AIRCRAFT NONLINEAR MODEL IN MULTIVARIABLE POLYNOMIAL FORM FOR STABILITY AND CONTROL ANALYSIS

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Abstract

The paper proposes a coherent methodology for the analysis of the aircraft dynamic behavior in complex maneuvers. The motion of the aircraft is described by a set of nonlinear differential equations. Apart from the previous models, the present one has only polynomial nonlinearities (by the use of quaternions as rotation parameters and of the polynomial approximation for the forces and moments acting on aircraft). On the base on this model, the aircraft maneuvers, with constant controls, are analyzed. For the stationary motion an analytical solution is found using classical series and perturbation procedures. In the process of developing the stationary solution a method, which allows the finding of all solutions of an algebraic nonlinear system of equations is adapted. The stability of the stationary solution is analyzed within the limits of Leapunov method. The eigenvalues of a linear system of differential equations with periodic coefficients that result in this stage are evaluated by asymptotic expansion up to the point that a clear statement on the stability can be made. Finally a method for testing the belonging of the initial conditions to the domain of attraction of the stationary solutions is described

Introduction

Aircraft dynamic behavior was of a prime interest from the beginning of aviation. In the past the possibilities to analyze an aircraft and its motions were reduced due to the lack of experimental data and mathematical methods disposable for analyses. The accumulation of pertinent experimental data has permitted a complex modeling of aircraft motions but they lacked analysis methods. The complex maneuvers proved from the beginning the highly nonlinear aspect of aircraft motions. The first analysis considered that a highly nonlinear maneuver is nothing but a perturbed motion of a linear one. A first investigation in the line of a mathematically sound analysis of a maneuvering aircraft was performed in[5] by Hacker and Oprisiu. Lately a lot of papers concerning this problem were performed[7-9]. One of the difficulties that have resulted was the working with the nonlinear aspects of the mathematical of the aircraft motion (singularities arising from attitude angles representation, trigonometric nonlinearities, etc.). The subsequent paper intends to elude these problems submitting a mathematical model of the aircraft motions that considers some new results from the qualitative theory of differential equations and facilities of symbolic processors.

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The mathematical model of the aircraft motion

The motion of an aircraft is described in the most general form by a system of differential equations. On the basis of the Newton's laws of dynamics and with a minimum of simplifying hypothesis[1]:

- -the aircraft is moving in an inertial frame of reference fixed on a plane earth
- -the atmosphere is at rest relative to the earth
- -the aircraft is a rigid body with no rotating parts

the mathematical model of a maneuvering aircraft is the following (the motion around its center of mass):

$$\begin{aligned}
\dot{u} &= \frac{\mathcal{X}_a}{m} - qw + rv + \mathcal{X}_g \\
\dot{v} &= \frac{\mathcal{Y}_a}{m} - rv + pw + \mathcal{Y}_g \\
\dot{w} &= \frac{\mathcal{Z}_a}{m} - pv + qu + \mathcal{Z}_g \\
\dot{p} &= \frac{\mathcal{L}}{A} - qr\frac{C-B}{A} \\
\dot{q} &= \frac{\mathcal{M}}{B} - rp\frac{A-C}{B} \\
\dot{r} &= \frac{\mathcal{N}}{A} + pq\frac{A-B}{A}
\end{aligned} \tag{1}$$

where:

V	- is the velocity of aircraft
u, v, w	- are the components of **V**
p, q, r	- are the components of the rotation velocity
$\mathcal{X}_a, \mathcal{Y}_a, \mathcal{Z}_a$	- are the components of aerodynamic forces and thrust
$\mathcal{L}, \mathcal{M}, \mathcal{N}$	- are the components of aerodynamic moments and of the thrust
A, B, C	- are the moments of inertia around the principal axis of inertia of the aircraft

As it can be easily seen the equations (1) have a polynomial form with constant coefficients if the forces and moments have one. The aerodynamic forces and moments, at least in the form used in flight mechanics, can be very well approximated in polynomial form by Cebishev polynomials, spline functions, etc. The components of the weight may be introduced in a polynomial form if we chose a special parametrisation of the rotation group.

To complete the description of the maneuvering aircraft we have to add the so-called cinematic equations of motion. These equations have a form that depends of the parametrisation of the rotation group.

The most common rotation group used is that of Eulerian angles. This rotation group has the advantage of having a very clear physical interpretation that is amenable to engineering insight. The disadvantages of this rotation group are:

- the existence of singularities in certain orientation
- the nonlinearities are trigonometric in nature

To avoid the existence of singular points and to deal only with polynomial forms we will introduce a rotation group with 4 dimension the so called "quaternion method" (first used by Lord Hamilton[(2)]).

To facilitate the understanding of the physical meaning of this rotating group we will remind the Euler theorem that states the following:

Two reference systems with common origin (S_0 and S) may be superposed by a single rotation around a unique given axis.

