APPLICATION OF MARKOV PROCESS THEORY TO INVESTIGATION OF AIRCRAFT OPERATIONAL PROCESSES

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Abstract

Aircraft operation is a stochastic process that can be divided into well-defined, discrete states of operation. This process can be represented with a graph model, and approximated with a continuous-time, discrete state space Markov process. Our present paper deals with Markov-, or Semi-Markov models of aircraft operation process and describes the possibility of their use for one specific purpose - for wartime aircraft operation.

Introduction

Stochastic processes whose development in the future is influenced by their development in the past only through their development in the present, that is stochastic processes without after-effects, are called Markov-processes.[5] The history of studying such processes started with the activity of Andrei Andreievitch Markov (1856-1922) the Russian mathematician.

Wartime aircraft operation - as usually operation itself - is a stochastic process based upon the aircraft, their maintenance, their preparation for combat mission, and also based upon the personnel carrying out aircraft repair, and upon the regulations.

This process, which is in fact the complex of events that happen to the aircraft, or to one of its systems, or its equipment (that is to the object of operation) from its manufacturing to its discarding, is a random in time and frequency succession of states of operation.

As leaving a certain state of operation does not depend on previous states, or their succession, that is the process has no after-effects, operation can be considered as a mathematically continuous time, discrete state space Markov-process. Such stochastic process can be approximated with a Markov-chain.

Decisions concerning an operation system, and its control-efficiency respectively, can be made on the basis of certain characteristic features. In our case such characteristic feature can be the number of combat-ready aircraft of a flying unit. During the study of the operation process such characteristic features can be established through the system approach, by means of the continuous-time, discrete-state space Markov-, and Semi-Markov models of the operation process.

Markov-Processes

The mathematically described probability process $\eta(t)$ is called Markov-process if the equation of hypothetical probabilities

$$P\left\{\eta(t_{n+1})=X_{n+1} \middle| \eta(t_1)=X_1 \ldots \eta(t_n)=X_n\right\}= =P\left\{\eta(t_{n+1})=X_{n+1} \middle| \eta(t_n)=X_n\right\} \quad (1)$$

proves to be true with the probability 1 for each $t_1<t_2<\ldots<t_n$ real number.[3]

If process $\eta(t)$ during the study period can have an X value at any moment, it is called a continuous-time process. If η can only have some value at certain moments, the process is called a discrete-time process. A stochastic process is considered to be of discrete state space, if the possible values of variate η constitute a finite set or a count non-finite set.

Finite or count non-finite stochastic processes, that is the discrete state space ones with no after-effects, are called Markov-chain.[1] In this case, the value established in the equation (1) is called the transition probability:

$$P_{ij}^{n,n+1}=P\left\{\eta(t_{n+1})=X_j \middle| \eta(t_n)=X_i\right\} \quad . \quad (2)$$

The transition probability expresses that $\eta(t_{n+1}) = X_j$; which in our case can be interpreted in the following way: the object of operation at t_{n+1} moment can be found in the j-th state, supposing that $\eta(t_n) = X_i$.

$P_{ij}^{n,n+1}$ marking above also shows that the transition probability is the function of not only the i beginning state and of the j end state, but it is also the function of t_n time. In order to have a simpler marking we are going to use the formula as follows:

$$P_{ij}^{n,n+1}=P_{ij}(t_n)=P_{ij}(t) \quad . \quad (3)$$

Having N number of states, P_{ij} transition probabilities are usually arranged in matrix.

$$\underset{=N\times N}{P}(t) = \left[P_{ij}(t) \right] \quad (4)$$

matrix is called the Markov-matrix of the process, or the transition probability matrix.

If the one-step transition probabilities are not time-dependent, we call the Markov-process stationary. In this case we can state that

$$P_{ij}^{n,n+1} = P_{ij} \quad , \quad (5)$$

or

$$\underset{=N\times N}{P} = \left[P_{ij} \right] \quad (6)$$

as it does not depend on the value of n, and P_{ij} means that the value of $\eta(t)$ is probably transiting from X_i to X_j during the $(t_n;t_{n+1})$ time interval.

A Markov-process can be characterized unambiguously by supplying the transition probabilities, and the distributions of leaving different states. If distribution of leaving different states are not of the same character - at least of them is different from the others, the stochastic process is called Semi-Markov process, as an example will illustrate it later on in this paper.

Mathematical Model of Operational Process

Operational process for each aircraft can be described with the so called operational chain, which is, from mathematical point of view a Markov-chain (Fig. 1).

