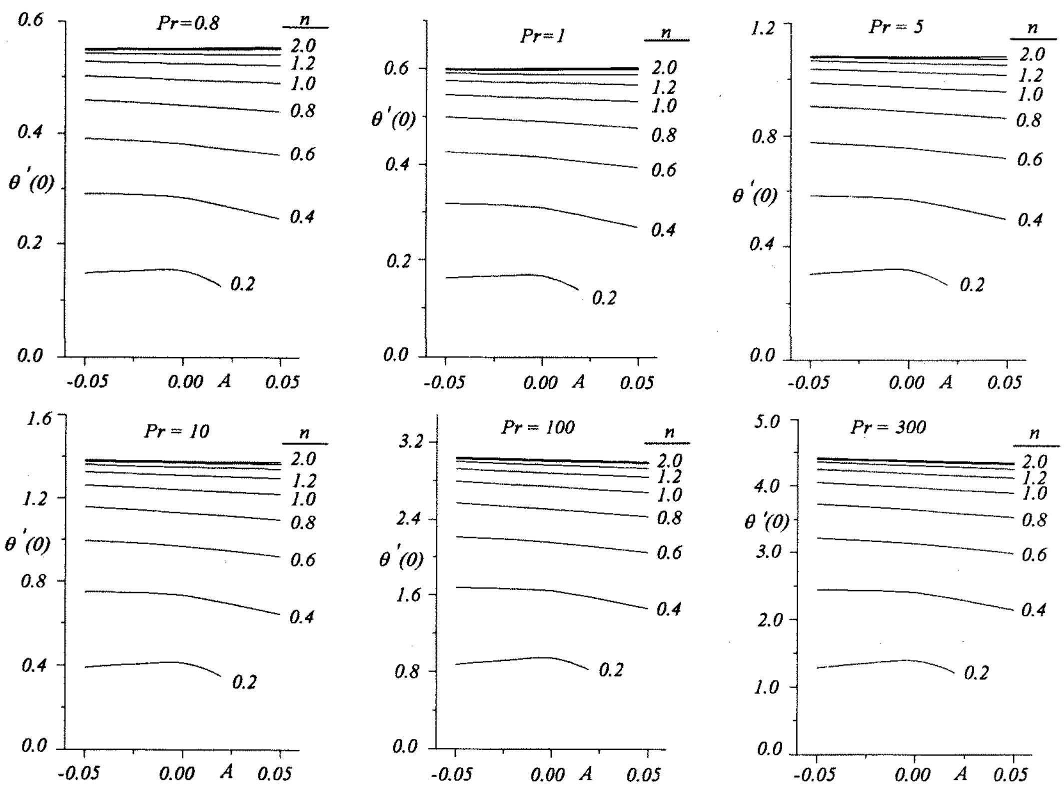


Figure 7 : Effect of curvature on heat transfer for various values of power-law index and Prandtl numbers

values of the power-law index, for different values of the curvature parameter are presented in Figures 5 and 6, respectively. They show that the influence of Prandtl number on heat transfer for pseudoplastic fluids $(n<1)$ is more severe than that of dilatant fluids $(n>1)$. Furthermore, for the dilatant fluids, the curves $\theta'(0)$ versus n for a fixed Pr varies so slowly that they can be considered as constant values.

Figures 7 shows that the heat transfer decrease linearly while curvature parameter A increases algebraically and the effect of the curvature on heat transfer is small. In addition, the quantities of heat transfer for convex surfaces are greater than that of concave surfaces. Furthermore, the curvature effect on heat transfer for pseudoplastic fluids $(n<1)$ is more severe than that for dilatant fluid $(n>1)$.

5. Conclusions

In this paper, the energy equation for non-Newtonian power-law fluids is obtained in a non-orthogonal accelerated curved coordinates system. The resulting equations consist of a generalized form of the equations due to Roberts and Grundman [9], for the non-Newtonian constant density fluids.

As an application in two-dimensional case, the possibility of similarity solutions for laminar velocity and thermal boundary layer equations including the surface curvature effects were investigated by using transformation group method for non-Newtonian power-law fluids. This investigation was made out in the cases of the wedge flow, Goldstein type flows and non-constant heat conductivity flow. It has been shown that the similarity solution of the energy equation exists only for the right angle wedge geometry. In the case of flat plate the solution is limited by local similarity. For Goldstein type of flows, a similarity solution was obtained for the equation of motion while similarity solution for energy equation existed only for Newtonian fluids.

Numerical solutions for right angle wedge flow have been carried out. Its results have shown that the temperature profile shape is modified by the curvature effects at the outer edge of the boundary layer and the thermal boundary layer thickness of the concave surface is smaller than that of the convex surface. The heat transfer decreases linearly while curvature parameter A is increasing algebraically and the effect of curvature on heat transfer is small. But curvature effect on heat transfer for pseudoplastic fluid $(n<1)$ is more severe than that for dilatant fluid $(n>1)$

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