The physical interpretation of the quaternions consists in the fact that they represent the direction cosines of the axis (α, β, γ) and the rotation angle around this axis(ω).

To simplify the calculations the rotation group we use a slightly modified form of these parameters:

It is easily found that one of the parameters is redundant ($\xi^2+\eta^2+\zeta\zeta^2+\eta^2+\zeta^2+\chi^2=1$). Details of this derivation of the rotation group may be found in reference [(3)].

The cinematic equations based on the quaternion method are:

$$\begin{vmatrix}\dot\xi\\ \dot\eta\\ \dot\zeta\\ \dot\chi\end{vmatrix}=\frac{1}{2}\begin{vmatrix}0 & r & -q & p\\ -r & 0 & p & q\\ q & -p & 0 & r\\ -p & -q & -r & 0\end{vmatrix}\begin{vmatrix}\xi\\ \eta\\ \zeta\\ \chi\end{vmatrix} \qquad (3)$$

In this formalism of the rotation group the component of gravity in force equations becomes:

$$\begin{aligned}\mathcal{X}_g &= 2gm(\xi\zeta-\eta\chi)\\ \mathcal{Y}_g &= 2gm(\eta\zeta+\xi\chi)\\ \mathcal{Z}_g &= mg(-\xi^2-\eta^2+\zeta^2+\chi^2)\end{aligned} \qquad (4)$$

Having in mind that the dependence of the aerodynamic forces and moments is approximated in a polynomial form the mathematical model of a maneuvering aircraft may be written:

$$\dot x = P(x,u) \qquad (5)$$

where:

$$\mathbf{x}^T = (u, v, w, p, q, r, \xi, \eta, \zeta, \chi)$$
$$\mathbf{u}^T = (\boldsymbol{\delta_a}, \boldsymbol{\delta_e}, \boldsymbol{\delta_r}) \qquad (6)$$

P (x, u) is a vector whose components are polynomials with the structure:

$$P_i = \sum_{j=1}^{j=10\prod_{i=1}^{i=n} k_i} c_j x_1^{p_1} x_2^{p_2} \dots x_{10}^{p_{10}} u_1^{p_{11}} u_2^{p_{12}} u_3^{p_{13}}$$
$$(p_1 = 1, k_1; p_2 = 1, k_2 \dots p_{13} = 1, k_{13}) \qquad (7)$$

This model is used in this form to simulate the motion of an aircraft according to a given law of the controls $\mathbf{u}^*=\mathbf{u}(t)$.

Another use of this system is for stability calculations, especially when the controls are constant.

To perform that task we need to:

-find the stationary solution corresponding to that control $\mathbf{u}^*$

-develop the variation (perturbed) system attached to (5)

-analyze the stability

This is a formidable task and has a lot of difficulties especially when the aerodynamics forces and moments are nonlinear. Apart from the very special cases, the fulfillment of the analysis requires a large set of simplifying hypothesis and thus reduces the reliability of predictions.

In order to have some previously available result to compare we will make some simplifications (these are not required by the method).

The most studied nonlinear model of the motion of an aircraft is that proposed by Rhoads and Schuler in reference [(4)] and analyzed in different modes by references [(5,..9)] and others.

The hypotheses taking into account are:

- u -is partially controlled, that is constant, and the first equation is neglected ;
- u -is great against v and w and thus:
- $u \cong V_0$, $\alpha \cong w / V_0$, $\beta \cong v / V_0$ (8)
- the aerodynamic forces and moments have the following polynomial approximation:

$$y_a = y_\beta\beta + y_p p + y_r r + y_{p\alpha} p\alpha + y_{\delta r}\delta_r$$

$$z_a = z_\alpha\alpha + z_q q + z_{\delta e}\delta_e \tag{9}$$

$$l_a = l_\beta\beta + l_p p + l_q q + l_r r + l_{\beta\alpha}\beta\alpha + l_{r\alpha} r\alpha + l_{\delta_a}\delta_a + l_{\delta r}\delta_r$$

$$m_a = m_\alpha\alpha + m_p p + m_q q + m_r r + m_{\dot\alpha}\dot\alpha + m_{\delta e}\delta_e$$

$$n_a = n_\beta\beta + n_p p + n_q q + n_r r + n_{p\alpha} p\alpha + n_{\delta_a}\delta_a + n_{\delta_r}\delta_r$$