FIGURE 1

When analysing operational processes with the system approach, the actual succession of single states for each aircraft is no concern of ours. It is rather complicated to describe the whole operational process with an operational chain. In order to achieve a clearer survey it is advisable to describe the operational process as a directed graph.[2]

Within the graph of operation states are represented by the angulare points of the graph, and transitions from one state to another are represented by the directed edges of the graph (see Fig. 2).

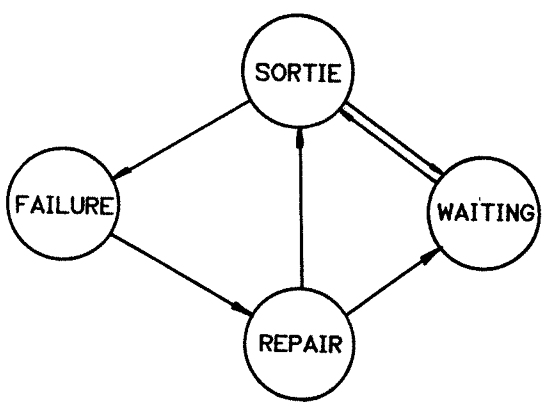


FIGURE 2

Analysing the operational chain or the type graph, we assume that states are clearly marked, and transitions occur during zero-time. For characterization of transitions from one state to another we use their transition probability.

The limit of transition probability P_{ij} below is called transition probability density, and marked with β_{ij}:

$$\beta_{ij} = \lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t)}{\Delta t} , \qquad (7)$$

where:

Δt – the length of the time interval.

Another characteristic feature is the relative frequency, that is the probability of staying in the i-th state:

$$P_i(\Delta t) \cong \frac{n_i(\Delta t)}{\sum_{j=1}^{N} n_j(\Delta t)} , \qquad (8)$$

where:

$n_i(\Delta t)$ – number of steps into the i-th state during Δt time.

The staying of the object of operation in the i-th state can also be characterized by the mean time of staying in given state, which is marked with t_i.

On the basis of which was said above, the operational process consisting of N states (in other words, which is divided into N states) can be characterized by the parameters below:

- N – number of states;
- $\underline{t}$ – vector of mean time periods spent in different states;
- $\underline{A}$ – probability vector of staying in different states;
- $\underline{\underline{P}}$ – transition probability matrix.

Naturally, instead of vector

$\underline{t}$, depending on the points of view of the analysis, vector $\underline{C}$ of cost of entering given state or vector $\underline{M}$ of cost of labour consumption can also be considered.

Knowing the characteristics above we are able to determine the change in time of probability of staying in different states, and the requirements for operational cost or for working hours of operation.

Then we have a system of equations consisting of as many equations, as many states we have within the operational process.[6]

Transition probabilities of changing states of the continuous-time process analysed with Δt time-shifting (which transforms it into a discrete-time process) can be determined with the help of equation (7) in the following way:

$$P_{ij}(t) = \beta_{ij}(t)\ \Delta t \quad . \qquad (9)$$

It is important to mention at this moment that we are supposed to choose such intervals during which the object of operation will perform only one change of states with the probability 1. The variates above can be arranged into the Markov-matrix which was introduced earlier.

For the sake of further analyses it is advisable for us to consider the case a change of states where after Δt time period the object of operation remains in the same state as prior to that time interval. So the determination of variates in the main diagonal of the matrix is carried out as follows:

$$P_{ii} = 1 - \sum_{j=1}^{N} P_{ji} \quad (\text{if } i \neq j) \ . \ (10)$$

As the total space means that the object of operation enters into a new state or it remains in the beginning state.

Using the Markov-matrix, the change in time of the probability of staying in different states can occur according to the equation

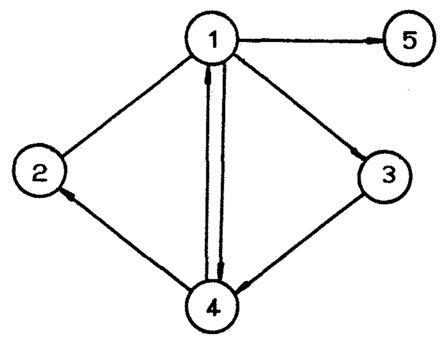
$$\underline{A}(t+\Delta t) = \underline{\underline{P}}^{*}(t)\ \underline{A}(t) \quad , \qquad (11)$$

where $\underline{\underline{P}}^{*}$ is the transposed matrix of $\underline{\underline{P}}$.