. It is noted:

$$x_a = \mathcal{X}_a / mV_0, \quad y_a = \mathcal{Y} / mV_0, \quad z_a = \mathcal{Z}_a / mV_0,$$

$$l_a = \mathcal{L}_a / A, \quad m_a = \mathcal{M}_a / B, \quad n_a = \mathcal{N}_a / C,$$

$$i_1 = (C - B) / A, \quad i_2 = (A - C) / B, \quad i_3 = (B - A) / C,$$

$$\varepsilon = g / V_0 \tag{10}$$

The original model (1) and (3) will become, after some algebra:

$$\dot\beta = y_\beta\beta + y_p p + (y_r - 1)r + (y_{p\alpha} + 1)p\alpha + y_{\delta_r}\delta_r + 2\varepsilon(\eta\zeta + \xi\chi)$$

$$\dot\alpha = z_\alpha\alpha + (1 + z_q)q - p\beta + z_{\delta_e}\delta_e + \varepsilon(-\xi^2 - \eta^2 + \zeta^2 + \chi^2)$$

$$\dot p = l_\beta\beta + l_p p + l_q q + l_r r + l_{\beta\alpha}\beta\alpha + l_{p\alpha} p\alpha + l_{r\alpha} r\alpha - i_1 qr + l_{\delta_a}\delta_a + l_{\delta_r}\delta_r$$

$$\dot q = (m_\alpha + m_{\dot\alpha} z_\alpha)\alpha + m_p p + (m_q + m_{\dot\alpha} + m_{\dot\alpha} z_q)q + m_r r - i_2 pr - \varepsilon m_{\dot\alpha}(-\xi^2 - \eta^2 + \zeta^2 + \chi^2) + m_{\delta_e}\delta_e$$

$$\dot\xi = \frac{1}{2}(p\chi - q\zeta + r\eta) \tag{12}$$

$$\dot\eta = \frac{1}{2}(p\zeta + q\chi - r\xi)$$

$$\dot\zeta = \frac{1}{2}(-p\eta + q\xi + r\chi)$$

$$\dot\chi = \frac{1}{2}(-p\xi - q\eta - r\xi)$$

We note that the systems (12) have the same structure as (5) that is $\dot x = P(x, u)$. For reason of further development we note that there is a small parameter ε who separates two groups of equations . This is emphasized by writing equations (12) in the following form:

$$\dot y_i = a_i Y + Y^T N_i Y + \varepsilon Z^T M_i Z + C_i u \quad (i = 1,5)$$

$$\dot z_j = Y^T Q_j Z \quad (i = 1,4) \tag{13}$$

where:

$$Y^T = (\beta, \alpha, p, q, r)$$

$$Z^T = (\xi, \eta, \zeta, \chi) \tag{14}$$

$$u^T = (\delta_a, \delta_e, \delta_r)$$

The signification of the other notation is given in the Appendix .

Response of the aircraft to a constant control

The response of the aircraft to a control $\mathbf{u}^*(t)$ is the integral of the system (13). If this control is constant $\mathbf{u}^*(t) = \mathbf{u_0}$ then the stationary solution may be found in a closed form.

The system of equations (13) is formed by two subsystems closed by a small parameter:

$$\dot Y = P_{10}(Y, u) + \varepsilon P_{11}(Z)$$

$$\dot Z = P_{20}(Y, Z) \tag{15}$$

This structure of the system (13) suggests a solution of the following form:

$$Y = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \varepsilon^3 Y_3 + \ldots + \varepsilon^n Y_n + \ldots$$

$$Z = Z_0 + \varepsilon Z_1 + \varepsilon^2 Z_2 + \varepsilon^3 Z_3 + \ldots + \varepsilon^n Z_n + \ldots \tag{16}$$

According to these solutions the system (14) splits into a set of subsystems, grouped upon the power of the small parameter ε

Thus for ε=0:

$$\dot y_{i0} = a_i Y_0 + Y_0^T N_i Y_0 + C_i u \quad (i = 1,5) \tag{17.1}$$

$$\dot z_{j0} = Y_0^T Q_j Z_0 \quad (j = 1,4)$$

- for the first approximation ε:

$$\dot y_{i1} = a_i Y_1 + 2Y_0^T N_i Y_1 + Z_0^T M_i Z_0 \tag{17.2}$$

$$\dot z_{j1} = Y_0^T Q_j Z_1 + Y_1^T Q_j Z_0$$

-for the second approximation

$$\dot y_{i2} = a_i Y_2 + 2Y_0^T N_i Y_2 + Y_1^T N_i Y_1 + 2Z_0^T M_i Z_1$$

$$\dot z_{j2} = Y_0^T Q_j Z_2 + Y_1^T Q_j Z_1 + Y_2^T Q_j Z_0 \tag{17.3}$$

and so on.

Thus the integration of the system (13) has been replaced with the solving of the equations (17). Under this form the task is greatly simplified because:

- -apart from (17.1) all the equations form linear non homogenous systems;
- -the homogenous parts have constant coefficients and are identical for all the systems;
- -the non homogenous parts are time functions obtained recurrently.

As an observation, we underline that the system (17.1) has been the common used model for the study of maneuvering aircraft with constant controls[5...9]. In our procedure this system is a first step into the process of finding the solutions and not a consequence of neglecting gravitational terms.

The first group of equations (17.1) is split at its turn in subsystems that can be solved separately. The stationary solutions presume that $\dot{Y}_0$ and $\dot{Z}_0$ are null and we may find them by solving the algebraic equations:

$$\begin{aligned} &a_i Y_0 + Y_0^T N_i Y_0 + C_i u = 0 \qquad (i = 1{,}5) \qquad (18)\\ &Y_o^T Q_j Z_0 = 0 \qquad (j = 1{,}4) \end{aligned}$$

The solving of system (18) was tried in a lot of papers[5...9] and has been demonstrated that there are a variable number of solutions (like all nonlinear algebraic systems) as a function of the value of the constant control u_0. A general method for solving system (18) is given in the Appendix. This method provides the finding of all the solutions of (18). It was also observed the existence of all phenomena of the nonlinear dynamic systems: bifurcation, jumps, etc. We will note this solution Y_{0s}.

With these results the system (18) becomes:

$$\dot{z}_{j0} = Y_{0s}^T Q_j Z_0 \qquad (j = 1{,}4) \qquad (19)$$

and because Y_{0s} is constant, (19) is a system of linear equations with constant coefficients:

$$\dot{Z}_0 = Q Z_0 \qquad (20)$$

where:

$$Q = \frac{1}{2}\begin{vmatrix} 0 & r_{0s} & -q_{0s} & p_{0s} \\ -r_{0s} & 0 & p_{0s} & q_{0s} \\ q_{0s} & -p_{0s} & 0 & r_{0s} \\ -p_{0s} & -r_{0s} & -q_{0s} & 0 \end{vmatrix} \qquad (21)$$

Matrix Q has double pure imaginary eigenvalues $\lambda_{1,4} = \pm\sqrt{-(p_{0s}^2 + q_{0s}^2 + r_{0s}^2)} = \pm i\omega$ and the solution of (19) is periodic $Z_{0s}(\omega t)$:

where:

$$Z_{0s}(\omega t) = \begin{vmatrix} \xi_0 & \dfrac{r_{0s}\eta_0 - q_{0s}\zeta_0 + p_{0s}\chi_0}{2\omega} \\ \eta_0 & \dfrac{-r_{0s}\xi_0 + p_{0s}\zeta_0 + q_{0s}\chi_0}{2\omega} \\ \zeta_0 & \dfrac{q_{0s}\xi_0 - p_{0s}\eta_0 + r_{0s}\chi_0}{2\omega} \\ \chi_0 & \dfrac{-p_{0s}\xi_0 - q_{0s}\eta_0 - r_{0s}\zeta_0}{2\omega} \end{vmatrix} \begin{vmatrix} \cos(\omega t) \\ \sin(\omega t) \end{vmatrix} \qquad (22)$$

Consequently the solution of (17.1) is

$$\begin{aligned} &Y_0 = Y_{0s} = \text{constant}\\ &Z_0 = Z_{0s} = Z_{01}\cos(\omega t) + Z_{02}\sin(\omega t) \qquad (23) \end{aligned}$$

Taking into account (23) and the system (17), after some algebraic manipulation the stationary solution of the system (13) becomes:

$$Y_s = Y_{0s} + \sum_{i=1}^{i=\infty} \varepsilon^i \left\{ Y_{i0} + \sum_{j=1}^{i} \left[Y_{i,(2j-1)} \cos(2j\omega t) + Y_{i,2j} \sin(2j\omega t) \right] \right\} \qquad (24)$$

$$Z_s = Z_{0s} + \sum_{i=1}^{i=\infty} \varepsilon^i \left\{ Z_{i0} + \sum_{j=1}^{i+1} \left[Z_{i,(2j-1)} \cos[(2j-1)\omega t] + Z_{i,(2j)} \sin[(2j-1)\omega t] \right] \right\}$$

Figures 1 and 2 shows the results of the numerical integration of the system (13) with a Runge-Kutta method in comparison with the results obtained with formulae (24) (the solid line is the numerical integration and the dotted lines are the first respectively the second approximation in ε

The stability of the stationary solutions

The analysis of the stability of a given motion, on the line of Leapunov methods requires the equations of the perturbed motion around that stationary solution.

Observing that the real solution of system (13) is:

$$\begin{aligned} y_i &= y_{si} + \Delta y_i \\ z_i &= z_{si} + \Delta z_i \end{aligned} \qquad (25)$$

and taking into account the stationary solutions (24) the equations of the perturbed motion attached to (13) may be written (after some algebra) as:

$$\begin{aligned} \Delta\dot{y}_i &= a_i\Delta Y + 2Y_0N_i\Delta Y + \Delta YN_i\Delta y + \varepsilon(2Y_1N_i\Delta Y \\ &\quad + 2Z_0M_i\Delta Z) + \varepsilon^2(2Y_2N_i\Delta Y + 2Z_1M_i\Delta Z) + \ldots \\ \Delta\dot{z}_i &= Y_0Q_i\Delta Z + Z_0Q_i\Delta Y + \Delta YQ_i\Delta Z + \varepsilon(Y_1Q_i\Delta Z \\ &\quad + Z_1Q_i\Delta Z) + \varepsilon^2(Y_2Q_i\Delta Z + Z_2Q_i\Delta Y) + \ldots \end{aligned} \qquad (26)$$