Modelling Wartime Aircraft Operation

Setting the graph model

On the basis of data available for us, we modelled wartime aircraft operation with a continuous-time stochastic process, which consists of five states. So the type graph of operation is a directed graph consisting of five angular points, which can be seen in figure 3.



3.FIGURE

The names of the states are:

1 - Sortie

The aircraft are accomplishing assigned mission, they are in combat-engagement, or approaching assigned airspace, respectively they are flying to the airfield assigned to them.

2 - Type 'A' repair, or waiting for repair

The aircraft is damaged to an extent which allows its repair at the flying unit. Mean time spent in this state: 3 hours.

3 – Type 'B' repair, or waiting for repair

The aircraft is damaged more seriously, but it still can be repaired at the unit. Mean time spent in this state: 8 hours.

4 – Ready for operation

The aircraft is ready for operation. It is either under pre-flight routine or is waiting for the next sortie in combat-ready state.

5 – Non-recoverable loss

The aircraft entering this stage are damaged to an extent which does not allow to use them again during the combat operation. These aircraft form two groups: to one of them belong the aircraft which are called non-recoverable loss, the other one consists of the aircraft that suffered damage of different degrees and cannot be repaired at the unit.

The Markov-model

Firstly we assumed that the time of leaving each state has exponential distribution, and then we determided the coefficient matrix of transition probabilities of changing states (table I.), and so we got the Markov-model of operation.

	1	2	3	4	5
1		$\frac{k_A}{t_1}$	$\frac{k_B}{t_1}$	$\frac{1-k_{fl}}{t_1}$	$\frac{k_{nr}}{t_1}$
2	0		0	$\frac{1}{t_2}$	0
3	0	0		$\frac{1}{t_3}$	0
4	$\frac{1}{t_c}$	0	0		0
5	0	0	0	0	

TABLE I.

This model fundamentally corresponds to the mathematical model of normal "peace time" operation process. It is useful for the modelling of the so called service of the continuous air combat activity. The activity of allied flying units inspecting the no-fly zone in South Iraq after the Gulf War is a good example of this type of combat mission. It is to be noted that in similar cases, when valuating the prospective factors of loss, besides the effectiveness of enemy's air defense, there should be considered the political "background" as well.

The disadvantage of this Markov-model is that it is not usable for the analysis with the system approach of the service process of combat activity when a flying unit is accomplishing mission with all its aircraft at a time. For instance, such activities can be series of air-raids (see Operation Desert Storm), air support of ground forces or air dropping on the enemy territory. The air transport and air drop of aid supplies in the Yugoslavian area can also be considered such mission.

The Semi-Markov Model

Because of the limitations of the Markov-model, for the modelling of the service of combat activity requiring recurrent sorties we set up the Semi-Markov model of wartime operation. After analysing each state and changes of states we determined the character of distributions of time for leaving each state. On the basis of the results of the analysis we determined the elements of the matrix in table II.

We considered changes of states due to faliure, damage or destory to be of normal distribution, where:
the prospective time of leaving is

$$m = \frac{t_f}{2};$$

its variance is:

$$\sigma = \frac{t_f}{6}$$

due to so called '3σ rule', according to which, the values of the variate of prospective value m, and of variance σ of normal distribution will fall "practically certainly" in the $(m-3\sigma$, $m+3)$ interval (its probability in fact is 0,9973).

The first and last leg of the sortie is actually a flight en route (approaching assigned airspace or airfield), that is why the probability of failure due to damage is much lower than it is during the air combat (the difference between the probabilities can be of one or more than one order).

On the basis of the Markov-model, of the experience and specialist literature, we assume that the times of staying in repair states are of exponential distribution.

	1	2	3	4	5
1		$k_A\ F(t)$	$k_B\ F(t)$	$t=b*t_c+t_1 \rightarrow 1$ $t\neq b*t_c+t_1 \rightarrow 0$	$k_{nr}\ F(t)$
2	0		0	$1-e^{\mu_A*t}$	0
3	0	0		$1-e^{\mu_B*t}$	0
4	$t=b*t_c \rightarrow 1$ $t\neq b*t_c \rightarrow 0$	0	0		0
5	0	0	0	0	

TABLE II.

We determined the transition between state 1 (sortie) and 4 (ready for operation) in the way described in table II. as if it was a step function. In that way I could model the case when the aircraft fly out for the sortie at the same time, or more precisely within a relatively short time, and come back the same way.