To develop a stability analysis with the Leapunov's first method we need the linear approximation of (26) which is:

$$\begin{aligned} \Delta\dot{y}_i &= (a_i + 2Y_0N_i)\Delta Y + 2\varepsilon(Y_1 + \varepsilon Y_2 + \varepsilon^2 Y_3 + \ldots \\ &\quad \ldots)N_i\Delta Y + 2\varepsilon(Z_0 + \varepsilon Z_1 + \varepsilon^2 Z_2 + \ldots)M_i\Delta Z \\ \Delta\dot{z}_i &= Y_0Q\Delta Z + Z_0Q\Delta Y + \varepsilon(Y_1 + \varepsilon Y_2 + \varepsilon^2 Y_3 \\ &\quad + \ldots)Q_i\Delta Z + \varepsilon(Z_1 + \varepsilon Z_2 + \varepsilon^2 Z_3 + \ldots)Q_i\Delta Y \end{aligned} \qquad (27)$$

or in a condensed form::

$$\Delta\dot{X} = P(t,\varepsilon)\Delta X \qquad (28)$$

where

$$\Delta X^T = (\Delta Y^T, \Delta Z^T) = (\Delta\beta, \Delta\alpha, \Delta p, \Delta q, \Delta r, \Delta\xi, \Delta\eta, \Delta\zeta, \Delta\chi)$$

and:

$$P(t,\varepsilon)=\begin{vmatrix} y_\beta & (1+y_{p\alpha})p_s & y_p+(1+y_{p\alpha})\alpha_s & 0 & y_r-1 & 2\varepsilon\chi_s & 2\varepsilon\zeta_s & 2\varepsilon\eta_s & 2\varepsilon\xi_s \\ -p_s & z_\alpha & -\beta_s & 1+z_q & 0 & -4\varepsilon\xi_s & -4\varepsilon\eta_s & 0 & 0 \\ l_\beta+l_{\beta\alpha}\alpha_s & l_{r\alpha}r_s+l_{\beta\alpha}\beta_s+l_{p\alpha}p_s & l_p+l_{p\alpha}\alpha_s & l_q-i_1r_s & l_r-i_1q_s+l_{r\alpha}\alpha_s & 0 & 0 & 0 & 0 \\ -m_{\dot\alpha}p_s & m_\alpha+m_{\dot\alpha}z_\alpha & m_p-i_2r_s-m_{\dot\alpha}\beta_s & m_q+m_{\dot\alpha}+m_{\dot\alpha}z_q & m_r-i_2p_s & -4m_{\dot\alpha}\xi_s & -4m_{\dot\alpha}\eta_s & 0 & 0 \\ n_\beta & n_{p\alpha}p_s & n_p-i_3q_s+n_{p\alpha}\alpha_s & n_q-i_3p_s & n_r & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\chi_s}{2} & -\frac{\zeta_s}{2} & \frac{\eta_s}{2} & 0 & \frac{r_s}{2} & -\frac{q_s}{2} & \frac{p_s}{2} \\ 0 & 0 & \frac{\zeta_s}{2} & \frac{\chi_s}{2} & -\frac{\xi_s}{2} & -\frac{r_s}{2} & 0 & \frac{p_s}{2} & \frac{q_s}{2} \\ 0 & 0 & -\frac{\eta_s}{2} & \frac{\xi_s}{2} & \frac{\chi_s}{2} & \frac{q_s}{2} & -\frac{p_s}{2} & 0 & \frac{q_s}{2} \\ 0 & 0 & -\frac{\xi_s}{2} & -\frac{\eta_s}{2} & \frac{\zeta_s}{2} & -\frac{p_s}{2} & -\frac{q_s}{2} & -\frac{r_s}{2} & 0 \end{vmatrix} \quad (29)$$

or, taking into account the form of $\mathbf{X}_s$ from (24)

$$P(t,\varepsilon)=P_0(t)+\sum_{i=1}^{i=\infty}\varepsilon^i P_i(t)=P_{00}+P_{01}\cos(\omega t)$$

$$+P_{02}(\sin(\omega t)+\sum_{i=1}^{i=\infty}\varepsilon^i\left[P_{i0}+\sum_{j=1}^{j=2i+1}(P_{i(2j-1)0}\cos(j\omega t)\right] \quad (30)$$

Note that matrix $P(t,\varepsilon)$ is periodic with a period $T=2\pi/\omega$. The analysis of the system (28) is completed upon the same method described in[6].

The fundamental matrix of the system (28) is searched for in the form $C(t,\varepsilon)$ with $C_0(0,\varepsilon)=I$. This matrix (having a small parameter ε) may be expanded in a series:

$$C(t,\varepsilon)=C_0(t)+\sum_{i=1}^{i=\infty}\varepsilon^i C_i(t) \quad (31)$$

with : $C_0(t)=I$ and $C_i(0)=0$ $(i=1,\ )$.

Solving (28) with the fundamental matrix $C(t,\varepsilon)$ and delimiting the terms after the powers of ε it is obtained the following system:

$$\dot C_0(t)=P_0(t)C_0(t)$$

$$\dot C_1(t)=P_0(t)C_1(t)+P_1(t)C_0(t) \quad (32)$$

$$\dot C_2(t)=P_0(t)C_2(t)+P_1(t)C_1(t)+P_2(t)C_0(t)$$

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The stability of the solution of the system (28) is appreciated in accordance with the eigenvalues of the matrix $C(T,\varepsilon)$:

$$\det|\lambda I-C(T,\varepsilon)|=0 \quad (33)$$

Changing $C(T,\varepsilon)$ with the approximations given by (32) a series of evaluation of the eigenvalues is obtained. The calculus is continued up to the point where we have a clear conclusion of the stability properties (that is no unitary eigenvalues).

In our example for the first approximation $\varepsilon=0$

$$\dot C_0(t)=P_0(t)C_0(t) \quad (34)$$

Note that $P_0(t)$ may be written in a block of matrix manner:

$$P_0(t)=\begin{vmatrix} A_{11} & 0 \\ A_{21}(t) & A_{22}\end{vmatrix} \quad (35)$$

with A_{11} and A_{22} constant matrix and A_{21} time dependent matrix. The solution of (34) is:

$$C_0(t)=\begin{vmatrix} e^{A_{11}t} & 0 \\ e^{A_{22}t}\int_0^t e^{-A_{22}\tau}A_{21}(\tau)e^{A_{11}\tau}d\tau & e^{A_{22}t}\end{vmatrix} \quad (36)$$

and its eigenvalues at the time $t=T$ are given by:

$$\det\left|\lambda I_5-e^{A_{11}T}\right|=0$$

$$\det\left|\lambda I_4-e^{A_{22}T}\right|=0 \quad (37)$$

Since A_{22} has pure imaginary eigenvalues the solution of the first approximation is inconclusive.

We will continue with the second approximation ε.

$$\dot C_1(t)=P_0(t)C_1(t)+\varepsilon P_1(t)C_1(t) \quad (38)$$

In this case the calculations are not so simply like for $\varepsilon=0$ but after some tedious operation (analytical and finally numerical one) we obtain some eigenvalues outside the unitary circle. Now we can conclude that the motion of an aircraft maneuvering with constant controls is always unstable (though its instability is mild and applies especially to attitudes).

Though the result is generally known the importance of the methods lays in the possibility to elude the large amount of numerical calculation required by the stability analysis of an aircraft. Also the method permits the underlining of the contribution of the different aerodynamic and mass characteristics of the aircraft to the stability parameters.

Another advantage of the method consists in the possibility of having a methodical way to the design of the stability augmentation systems

Domain of attraction of the stationary stable solutions

It was demonstrated in previous works[5..9] that the analysis of nonlinear models of the motion results in multiple stable solutions with obvious consequences (bifurcation, jumps, etc.)

Except from Hacker[8], nobody emphasized the importance of the estimation of the domain of attraction. There is also no general method for a valuable estimation. The present model of aircraft motion that includes only polynomial forms permits the application of some recent results in the domain of differential equations[10,13].

The analysis is based on the following theorem demonstrated in[12]:

If an analytic function $f: R^n \to R^n$ has the properties:

- $f(0)=0$;
- *the real parts of the eigenvalues of the matrix* $\left|\frac{\partial f_i}{\partial x_j}(0)\right|$ *are negative;*

then the domain of attraction of the null solution of the system :

$$\dot{x} = f(x) \tag{39}$$

is the same with the domain in which the function V(x) is analytic. The function V(x) is the solution of the equations:

$$\langle \mathrm{grad}V(x) | f(x) \rangle = -\|x\|^2$$
$$V(0) = 0 \tag{40}$$

The solution of the equations (40) is denominated the optimal Leapunov function of the system (39).

In[13] the recurrence formulae for the series development of V were found if the matrix $\left|\frac{\partial f_i}{\partial x_j}(0)\right|$ is diagonal. There is also demonstrated that if the matrix is not diagonal there is a transformation $s=g(x)$ with which the matrix $\left|\frac{\partial f_i}{\partial x_j}(0)\right| S$ is diagonal. Therefore the problem of finding V becomes the problem of finding W=VS in the same conditions as in [13]. Finally a relationship between the coefficients of W and V was established.