Markings used in the tables and equations:

k_A - factor of sending the aircraft to type 'A' repair;

k_B - factor of sending the aircraft to type 'B' repair;

k_{nr} - factor of non-recoverable loss;

k_{fl} - full-loss factor:

$$k_{fl} = k_A + k_B + k_{nr} ; \qquad (12)$$

t_2 - mean time of type 'A' repair;

t_3 - mean time of type 'B' repair;

t_1 - mean time of sortie;

t_c - cycle time of sortie (assuming cyclical sorties);

b - serial number of the sortie;

$$F(t) = \frac{6}{t_1\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{18\left(\tau-\frac{t_1}{2}\right)^2}{t_1^2}} d\tau ; \qquad (13)$$

$$\mu_A = \frac{n_A \ \eta_w \ \eta_p^k}{t_2 \ C_2} ; \qquad (14)$$

$$\mu_B = \frac{n_B \ \eta_w \ \eta_p^k}{t_3 \ C_3} ; \qquad (15)$$

n_A - number of teams doing type 'A' repair;

n_B - number of teams doing type 'B' repair;

η_w - factor characterising working time loss;

η_p - factor representing loss in personnel;

k - the serial number of the day in question;

C_i - number of aircraft staying in i-th state.

Comparison of Models

On the basis of two tables above, using both models, we determined the change of probabilities of staying in different states depending on time. These results are shown in figure 4 (Markov-model), and figure 5 (Semi-Markov model). Naturally, starting data in both cases are the same.

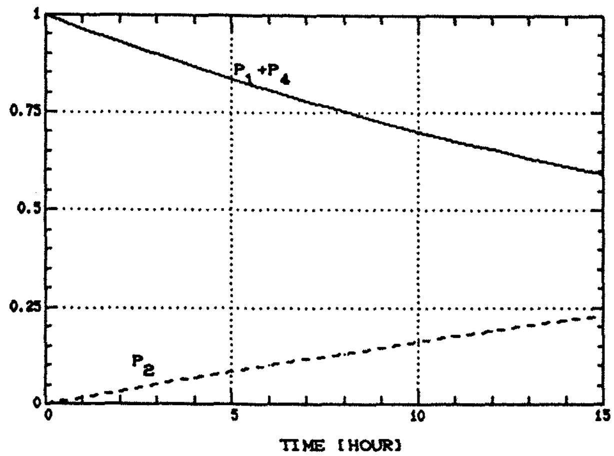


FIGURE 4

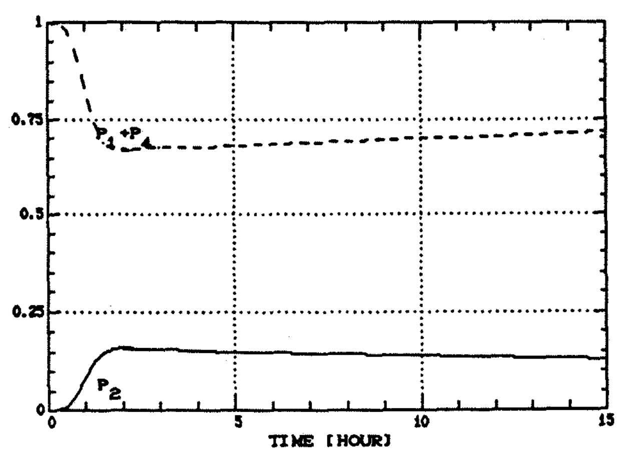


FIGURE 5

Using these models different varieties of damage can be modelled with the change of loss factor.[4] For example, if the enemy expectedly is going to use small arms - naturally, it is not the Stinger rocket missile what we have in mind here -, or we have strong air superiority, then, supposingly, after the sorties, we are going to have an increased number of aircraft to be sent to type 'A' repair. But if the enemy has a highly developed air defense system, or we do not have the necessary air superiority, in that case the amount of non-recoverable loss will increase in comparison with the former case.

An Example for the Use of Semi-Markov Model

In order to demonstrate the possibilities of use of the Semi-Markov model, we made it operate with different starting data.

As the first step, with the help of the data taken down by us, we determined the change in the number of aircraft during a 16-hour combat activity. The aircraft carry out four missions during this period, the first one between 00.00 - 02.00, the second one between 04.00 - 05.00, the third one between 07.00 - 08.00, and the forth one between 10.00 - 11.00 hours. On the base airfield there are 6 teams working for type 'A' repair, and 4 teams for type 'B' repair. Graphs based on the list of results can be seen in figure 6.