On the basis of these results the belonging to an initial condition of the flight to the domain of attraction of a stationary solution (corresponding to a constant control) is investigated upon the following procedure:

1) the stationary solution x_{sj} (j is the number of the solutions) of the system (5) is found with the method described in the Appendix;

2) the equations of the perturbed motion around this solution are developed:

$$\Delta\dot{X} = R(\Delta x, u) \tag{41}$$

where: $x_j = x_{sj} + \Delta x_j$;

3) the stability of the solution x_{sj} is analyzed (by finding the eigenvalues ($\lambda_1 \ \ \lambda_{10}$) of the matrix $\left|\frac{\partial R}{\partial x_j}(0)\right|$)

4) if the solution x_{sj} is stable (and in the general case there are more than one stable solutions) a transformation nonsingular matrix S which makes $\left|\frac{\partial R}{\partial \Delta x}(0)\right|$ diagonal is searched for by solving:

$$S^{-1}\left|\frac{\partial R}{\partial \Delta x}\right| S = \mathrm{diag}|\lambda_1, \ldots, \lambda_{10}| \tag{42}$$

5) the function:

$$W = VS = \sum_{m=2}^{m=\infty}\left(\sum_{j=1}^{j=m} B_{j_1 j_2 \ldots j_n} z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}\right)$$

is determined by solving the equations:

$$\sum_{i=1}^{i=10} \frac{\partial W}{\partial z_i} g_i = -\sum_{i=1}^{i=10}\left(\sum_{j=1}^{j=10} s_{ij} z_j\right)^2 \tag{43}$$

$$W(0) = 0$$

where:

$$z = S^{-1}\Delta x$$

$$g_i = \lambda_i z_i + \sum_{j=2}^{j=10} b^i_{j_1 j_2 \ldots j_{10}} z_1^{j_1} z_2^{j_2} \ldots z_{10}^{j_{10}}$$

$$b^i_{j_1 j_2 \ldots j_{10}} = \sum_{r=1}^{r=10} s'_{ir} \sum_{j=1}^{j=10} s_{jp} \sum_{q=j}^{q=10} a^r_{0,0..1+\sigma^j_g, \sigma^{j+1}_q, \sigma^{j+2}_q, \ldots \sigma^{10}_q} s_{qp}$$

if $j_p=2$ (44)

$$b^i_{j_1 j_2 \ldots j_{10}} = \sum_{r=1}^{r=10} s'_{ir} \sum_{j=1}^{j=10} \sum_{q=j}^{q=10} a^r_{0,\ldots 1+\sigma^j_{qj}, \sigma^{j+1}_q, \ldots \sigma^{10}_q}(s_{jk}s_{ql} + s_{jl}s_{qk})$$

if $j_k=1$ and $j_l=1$

$$\sigma^j_q = \begin{cases} 1 & \text{if } j = q \\ 0 & \text{if } j \neq q \end{cases}$$

s_{ij} the elements of the matrix S

s'_{ij} the elements of the matrix S^{-1}

The recurrence formulae for the terms B is:

$$B_{j_1 \ldots j_{10}} = -\frac{1}{2\lambda_{i_0}} \sum_{k=1}^{k=10} s^2_{kj_{i_0}} \quad \text{if } |j| = 2 \text{ and } j_{i_0} = 2$$

$$B_{j_1 \ldots j_{10}} = -\frac{2}{\lambda_p + \lambda_q} \sum_{k=1}^{k=10} s_{kj_p} s_{kj_q} \quad \text{if} \quad |j| = 2 \quad \text{and}$$

$$j_p = j_q = 1 \tag{45}$$

$$B_{j_1 j_2 \ldots j_{10}} = -\frac{1}{\sum_{i=1}^{i=10} j_i \lambda_i} \sum \sum (j_i - l_i + 1) b^i_{l_1 \ldots l_{10}} B_{j_1 - l_1 \ldots j_i - l_i + 1 \ldots j_n - l_n}$$

6)the belonging of an initial point $X_0^T = (u_0, v_0, w_0, p_0, q_0, r_0, \xi_0, \eta_0, \varsigma_0, \chi_0)$ to the domain of attraction to the stationary solution $X_s^T = (u_s, v_s, w_s, p_s, q_s, r_s, \xi_s, \eta_s, \zeta_s, \chi_s)$ is confirmed if the relation:

$$\sqrt{\sum_{i=1}^{i=10} |z_i|^2} < \frac{1}{\overline{\lim_m} \sqrt[m]{\sum_{|j|=m} \left| B_{j_1 \ldots j_{10}} (z_1^0)^{j_1} \ldots (z_{10}^0)^{j_{10}} \right|}} \tag{46}$$

is fulfilled.

Note that: $(z_1^0, z_2^0 \ldots z_{10}^0) = S^{-1}(u_0, v_0 \ldots \chi_0)$

This method was tested for the aircraft from[4]. For the calculation was proposed only the system with $\varepsilon=0$ approximation (equations 17.1) because the complete model has no asymptotically stable stationary solutions

It was found that such initial points as $(1,0,0,0,0)$; $(0,1,0,0,0)$; $(0,0,1,0,0)$; $(0,0,0,1,0)$ and $(0,0,0,0,1)$ belong to the domain of attraction of the principal tree of stable stationary solutions (that which begin in origin). It is very interesting that the initial conditions far from the stationary stable solution belong to their domain of attraction. Also this result gives a good mark to the method because another method[8] permits to test only initial conditions very closed to the stationary solutions.

Conclusion

The model of the motion of a maneuvering aircraft proposed here contains only nonlinearities in polynomial form. This allows an analytical estimation of the response of the aircraft to a constant control and a very good approximation of the domain of attraction of the stable stationary solutions. The method was tested for a very well known model of aircraft[4] and the results are in good agreement with those also known.

Though the first steps are cumbersome (there are a lot of algebraic calculations) the uses of symbolic processors make the task easy.

Some points must be checked in the future to have a pertinent response of the compatibility of the model with the real aircraft:

-the accuracy of the polynomial representation of the forces and moments acting on the aircraft;

-the real existence of some stationary solutions that result in the process of solving the nonlinear algebraic equations;

There will be also of interest a method that can visualize the borders of the domain of attraction or to make them intuitive (a domain in a 10 dimension space is difficult to understand).

Appendix

The signification of some vector and matrix notation

The signification of the notations used in the paper is the following:

$$a_1^T = (y_\beta, 0, y_p, 0, y_r - 1)$$

$$a_2^T = (0, z_\alpha, 0, z_q + 1, 0)$$

$$a_3^T = (\ell_\beta, 0, \ell_p, \ell_q, \ell_r)$$

$$a_4^T = (0, m_\alpha + m_{\dot\alpha} z_\alpha, m_p, m_q + m_{\dot\alpha} + m_{\dot\alpha} z_q, m_r)$$

$$a_5^T = (n_\beta, 0, n_p, n_q, n_r)$$

$$N_1 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{p\alpha} + 1 & 0 & 0 \\ 0 & y_{p\alpha} + 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$N_2 = \begin{vmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$N_3 = \begin{vmatrix} 0 & \ell_{\beta\alpha} & 0 & 0 & 0 \\ \ell_{\beta\alpha} & 0 & \ell_{p\alpha} & 0 & \ell_{r\alpha} \\ 0 & \ell_{p\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i_1 \\ 0 & \ell_{r\alpha} & 0 & -i_1 & 0 \end{vmatrix}$$

$$N_4 = \begin{vmatrix} 0 & 0 & -m_{\dot\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -m_{\dot\alpha} & 0 & 0 & 0 & -i_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i_2 & 0 & 0 \end{vmatrix}$$

$$N_5 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_{p\alpha} & 0 & 0 \\ 0 & n_{p\alpha} & 0 & -i_3 & 0 \\ 0 & 0 & -i_3 & 0 & 0 \end{vmatrix}$$

$$M_1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \quad M_2 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$M_3 = \varnothing \ ; \ M_4 = \begin{vmatrix} -m_{\dot\alpha} & 0 & 0 & 0 \\ 0 & -m_{\dot\alpha} & 0 & 0 \\ 0 & 0 & m_{\dot\alpha} & 0 \\ 0 & 0 & 0 & m_{\dot\alpha} \end{vmatrix} ; \ M_5 = \varnothing$$

$$Q_1 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \quad Q_2 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$

$$Q_3 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad Q_4 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$C_1 = (0,0,y_{\delta r}) \quad C_2 = (0,z_{\delta e},0) \quad C_3 = (l_{\delta a},0,l_{\delta r})$$
$$C_4 = (0,m_{\delta e},0) \quad C_5 = (n_{\delta a},0,n_{\delta r})$$

The roots of polynomial equations

The method of solving a system of equations

$$P(x)=0 \tag{47}$$

where $P(x)$ consists only of polynomials is based on the theorem from[14]:

If the system of equations $P(x)=0$ *has only polynomial components:*

$$P_1(x_1,x_2,\ldots x_n)=0$$
$$P_2(x_1,x_2,\ldots x_n)=0 \tag{48}$$
$$\ldots\ldots$$
$$P_n(x_1,x_2,\ldots x_n)=0$$

then there always exists a set of polynomials $Q(x)$ *whose components are:*

$$Q_1(x_1,x_2,\ldots x_n)=0$$
$$Q_2(x_2,x_3,\ldots x_n)=0$$
$$Q_3(x_3,x_4,\ldots x_n)=0 \tag{49}$$
$$\ldots\ldots$$
$$Q_n(x_n)=0$$

and the solutions of $P(x)=0$ *are identical with those of* $Q(x)=0$.

The procedure for that transformation is something similar to the method of Gauss for linear equations and is described in[15].

In the form (49) the system is easily solved by classical methods. The transformation from $P(x)=0$ to $Q(x)=0$ is cumbersome but with the symbolic processors is quite rapid.

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