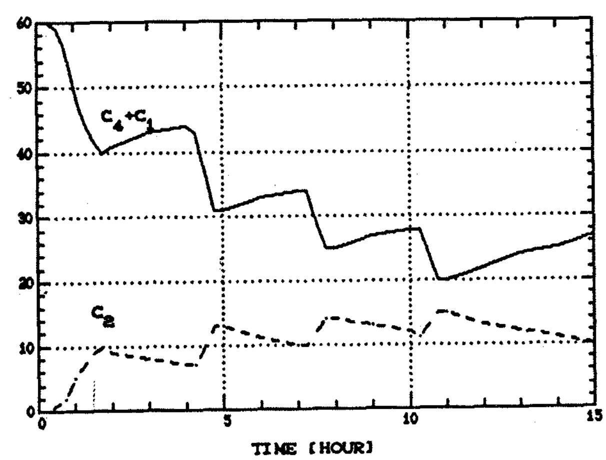


FIGURE 6

The diagrams contain the expected number of aircraft being in type 'A' repair state or those of waiting for type 'A' repair

(C_2), and also those of ready for operation and being out for mission (C_4+C_1).

Our next step was modelling the fact that enemy has mostly small arms. So we increased the factor of sending aircraft into type 'A' repair, while the other loss factors were reduced. On the basis of running results we got figure 7.

Comparing the two graphs we can state that without any structural change, by the end of the period (with the same characteristics) it is only the proportion of the number of aircraft waiting for repair of different levels that has changed. The total number of sorties has not changed, neither has the number of aircraft ready for operation changed significantly (28 instead of 29) by the end of the period.

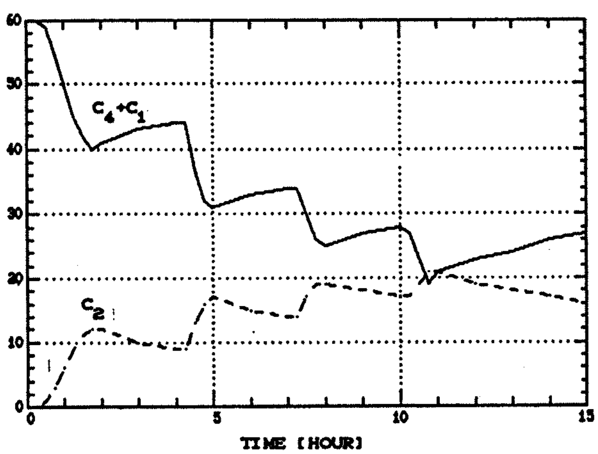


FIGURE 7

According to these results we changed the number of teams performing repairs. We increased the number of teams performing type 'A' repair from 6 to 8, and reduced those of performing type 'B' repair from 4 to 3. In our present paper, while changing the number of repair teams we disregarded the aspect of how many persons the teams need in each specialty. So we do not doubt that there might be an error in the modification modelled by us, we would only like to underline that our main objective here is to demonstrate one possible use of the model.

In this case the total number of sorties has increased by 3, and the number of aircraft ready for operation by the end of the period increased by 2 (see Fig. 8). But due to the increased number of sorties, unfortunately, the number of total loss has also increased by 2. Considering the results, if such loss can be expected, it seems to be advisable to modify the structure of technical personnel in the way described above.

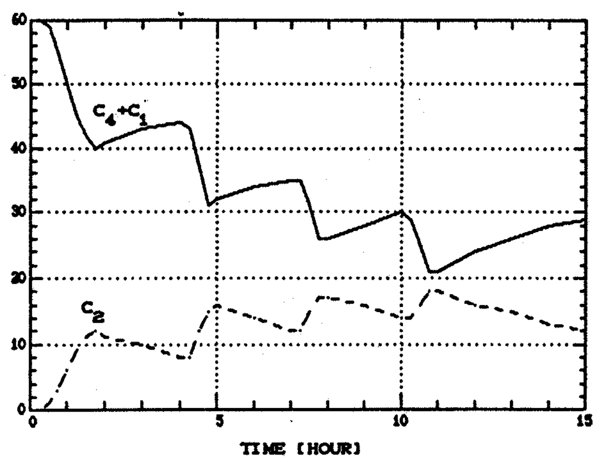


FIGURE 8

Using the models above there can be studied the effects of forming repair and preparation teams in the case of different damage probabilities, that is, in the case of different enemy weapon systems or of air superiority of different level. The results can be used by the flight commander as information data, when planning the combat activity of his unit. The model can also be used during the preliminary planning of air operations of different level for predicting the volume and distribution in time of expected technical service capacity and material needs.

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